Notes

95.01 The rational distance problem

Introduction

It is not known whether there is a point in the plane of a unit square, that is at a rational distance from each of the four corners. See [1, 2, 3]. Here, we give a negative answer for boundary points, using the non-existence of particular Pythagorean triangles.

Throughout, variables will denote positive integers, and (a, b), (a, b, c) will denote HCFs. A Pythagorean triangle is a triple $T = [a, b, c]$ satisfying $a^2 + b^2 = c^2$. T is called primitive (PPT) if further $(a, b, c) = 1$. Recall that:

(R0) If $[a, b, c]$ is a PPT, then, for some $p > q$, with $(p, q) = 1$ and $p + q$ odd, we have

$$\{a, b\} = \{p^2 - q^2, 2pq\} \text{ and } c = p^2 + q^2.$$

(R1) In a PPT, the hypotenuse is odd, and the even side is divisible by 4.

The following is left as an exercise:

(R2) Suppose that $xy = pq$ with $(x, y) = 1$ and $(p, q) = 1$. Then $x = p_1q_1$, $y = p_2q_2$, where $p_1p_2 = p$, $q_1q_2 = q$ and p_1, p_2, q_1, q_2 are pairwise coprime.

(R3) A system of the form $x^2 - y^2 = z^2$, $4x^2 + y^2 = t^2$, where x, y, z, t are pairwise coprime, is impossible.

Proof: Otherwise, $[y, z, x]$ is a PPT, so x is odd. Hence $2x \equiv 2 \pmod 4$. But $[2x, y, t]$ is a PPT with $2x$ even. Hence, by (R1), $2x \equiv 0 \pmod 4$. We obtain a contradiction.

The results

Proposition 1: There is no Pythagorean triangle of the form $T = [a^2 + b^2, ab, c]$. Stated otherwise, the Diophantine equation $X^4 + Y^4 + 3X^2Y^2 = Z^2$ has no solution in non-zero integers.

Lemma: Suppose that $t^2(x^2 - y^2) = z^2(4x^2 + y^2)$, where x, y, z, t are pairwise coprime. Then $[y, 2z, t]$ and $[t, z, x]$ are both PPT's.

Proof of the Lemma: Set $N = x^2 - y^2$, $D = 4x^2 + y^2$. Rewrite the relation as $\dfrac{N}{D} = \dfrac{z^2}{t^2}$ (1) where the fraction $\dfrac{z^2}{t^2}$ is in lowest terms. Set $d = (N, D)$. Because d divides $N + D = 5x^2$, d divides $D - 4N = 5y^2$, and $(x, y) = 1$, we easily find that $d = 1$ or 5. If $d = 1$, N/D is in lowest terms, hence, $N = 5z^2$ and $D = t^2$, which is impossible by (R3). Hence $d = 5$ and hence $N = 5z^2$ and $D = 5t^2$, that is, $x^2 - y^2 = 5z^2$ and $4x^2 + y^2 = 5t^2$. Eliminating first x^2, we get $y^2 + (2z)^2 = t^2$. Hence, as $(y, t) = 1$, $[y, 2z, t]$ is a PPT. Eliminating next y^2, we get $t^2 + z^2 = x^2$. Hence, clearly, $[t, z, x]$ is a PPT.

Proof of Proposition 1:

We use infinite descent: suppose that $T = [a^2 + b^2, ab, c]$ is a Pythagorean triangle. By factoring out $\delta = (a, b)$, we may assume $(a, b) = 1$. This operation is non-increasing for T. Now, clearly, $(a^2 + b^2, ab) = 1$, so T is a PPT. By (R0), for some coprime p and q $(p > q)$ with $p + q$ odd, we have

$$\{a^2 + b^2, ab\} = \{p^2 - q^2, 2pq\} \text{ and } c = p^2 + q^2.$$

If we had $a^2 + b^2 = 2pq$, $a + b$ would be even. Since $(a, b) = 1$, a and b would be odd and we would get $a^2 + b^2 \equiv 2 \pmod 4$, which contradicts with $2pq \equiv 0 \pmod 4$. We conclude that

$$a^2 + b^2 = p^2 - q^2 \qquad \text{and} \qquad ab = 2pq \qquad (*).$$

Now, a, b have opposite parity and play symmetric roles. We may assume a even. Set $a = 2\omega$. Relations (*) become

$$4\omega^2 + b^2 = p^2 - q^2 \quad (1) \qquad \omega b = pq \quad (2)$$

with $(\omega, b) = 1$. From (2) and (R2) we get $\omega = xz$, $b = ty$, $p = xt$, $q = zy$, x, y, z, t pairwise coprime. (1) can be rewritten as $t^2(x^2 - y^2) = z^2(4x^2 + y^2)$. By the lemma, $[y, 2z, t]$ and $T_1 = [t, z, x]$ are both PPTs. In particular, by (R0),

$$y = r^2 - s^2, \qquad 2z = 2rs, \qquad t = r^2 + s^2.$$

Finally, the Pythagorean triangle T_1 has the same form as T, namely

$$T_1 = [t, z, x] = [r^2 + s^2, rs, x]$$

where $r^2 + s^2 = t \leqslant ty = b < a^2 + b^2$ and $rs = z \leqslant xz = \omega = \frac{1}{2}a < a \leqslant ab$. That is, T_1 is strictly smaller than T.

Corollary: There is no Pythagorean triangle of the form $T = [a^2 - b^2, c, ab]$. Stated otherwise, the Diophantine equation $X^4 + Y^4 + Z^2 = 3X^2Y^2$ has no solution in non-zero integers with $X^2 \neq Y^2$.

Proof: Otherwise, there is such a triangle T with $(a, b) = 1$, so $(a^2 - b^2, ab) = 1$. Hence T is a PPT. By (R1), ab is odd, so $a^2 - b^2$ is even. By (R0), $a^2 - b^2 = 2pq$ and $ab = p^2 + q^2$. Hence $a^2(-b^2) = -(p^2 + q^2)^2$. Now, a^2 and $-b^2$ are the roots of $t^2 - 2pqt - (p^2 + q^2)^2 = 0$. The (reduced) discriminant must be a perfect square. That is, $(p^2 + q^2)^2 + (pq)^2 = r^2$, where clearly $r \neq 0$. This contradicts Proposition 1.

Proposition 2: There is no point on the perimeter of a unit square that is at rational distance from the four corners.

Proof: Let $A_1A_2A_3A_4$ be a unit square (in cyclic order), and M the midpoint of A_1A_2. Let P be a point on A_1A_2, say on A_1M. Set $d_i = PA_i$, $i = 1, \dots, 4$ and suppose the d_i are all rational. We multiply by an appropriate integer so all the d_i are even (positive) integers. Set $A_1A_2 = 2m$, $d_3 = 2p$, $d_4 = 2q$, $PM = n$. The Pythagorean relations are $4m^2 + (m+n)^2 = 4p^2$ (3) and $4m^2 + (m-n)^2 = 4q^2$. Subtracting, we get $p^2 - q^2 = mn$. Hence $(5m^2)n^2 = 5(p^2 - q^2)^2$. Replacing mn by $p^2 - q^2$ in (3), we get $5m^2 + n^2 = 2(p^2 + q^2)$. Now, $5m^2$ and n^2 are the roots of

$$t^2 - 2(p^2 + q^2)t + 5(p^2 - q^2)^2 = 0.$$

Since $5m^2 = n^2$ is impossible, the (reduced) discriminant

$$\delta = (p^2 + q^2)^2 - 5(p^2 - q^2)^2 = 4(3p^2q^2 - p^4 - q^4)$$

is a non-zero perfect square, and so is $\delta / 4$. Therefore, for some $r \neq 0$, we have $p^4 + q^4 + r^2 = 3p^2q^2$, which contradicts the Corollary.

References

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2. J. H. J. Almering, Rational quadrilaterals, *Indag. Mat.* **25** (1963), pp. 192-199.
3. T. G. Berry, Points at rational distance from the corners of a unit square, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **17** (1990), pp. 505-529.

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95.02 Commensurable triangles

The problem I would like to address is the following. It was posed and solved recently by Richard Parris [1], so what I have to say was inspired by him. But my solution is more direct.

We are given two positive integers h and k, which we may suppose are relatively prime, and we want to construct triangles in which the ratio of one angle to another is $h : k$.

This problem was motivated in part by the 4-5-6 triangle, in which one angle is twice amother.

Let $\frac{p}{q}$ be any rational with p and q relatively prime, with

$$1 > \frac{p}{q} > \cos\frac{\pi}{h + k}.$$

Define α by

$$\cos\alpha = \frac{p}{q}.$$