# NONLINEAR ELECTROMAGNETIC–SPIN WAVES


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*Abstract*:

Nonlinear wave excitations of ferromagnets, caused by the interaction of the solenoidal electromagnetic field with magnetization oscillations, i.e., coupled electromagnetic–spin waves, are investigated. The analysis is carried out in a framework of a phenomenological approach with the use of the complete set of Maxwell equations and the magnetization dynamics (Landau–Lifshitz) equation. A broad class of nonlinear waves is studied, including monochromatic, quasimonochromatic, solitary, Riemann and shock waves in ferromagnets and antiferromagnets. The effect of an external magnetic field, magnetic anisotropy, energy dissipation and finite dimensions of the sample on the propagation of nonlinear electromagnetic–spin waves is considered.

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## Preface

Investigations of nonlinear waves in magneto-ordered substances started long ago, and since then considerable results have been obtained. Yet, as a rule, principal attention has been paid to the spin branch, both in insulating ferromagnets and antiferromagnets. However, from the viewpoint of applications and from the viewpoint of theory, electromagnetic-spin waves (for brevity, EMS waves in the sequel) make as good an object of investigation as spin waves. Moreover, in the investigation of currently actual millimeter and submillimeter ranges EMS waves are preferable.

A sizeable stock of results on EMS waves has been accumulated up to now, but the articles describing them are scattered in various sorts of publications, not easily accessible. In order to take the next step in studying EMS waves, it is desirable to gather and sum up the results obtained thus far. This is the purpose of the present survey. From the material gathered in it one can see that EMS waves are rather diverse. They can be monochromatic but nonstationary, or stationary, shock, soliton-type, etc. We hope this survey may stimulate the activity in the field, since, despite the large number of similar results, a review of new types of waves is of interest; the reason being that the waves considered are essentially different in their nature and properties. The propagation of EMS waves in a waveguide is also examined in the survey. This problem is actual since the situation readily lends itself to technical applications. But this problem is the only one of its kind.

Now we will comment on material not present in the survey. They are some investigations which have not been carried out, though the need for them is pressing. First, the nonlinear wave excitation; in some cases the equation initially set up can be reduced to an integrable equation (Burgers, sine-Gordon, NSE, KdV). Here the problem can be treated with the inverse scattering method. In other cases one can use the automodelling technique, which however becomes rather difficult for an experimental realization.

The interaction of nonlinear waves with obstacles is of great interest. This interaction generates a specific emission which can be used for the nonlinear wave diagnostics. We hope the survey will also initiate such studies.

## 1. Introduction

Owing to the Zeeman energy of the magnetic moment in a magnetic field, the electromagnetic wave propagation can be accompanied by the excitation of magnetization oscillations. On the other hand, the wave-induced time-dependent magnetization variation leads, according to the electromagnetic induction law, to the generation of a solenoidal electromagnetic field. The interaction of the solenoidal field with the magnetization oscillations results in coupled electromagnetic–spin waves [1–3].

In contrast to slow spin and magnetostatic magnetization waves [4–6], EMS waves propagate with a velocity comparable to that of light, which means that electrodynamic effects must be accounted for in the description of such waves. Besides, the space scale of EMS waves substantially exceeds the characteristic length of the inhomogeneous exchange interaction [4] that determines the space scale of spin waves. This permits one to proceed with the treatment of EMS waves without taking into account the inhomogeneous exchange, which drastically simplifies the analysis [4].

Typical frequencies of EMS waves belong to the microwave range, so the devices whose operation is based on such wave processes are broadly used in modern microwave technology. Thus, practical requirements here served as a stimulus for the thorough experimental and theoretical research of the

EMS wave propagation in ferromagnetic media (see, e.g., monographs in refs. [6–9] and the literature cited there).

In the investigations of the physical processes and applications relating to EMS waves primary attention has been paid to studying weakly excited states of ferromagnets which can be described by using a linearized set of Maxwell and magnetization dynamics equations (the Landau–Lifshitz equations from ref. [10]). As the exitation energy grows, the EMS wave propagation is accompanied by new effects, such as monochromatic wave self-action, interaction of various elementary wave excitations, etc., whose description requires using a nonlinear approximation. Certain types of EMS waves (solitary or shock waves) are substantially nonlinear formations which can be described only by means of nonlinear equations.

It should be noted that nonlinear EMS waves are investigated far less than nonlinear spin waves (see, e.g., refs. [11–13]) though the first investigations of shock waves in ferromagnets [14] preceded the first attempt to analyze nonlinear spin waves [15].

In the present work we will consider a broad class of nonlinear EMS waves. In the second section of this paper we will investigate the properties of monochromatic finite-amplitude waves. Such waves which are rather similar to elementary wave excitations in a ferromagnet, i.e., to linear monochromatic waves, can propagate along a d.c. magnetic field in an isotropic ferromagnet or along the anisotropy axis in a one-axis ferromagnet. The space structure of monochromatic waves will be considered both in ferromagnets and antiferromagnets; penetration of a wave into a ferromagnet and the resulting effect of nonlinear self-brightening of the ferromagnet will be also studied, as well as the structure of the two-dimensional weakly nonlinear EMS waves in a ferromagnet plate.

Section 3 of the work is devoted to studying solitary EMS waves in ferromagnetic media. These waves differ from magnetostatic magnetization solitons (the latter have been actively studied) in that they have a greater propagation velocity in comparison with that of light. We will consider waves both in the isotropic and anisotropic ferromagnets, as well as in an antiferromagnet. The properties of two-dimensional solitary waves in a ferromagnet plate will be examined.

In section 4 nonlinear EMS waves will be investigated, which are waves whose existence is caused by the dissipative nature of the magnetization dynamics, i.e., shock waves. We will study the evolution of simple waves in a ferromagnet and an antiferromagnet. The front structure of stationary shock waves will be studied in one- and two-sublattice ferromagnets.

The article will be concluded with an analysis of nonstationary shock waves in a nonsaturated ferromagnet.

## 2. Monochromatic and quasimonochromatic waves

### 2.1. Interaction of electromagnetic and spin waves

Within the framework of the phenomenological approach used in this work, EMS waves in a ferromagnetically ordered dielectric are described by the complete set of Maxwell equations,

$$\operatorname{rot}\operatorname{rot}\boldsymbol{H} + (\varepsilon_0/c^2)\partial^2(\boldsymbol{H} + 4\pi\boldsymbol{M})/\partial t^2 = 0\,, \qquad \operatorname{div}(\boldsymbol{H} + 4\pi\boldsymbol{M}) = 0\,, \tag{2.1}$$

and by the Landau–Lifshitz equation,

$$\frac{\partial \boldsymbol{M}}{\partial t} = -g\boldsymbol{M} \times \boldsymbol{H}_{\text{eff}} + \frac{\eta}{M_0}\boldsymbol{M} \times \frac{\partial \boldsymbol{M}}{\partial t}, \qquad \boldsymbol{H}_{\text{eff}} = \boldsymbol{H} + \beta \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{M}) + \alpha \nabla^2 \boldsymbol{M}, \tag{2.2}$$

where the dissipative term is given in the Gilbert form [16]. Here $\beta$ and $\alpha$ are the constants of the magnet anisotropy and of the inhomogeneous exchange, respectively, $\boldsymbol{n}$ is the unit vector directed along the anisotropy axis, $g$ is the gyromagnetic ratio, $M_0$ is the magnetic moment of saturation, $\eta$ is a dimensionless relaxation constant.

In case of small-amplitude waves the system (2.1), (2.2) can be linearized and thus reduced to the problem of an analysis of the dispersion relation [3] relating the frequencies and wave vectors of elementary wave excitations of a ferromagnet. The results of such an analysis are described in detail in the monograph of ref. [4].

In case of finite-amplitude waves there appear some singularities which will be considered for the simple example of circularly polarized plane monochromatic waves propagating along the anisotropy axis in a nondissipative light-axis ferromagnet. One can easily see that such waves are described by the solutions of the system (2.1), (2.2); the solutions must have the form

$$M_\perp = M_x + \mathrm{i}M_y = M_0 \sin\theta \exp(\mathrm{i}\omega t - \mathrm{i}kz), \tag{2.3}$$

with constant values for the parameters $\theta$, $\omega$ and $k$, $\theta$ being the deviation of the magnetization vector from the direction of wave propagation.

Substituting expression (2.3) into eqs. (2.1) and (2.2) yields a dispersion relation,

$$\omega_\mathrm{s}(k, \theta) - \omega = 4\pi g M_0 \cos\theta \, (\varepsilon_0 \omega^2/c^2)/(k^2 - \varepsilon_0 \omega^2/c^2), \tag{2.4}$$

where

$$\omega_\mathrm{s}(k, \theta) = g(H_0 + 4\pi M_0) + gM_0(\beta - 4\pi + \alpha k^2)\cos\theta \tag{2.5}$$

is the nonlinear spin wave frequency [4]. The quantity $H_0$ denotes the external magnetic field applied along the anisotropy axis.

Let us examine the dispersion and nonlinear properties of the waves defined by eq. (2.4). The substantially nonlinear wave processes in which we are interested occur in the frequency range $\omega \sim gM_0 \sim 10^{10}\ \mathrm{s}^{-1}$, which is typical for the phenomenon of ferromagnetic resonance. In this frequency range, if one takes into account the smallness of the coefficient $\omega\sqrt{\alpha\varepsilon_0}/c \sim \sqrt{\alpha\varepsilon_0}gM_0/c \sim 10^{-5}$, the general formula (2.4) implies a simple expression,

$$k^2(\omega, \theta) = \omega^2 \frac{\varepsilon_0}{c^2} \mu(\omega, \theta) = \omega^2 \frac{\varepsilon_0}{c^2} \frac{\omega_\mathrm{a}(\theta) - \omega}{\omega_\mathrm{s}(0, \theta) - \omega}. \tag{2.6}$$

For $\omega < 0$ the above describes the spectrum of waves with left rotation of the polarization plane, while in the frequency range $0 < \omega < \omega_\mathrm{s}(0, \theta)$ and for $\omega > \omega_\mathrm{a}(\theta) = \omega_\mathrm{s}(0, \theta) + 4\pi g M_0 \cos\theta$ it describes the spectrum of slow ($\omega/k < c/\sqrt{\varepsilon_0}$) and fast ($\omega/k > c/\sqrt{\varepsilon_0}$) waves with right polarization. It is essential that the inhomogeneous exchange interaction does not affect the spectrum of the waves described by relation (2.6). In the process of propagation of such waves, however, the excitation degree of the ferromagnet spin subsystem characterized by an angle of deviation of the magnetization from the equilibrium axis $\theta$ can be rather large. This fact justifies the introduction of the term coupled EMS waves, for it points out the self-consistent variation of the magnetization with the magnetic field in the process of wave propagation.

One can easily see that in the frequency range $|\omega| \gg \omega_a(\theta)$ the left-polarized waves of an arbitrary amplitude, as well as the right-polarized waves of the fast variety degenerate into purely electromagnetic waves with the phase velocity $\omega/k = c/\sqrt{\varepsilon_0}$ [4]. On the other hand, eq. (2.6) shows that the degeneration can also occur at an arbitrary frequency $\omega$ for all waves due to the substantially nonlinear saturation effect of the magnetic permeability, $\mu(\omega, \theta) \to 1$. The growth of the wave amplitude $H_\perp$ is related with the precession angle of magnetization by the relation

$$g|H_\perp| = g|H_x + \mathrm{i}H_y| = |\omega_s(0, \theta) - \omega| \tan\theta\,.$$

Affected by the field, the magnetic permeability $\mu(\omega, \theta)$ can change not only in value, but also in sign. If as the amplitude grows the function $\mu(\omega, \theta)$ changes from negative to positive, then the ferromagnet undergoes self-brightening; otherwise it becomes opaque (undergoes self-dimming).

For instance, let $\beta > 4\pi$ while the frequency $\omega$ is in the opaque region for linear right-polarized waves,

$$g(H_0 + 4\pi M_0) < \omega < g[H_0 + (\beta + 4\pi)M_0]$$

[according to the dispersion relation (2.6) $k^2 \sim \mu(\omega, 0) < 0$ in the given frequency range]. Equation (2.5) implies that the ferromagnet becomes transparent in the considered case if the wave amplitude exceeds a critical value determined by the condition

$$\theta > \arccos\{[\omega - g(H_0 + 4\pi M_0)]/g\beta M_0\}\,.$$

On the other hand, a ferromagnet transparent to linear waves in the frequency range $g(H_0 + 4\pi M_0) < \omega < g(H_0 + \beta M_0)$ becomes opaque to waves with an amplitude satisfying the condition,

$$\theta > \arccos\{[\omega - g(H_0 + 4\pi M_0)]/g(\beta - 4\pi)M_0\}\,.$$

Equation (2.6) correctly describes the dispersion of the right-polarized waves of the slow branch in the wave number domain

$$k = k_s = [\omega_s(0, \theta)/c]\sqrt[4]{4\pi\varepsilon_0 c^2/\alpha\omega_s^2(0, \theta)} \sim 10^2\, gM_0/c\,.$$

In the domain $k > k_s$ the slow EMS waves degenerate, becoming nonlinear spin waves with the dispersion law

$$\omega = \omega_s(k, \theta)\,, \tag{2.7}$$

where the functional form of $\omega_s(k, \theta)$ is given by formula (2.5). Comparing the results (2.6) and (2.7), one can conclude that the inhomogeneous exchange interaction is able to essentially affect the EMS wave dispersion only for such $\omega$ and $k$ for which these waves differ only slightly from spin waves [4].

### 2.2. Self-action of a monochromatic wave in a ferromagnet

Before analyzing the solutions of the system (2.1), (2.2) it is convenient to make the variables entering these equations dimensionless. So we will set

$$\boldsymbol{H} = 4\pi M_0 \boldsymbol{h}\,, \qquad \boldsymbol{M} = M_0 \boldsymbol{m}\,, \quad |\boldsymbol{m}| = 1\,, \qquad \tau = 4\pi g M_0 t\,, \qquad c\xi = 4\pi g M_0 \sqrt{\varepsilon_0}\, z\,; \qquad \beta = 4\pi\beta_0\,.$$

For the sake of simplicity we will restrict ourselves to the case of an isotropic ferromagnet magnetized by a field $h_0$ in the direction of the wave propagation. The solutions of the system (2.1)–(2.2) will be sought in the long-wave limit when the effect of the inhomogeneous exchange interaction may be neglected. One can easily see that the plane wave (2.3) is a particular case of monochromatic magnetization waves of the form [17, 18]

$$m_\perp = \sin\theta(\xi)\exp[\mathrm{i}\omega\tau - \mathrm{i}\Psi(\xi)]\,. \tag{2.8}$$

The quantities $\theta(\xi)$ and $\Psi(\xi)$ determining the magnetization state vary self-consistently with the magnetic field of the propagating wave (for the self-action effect see refs. [19, 20]).

By substituting expression (2.8) into the initial equations we get a nonlinear wave equation,

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} + \omega^2[\mu'(\omega,\theta) - \mathrm{i}\mu''(\omega,\theta)]W = 0\,, \tag{2.9}$$

where

$$h_\perp = W\,\mathrm{e}^{\mathrm{i}\omega\tau}\,, \quad W = R(\xi)\,\mathrm{e}^{-\mathrm{i}\varphi(\xi)}\,, \qquad R = \sqrt{A^2 + B^2}\,, \quad \varphi = \Psi - \arctan(B/A)\,,$$

$$\mu' - \mathrm{i}\mu'' = 1 + (\sin^2\theta/\cos\theta\, R^2)(A - \mathrm{i}B)\,, \tag{2.10}$$

$$A = h_0 + 1 - \cos\theta - \omega/(1+\eta^2)\,, \quad B = \eta\omega\cos\theta/(1+\eta^2)\,.$$

The quantity $\mu' - \mathrm{i}\mu''$ entering eq. (2.9) plays the role of the nonlinear magnetic permeability of the ferromagnet. When dissipation is absent, $\mu'(\omega,\theta)$ coincides with the function $\mu(\omega,\theta)$ defined by formula (2.6).

Equation (2.9) is equivalent to the set of equations

$$\ddot{R} - \varphi^2 R + \omega^2\mu'(\omega,\theta)R = 0\,, \qquad \mathrm{d}(\dot{\varphi}R^2)/\mathrm{d}\xi = -\omega^2\mu''(\omega,\theta)R^2\,. \tag{2.11}$$

Note that the quantity $\dot{\varphi}R^2$ is proportional to the flow energy density of the wave, i.e., the second of the above equations describes the wave energy absorption.

Let us first neglect the effect of magnetization relaxation on the wave propagation process. Then $\mu'' = 0$ and the system (2.9) coincides formally with the equations describing the motion of a material point in the field of central forces (here $R$ plays the role of the radius, $\varphi$ is the angle and $\xi$ is the time). The well-known formulas of classical mechanics [21],

$$\xi = \pm \int_{R_i}^{R} \frac{\mathrm{d}R}{\Phi(R)}\,, \qquad \varphi = \varphi_i \pm M \int_{R_i}^{R} \frac{\mathrm{d}R}{R^2 \Phi(R)}\,,$$
$$\Phi^2 = R_i^2 + M^2/R_i^2 - M^2/R^2 - \omega^2(h_0 + 1 - \omega)(\tan^2\theta - \tan^2\theta_i) + 2(\cos\theta - \cos\theta_i)\,, \tag{2.12}$$

determine an implicit dependence of the amplitude and the wave magnetic field phase on the coordinate $\xi$. The subscript i in the above expressions denotes the values of the appropriate quantities at $\xi = 0$, $M = \dot{\varphi}_i R_i^2$.

Let us dwell on the classification of the solutions of eq. (2.12). Since $\xi$ is real-valued, the expression $\Phi^2(R)$ must obviously be nonnegative for all physically meaningful $R$. It is evident that $\Phi^2(R_i) \geq 0$. Let $\Phi^2(R_i) > 0$, which can be achieved by a proper choice of the integration constant $\dot{R}_i \neq 0$. Then for $M \neq 0$ the equation $\Phi^2(R) = 0$ must have at least two roots $R_1 < R_i < R_2$ since $\Phi^2(R) \to \infty$ when $R \to 0$ and $R \to \infty$, according to the coupling relation from eq. (2.10). If $R_1$ and $R_2$ are simple roots of the equation $\Phi^2(R) = 0$, then the function $R(\xi)$ will oscillate with a period $\lambda_0$ defined by the relation

$$\lambda_0 = 2 \int_{R_1}^{R_2} \frac{\mathrm{d}R}{\Phi(R)}\,, \tag{2.13}$$

i.e., the self-action effect of the EMS waves considered leads to establishing a periodic structure of the electromagnetic field and magnetization in a ferromagnet. In the particular case with $\dot{R}_i = 0$ and $M^2 = \omega^2 \mu(\theta_i, \omega_i) R_i^4$ the equation $\Phi^2(R) = 0$ has a multiple root $R = R_i$, while solution (2.12) describes magnetization waves of the form (2.3).

In the case of $M = 0$ solutions (2.12) are of physical interest, for they correspond to standing waves and waves decaying exponentially when $\xi \to \infty$. While analyzing the standing wave structure it is convenient to place the origin $\xi = 0$ at the point where the wave amplitude reaches its maximum. Then the expression $\Phi^2(R)$ assumes the form $\Phi^2 = (\cos\theta - \cos\theta_i) f(\theta)$, and the function $f(\theta)$ vanishes only in that domain of parameter values where the waves are unstable. That is why the precession angle of magnetization, $\theta(\xi)$, for standing EMS waves is a periodic function passing through zero, unlike the case of the waves propagating with $M \neq 0$.

Solutions (2.12) corresponding to the exponentially decaying waves exist only for the waves with the right rotation of the polarization plane in the frequency range $h_0 < 0 < 1 + h_0$.

In connection with the above-mentioned instability of EMS waves we will examine in more detail the functional dependence of the wave magnetic field amplitude, $R$, on the magnetization precession angle $\theta$. This dependence implied by the corresponding formula (2.10) is schematically shown in fig. 1. Curve I is typical for the left-polarized waves with an arbitrary value of $\omega$ and for the right-polarized waves whose frequency lies outside the range $h_0 < \omega < 1 + h_0$. Curve II corresponds to the dependence $R(\theta)$ for the right-polarized waves with the frequency $\omega$ inside the indicated range. One can see that in the latter case the function $\theta = \theta(R)$ becomes multiple-valued if the amplitude $R$ exceeds the critical value determined by the formula

$$R_c = \eta\omega\sqrt{1 - (1 + h_0 - \omega)^2}\,. \tag{2.14}$$

This ambiguity was first established by Suhl [22] in a particular case of homogeneous magnetization

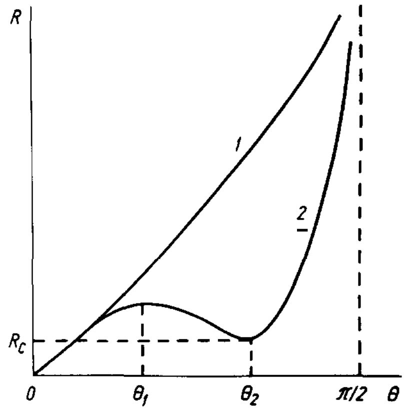


Fig. 1.

precession; Suhl showed that this ambiguity causes instability of magnetization oscillations. Let us examine the stability of the EMS wave (2.12) with respect to the generation of magnetostatic magnetization disturbances in the wave number domain $gM_0/c \ll k \ll 1/\alpha$. The magnetic field of such disturbances is generated by the magneto-dipole interaction, and in the one-dimensional case considered is a constituent part of the longitudinal field $h_z$. Substituting into the initial set of equations the disturbed transverse magnetization in the form

$$m_\perp = (\sin\theta + \tilde{m})\, e^{i\omega\tau - i\psi(\xi)} , \tag{2.15}$$

we will obtain, neglecting dissipation, a linearized equation to determine the disturbance amplitude $\tilde{m}$,

$$(1 + h_0 - \omega + i\,\partial/\partial\tau)\tilde{m} + \tfrac{1}{2}\tan^2\theta\,(1 + h_0 - \omega)(\tilde{m} + \tilde{m}^*) = 0 . \tag{2.16}$$

For the disturbances whose dependence on time is given by the factor $e^{\nu\tau}$ the dispersion relation follows from (2.16),

$$\cos^2\theta\,\nu^2 = -(1 + h_0 - \cos\theta - \omega)(1 + h_0 - \cos^3\theta - \omega) . \tag{2.17}$$

One can see that the quantity $\nu^2$ is positive for the angles

$$\theta_1 = \arccos\sqrt[3]{\Delta_0} < \theta < \theta_2 = \arccos\Delta_0 = \arccos(1 + h_0 - \omega) . \tag{2.18}$$

Thus, the part of curve II in fig. 1 which corresponds to the angles given by inequalities (2.18) depicts unstable EMS waves. This instability is similar to another one which was studied in detail in the theory of forced nonlinear oscillations [23]; this can be easily shown if one regards the relation $R = R(\omega, \theta)$ as giving the dependence of the force amplitude $R$ on $\theta$, the amplitude of forced oscillations with the frequency $\omega$.

Let us pass now to the analysis of decaying EMS due to magnetization relaxation. We will assume that the decay is weak ($\mu'' \ll \mu'$). The approach is justified in most cases since the ratio $\mu''/\mu'$ is proportional to a small value $\eta \sim 10^{-2}$.

The absorption of the plane monochromatic wave presents the simplest case. When there is no damping, the plane wave amplitude remains constant, while when the damping is small, the amplitude slowly decreases. That is why one can neglect the small quantity $\ddot{R}$ as compared to $\dot{\varphi}^2 R$ in the first equation of (2.11) [24]. By solving the resulting system of equations we find the function $\theta(\xi)$ as an implicit dependence,

$$\int_{\theta}^{\theta_i} \frac{\mathrm{d}[\sqrt{\mu'(\theta, \omega)}R^2(\theta)]}{\mu''(\theta, \omega)R^2(\theta)} = \omega\xi . \tag{2.19}$$

Calculating the $\theta(\xi)$ asymptotically when $\xi \to \infty$ we arrive at the formula

$$\theta(\xi) \to \theta_i S(\theta_i) \exp\{-\tfrac{1}{2}\omega[\mu''(0, \omega)/\sqrt{\mu'(0, \omega)}]\xi\} . \tag{2.20}$$

This differs from the corresponding formula of the linear theory only in the self-action factor $S$ [20],

$$S(\theta_i) = \frac{\tan(\theta_i)}{\theta_i} \exp\left[\sqrt{\frac{\mu'(\theta_i, \omega)}{\mu'(0, \omega)}} \frac{A^2(\theta_i)}{A^2(0)} \frac{1}{2\cos^2\theta_i} - \frac{1}{2} + \int_0^{\theta_i} \left(\sqrt{\frac{\mu'(\theta, \omega)}{\mu'(0, \omega)}} \frac{A^2(\theta)}{A^2(0)} - 1\right) \frac{\mathrm{d}\theta}{\sin\theta\cos\theta}\right], \tag{2.21}$$

which regains the information on the nonlinear stage of the wave evolution. The coefficient $A(\theta)$ in (2.21) has been defined in (2.10).

It is much more difficult to investigate how the absorption affects the propagation of the waves with amplitudes oscillating in space. In the general case one has to apply the averaging method developed in ref. [25], but this method enables one to obtain an analytical solution for rather special forms of the function $\mu(k)$. If the amplitude of oscillations is not large as compared to its mean value, the problem can be solved by using a method similar to the one considered above for the plane wave case.

Let us use once more the formal similarity of the system (2.11) with $\mu''(0)$ to the equations describing the motion of a material point in a field of central forces. One can easily see that a plane wave corresponds to the motion of a material point along a circumference. In view of the small damping the curve radius $R$ becomes a slowly decreasing function of $\xi$. A rosette-type trajectory [21] corresponds to a wave with an oscillating amplitude; thus, the quantity $R(\xi)$ can be represented as a sum which varies slowly under the action of absorption of the function $r(\xi)$ and oscillates with the period of the amplitude oscillation of the additional wave $\tilde{u}(\xi)$ that contributes the difference between the wave considered and the plane one.

At $\mu''=0$ (2.11) implies conservation of the momentum $M = \dot{\varphi}R^2$. Due to the absorption $M$ decreases slowly. Following ref. [25], we write an equation to determine $M$ by averaging over the second of eqs. (2.11),

$$\dot{M} = -\omega^2\langle \mu''(r + \tilde{u})(r + \tilde{u})^2\rangle . \tag{2.22}$$

In the first of eqs. (2.11) we may neglect the quantity $\ddot{r}$ as being small in comparison with $\ddot{\tilde{u}}$. The resulting equation,

$$\ddot{\tilde{u}} = -\omega^2\mu'(r+\tilde{u})(r+\tilde{u}) + M^2/(r+\tilde{u})^3 , \tag{2.23}$$

can be regarded as an equation for one-dimensional oscillations of a material point in a potential well [the function $\tilde{u}(\xi)$ is, by the statement of the problem, periodic at $\mu''=0$] with slowly varying parameters depending on the function $r(\xi)$. When $\tilde{u}\to 0$, i.e., in the case of a plane wave, eqs. (2.22) and (2.23) must result in (2.19). Combining this with eq. (2.23), we obtain a functional definition of the momentum

$$M = \omega\sqrt{\mu'(r)}r^2 . \tag{2.24}$$

In the considered case with $\tilde{u} \ll r$ the right-hand sides of eqs. (2.22) and (2.23) can be expanded in powers of $\tilde{u}$. For the sake of simplicity, we will retain only the principal terms of the expansion and, using (2.24), we arrive at the system of equations,

$$\begin{gathered} \ddot{\tilde{u}} = -\Omega^2(r)\tilde{u} = -\omega^2\left[4\mu'(r) + r\frac{\mathrm{d}}{\mathrm{d}r}\mu'(r)\right]\tilde{u} , \\ \frac{\mathrm{d}}{\mathrm{d}\xi}[\sqrt{\mu'(r)}r^2] = -\omega\mu''(r)r^2 - \omega\frac{\mathrm{d}^2}{\mathrm{d}r^2}[\mu''(r)r^2]\langle\tilde{u}^2\rangle , \end{gathered} \tag{2.25}$$

which describes the self-consistent variation of the functions $r(\xi)$ and $\tilde{u}(\xi)$. One can easily see that the equation determining the dependence $r(\xi)$ differs from the equation corresponding to the integral (2.19) only in one term proportional to $\langle\tilde{u}^2\rangle$ which accounts for the effect of wave amplitude oscillations on the absorption process. In the domain where waves with oscillating amplitude exist the conditions $\mu'>0$ and $\mathrm{d}r(\theta)/\mathrm{d}\theta>0$ are satisfied; these conditions can be used for demonstrating that the quantity $\Omega^2(r)$ is positive. By using an adiabatic version of the harmonic oscillator equation, we arrive at the equation

$$\langle\tilde{u}^2\rangle = \tilde{u}_\mathrm{i}^2\Omega(r_\mathrm{i})/2\Omega(r) , \tag{2.26}$$

where $\tilde{u}_\mathrm{i}$ is the initial value of the oscillation amplitude $\tilde{u}(\xi)$. Substituting relation (2.26) into the second equation of set (2.25) leads to a nonlinear first order equation which is integrated in the same manner as (2.19).

As we have noted, in the case of $M\neq 0$ solutions (2.12) describe monochromatic waves with a constant or spatially oscillating amplitude, while in the case of $M=0$ they describe standing waves and waves exponentially decaying when $\xi\to\infty$. The latter can be realized in the process in which a right-polarized magnetic wave penetrates into a ferromagnet, provided that the wave frequency $\omega$ belongs to the opacity region for linear EMS, $h_0<\omega<1+h_0$. Let us dwell on the corresponding boundary problem, since it will reveal an interesting effect of the nonlinear self-brightening of a ferromagnet.

Let a polarized electromagnetic wave $h_\mathrm{p}\exp[\mathrm{i}\omega\tau - \mathrm{i}(\omega/\sqrt{\varepsilon_0})\xi]$ fall on the surface of a ferromagnet filling the half-space $\xi>0$. The field in this half-space is described by a solution from (2.12) where the quantity $M$ which is proportional to the energy flow must be set, in the considered case of the complete internal reflection, to zero.

Expressions (2.12) corresponding to exponentially decaying waves must, in the asymptotic region

$\xi \to \infty$, turn out to be expressions of the linear theory. This determines the integration constant $\dot{R}_i$. We will find the two remaining constants $\theta_i$ and $\varphi_i$ with the help of the usual boundary conditions that require the continuity of tangential field components on the ferromagnet surface. It is easy to show that $\varphi_i = \arctan(\dot{R}_i/\omega\sqrt{\varepsilon_0}R_i)$ while the quantity $\theta_i$ is related with the incident wave amplitude $h_p$ by the relation

$$4h_p^2 = F^2(\theta_i) \equiv \sin^2\theta_i - 2\Delta_0(1-\cos\theta_i)/\cos\theta_i - 2\Delta_0(1-1/\varepsilon)(1-\cos\theta_i - \tfrac{1}{2}\Delta_0\tan^2\theta_i)\,, \tag{2.27}$$

where the quantity $\Delta_0$ was defined in formula (2.18).

Let us examine now the function $F(\theta_i)$. Note that the variation range of the angle $\theta_i$ in (2.27) is bounded from above by the condition

$$\theta_i < \theta_s = \arccos[\tfrac{1}{4}\Delta_0(1+\sqrt{1+8/\Delta_0})]\,, \tag{2.28}$$

which follows from the requirement that the quantity $\Phi^2(\omega,\theta)$ in (2.12) should be positive. By comparing eqs. (2.18) and (2.28), we arrive at the inequality $\theta_1 < \theta_s < \theta_2$. One can easily see that the function $F(\theta_i)$ reaches its maximum at the point $\theta_i = \theta_1$. Thus taking into account the wave instability in the region $\theta_1 < \theta_i < \theta_s$ of magnetization precession angles, eq. (2.27) determines a decaying wave in a ferromagnet in a unique way if the incident wave amplitude $h_p$ does not exceed the critical value $h_k = \tfrac{1}{2}F(\theta_1)$ that depends on the wave frequency as

$$h_k = \tfrac{1}{2}(1-\Delta_0^{1/3})[(1-\Delta_0^{1/3})(1+\Delta_0^{1/3})^3 + (\Delta_0/\varepsilon_0)(2+\Delta_0^{1/3})]^{1/2}\,. \tag{2.29}$$

For $h_p > h_k$ the wave begins to penetrate into a ferromagnet, bringing about the effect of the nonlinear self-brightening of the medium. The self-brightening threshold becomes lower as the incident wave frequency decreases; in the vicinity of $\omega \approx h_0$ it may be small,

$$h_k \approx (\sqrt{2}/3\sqrt{3})(\omega - h_0)\sqrt{\omega - h_0 + 9/8\varepsilon_0} \ll 1\,.$$

The graph of the function $h_k = h_k(\Delta_0)$ obtained according to formula (2.29) for the value of the dielectric constant $\varepsilon = 8$ (typical for ferromagnets) is shown in fig. 2. The asymptote of the function $\theta(\xi)$ in the considered case of complete internal reflection of the wave has the form

$$\begin{aligned} &\theta(\xi) = \theta_i S(\theta_i,\omega)\exp[-\omega\sqrt{\Delta_0/(\omega-h_0)}\,\xi]\,, \\ &S(\theta_i,\omega) = \frac{4}{\theta_i}\tan(\tfrac{1}{4}\theta_i)\exp\Bigg[\frac{1}{\sqrt{2(\omega-h_0)}} \\ &\qquad\times\int_0^{\theta_i}\left(\sqrt{(\omega-h_0)} - \frac{\cos^3\theta - \Delta_0}{\cos\theta\,[\cos^2\theta - \frac{1}{2}\Delta_0(1+\cos\theta)]^{1/2}}\right)\frac{\mathrm{d}\theta}{\sqrt{1-\cos\theta}}\Bigg]\,. \end{aligned} \tag{2.30}$$

Let us consider now the excitation in a ferromagnet caused by propagating EMS waves. The case of a plane wave is the simplest. For a plane wave $R(\theta) = R(\theta_i) = \text{const.}$ and $\dot{\varphi} = \omega\sqrt{\mu(\omega,\theta_i)} = \text{const.}$, so the boundary conditions are reduced to the conventional Fresnel formulas,

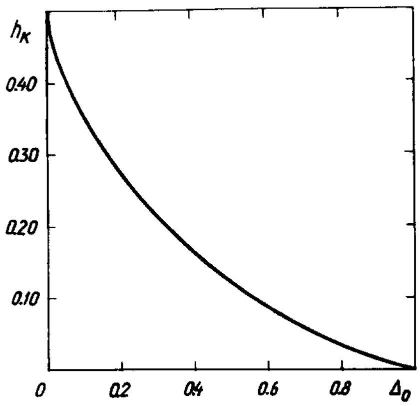


Fig. 2.

$$2h_p = [1 + \sqrt{\mu(\omega, \theta_i)/\varepsilon_0}]R(\theta_i)\,. \tag{2.31}$$

Taking into account formulas (2.10) which determine the quantities $\mu(\omega, \theta)$ and $R(\theta)$, we see that the function $\theta(h_p)$ following from (2.31) for left-polarized waves is a monotonically increasing function for any negative values of the frequency lying outside the opacity region for linear waves, $h_0 < \omega < 1 + h_0$. If $\omega$ belongs to the opacity region, then the variation of $h_p$ within the interval $(0, \infty)$ corresponds to the variation of the angle $\theta_i$ in the interval $\theta_2 = \arccos(\Delta_0) \le \theta_i < \pi/2$, with $R(\theta_2) = 0$ if the magnetization relaxation is not accounted for. In the case considered we encounter the hysteresis phenomenon, as is the case in many nonlinear problems. In order to excite a propagating right-polarized wave in the frequency range $h_0 < \omega < 1 + h_0$ the incident wave amplitude $h_p$ must exceed the critical value $h_k$ determined by formula (2.29). The resulting expression (2.31) proves that a ferromagnetic state exists in which a plane monochromatic wave is excited by an electromagnetic wave with $h_p < h_k$, the wave being directed from the vacuum. A ferromagnet can acquire such a state from a similar state with $h_p > h_k$ by adiabatically decreasing the incident wave amplitude.

A plane wave is a particular case of a monochromatic wave with an amplitude periodically varying in space. Such waves arise under the action of an electromagnetic field on a ferromagnet plate whose thickness does not exceed the EMS penetration depth. Let us examine the resonance on standing EMS waves in the simplest linear resonator formed by a layer of a ferromagnet of thickness $L$ sandwiched between two metallic plates. The function $R$ in the considered case must satisfy the obvious boundary conditions $\mathrm{d}R(\xi = 0)/\mathrm{d}\xi = \mathrm{d}R(\xi = L)/\mathrm{d}\xi = 0$; taking these conditions into account we will transform the function $\Phi^2(\theta, \omega)$ from (2.12) into the form

$$\Phi^2(\theta, \omega) = \omega^2 \Delta_0 (\cos\theta - \cos\theta_i)\left(\frac{\Delta_i(\cos\theta + \cos\theta_i)}{\cos^2\theta \cos^2\theta_i} - 2\right), \tag{2.32}$$

where $\theta_i = \theta(\xi = 0)$ is the value of the magnetization precession angle on one of the resonator boundaries. The energy flow in the considered case equals zero, when the absorption is not taken into account. The frequency $\omega$ is assumed to lie in the transparency region for linear waves, so that a stable field structure is realized within the resonator. So the expression in large parentheses in eq. (2.32) does not vanish in the angular range $0 \le \theta \le \pi/2$.

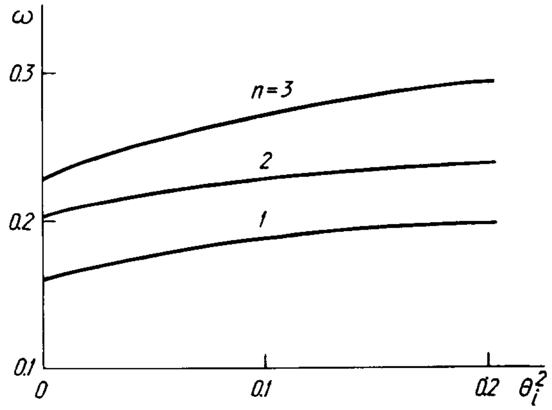


Fig. 3.

Let us find the eigenvalue spectrum of the resonator $\omega_n(\theta_i)$. Having noted that the angle $\theta$ varies from zero to $\theta_i$ at a distance equal to a quarter of the wavelength of its own oscillation, we will obtain the equation for determining $\omega_n(\theta_i)$ by equating the ratio of the resonator length $L$ to half of the oscillation wavelength (see eq. 2.13), to an integer $n$,

$$\frac{L}{2n} = \int_0^{\theta_i} \frac{\mathrm{d}R(\theta, \omega_n)}{\Phi(\theta, \omega_n)} \,. \tag{2.33}$$

In the linear approximation eq. (2.33) implies a well-known formula

$$\omega_n(0)\left(\frac{1 + h_0 - \omega_n(0)}{h_0 - \omega_n(0)}\right)^{1/2} = \frac{\pi n}{L} \,.$$

When the values of $\theta_i$ are finite, a nonlinear shift of the resonator eigenfrequencies occurs. The nonlinearity is especially pronounced for the right-polarized oscillations with a frequency close to that of the homogeneous ferromagnetic resonance. When $h_0 - \omega \ll 1$, the nonlinear frequency shift $\omega_n(\theta_i) - \omega_n(0)$ becomes sizeable even for small $\theta_i$ which one can see by analyzing the relation

$$(1/\omega_n)(\tfrac{3}{2}\theta_i^2 + h_0 - \omega_n)^{1/2}\{2E(\theta_i[2\theta_i^2 + \tfrac{4}{3}(h_0 - \omega_n)]^{-1/2}) - K(\theta_i[2\theta_i^2 + \tfrac{4}{3}(h_0 - \omega_n)]^{-1/2})\} = L/2n \,, \tag{2.34}$$

which in the case considered follows from the general formula (2.33). In this formula $E$ and $K$ are complete elliptic integrals. The functional dependence of $\omega_n(\theta_i)$ for several initial values of $r$ is shown in fig. 3.

### *2.3. A monochromatic wave in an antiferromagnet*

Equations (2.1), (2.2) and the solutions obtained from them describe monochromatic electromagnetic waves in a ferrodielectric with one magnet sublattice being characterized by the magnetization vector $\boldsymbol{M}$. Let us pass now to the analysis of nonlinear monochromatic EMS waves in two-sublattice magnets. We will restrict ourselves to the case of a one-axis antiferromagnet whose magnetic properties are characterized by two equivalent magnetic sublattices with the magnetizations $M_1$ and $M_2$ related by antiferromagnetic exchange [4]. Linear EMS waves in such a magnet were studied in ref. [26] while nonlinear ones were studied in ref. [27].

The magnetization dynamic equations in the case under consideration have the form [4],

$$\begin{aligned} &\partial \boldsymbol{M}_{1,2}/\partial t = -g\boldsymbol{M}_{1,2} \times \boldsymbol{H}_{\mathrm{eff}\,1,2}\,, \\ &\boldsymbol{H}_{\mathrm{eff}\,1,2} = \boldsymbol{H} - A\boldsymbol{M}_{2,1} + \beta \boldsymbol{n}\,\boldsymbol{n}\cdot\boldsymbol{M}_{1,2} + \beta' \boldsymbol{n}\,\boldsymbol{n}\cdot\boldsymbol{M}_{2,1} + \alpha\nabla^2\boldsymbol{M}_{1,2} + \alpha'\nabla^2\boldsymbol{M}_{2,1}\,, \end{aligned} \tag{2.35}$$

where $A$ is the constant of the homogeneous intersublattice exchange, and $|\boldsymbol{M}_1| = |\boldsymbol{M}_2|$.

Let us examine the solution of the systems of equations (2.1) and (2.35) in the long-wave limit when one may neglect several terms in $\boldsymbol{H}_{\mathrm{eff}}$ which are responsible for the inhomogeneous exchange interaction. As usual, we introduce the vectors of ferromagnetism $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2$ and antiferromagnetism $\boldsymbol{L} = \boldsymbol{M}_1 - \boldsymbol{M}_2$. Expressing eqs. (2.35) in terms of dimensionless variables we have,

$$\begin{aligned} &\partial \boldsymbol{m}/\partial\tau + \boldsymbol{m}\times(\boldsymbol{h} + \beta_0^+ \boldsymbol{n}\,\boldsymbol{n}\cdot\boldsymbol{m}) + \beta_0^-\,\boldsymbol{n}\cdot\boldsymbol{l}\,\boldsymbol{l}\times\boldsymbol{n} = 0\,, \\ &\partial \boldsymbol{l}/\partial\tau + \boldsymbol{l}\times(\boldsymbol{h} - A_0\boldsymbol{m} + \beta_0^+ \boldsymbol{n}\,\boldsymbol{n}\cdot\boldsymbol{m}) + \beta_0^-\,\boldsymbol{n}\cdot\boldsymbol{l}\,\boldsymbol{m}\times\boldsymbol{n} = 0\,, \end{aligned} \tag{2.36}$$

where $8\pi\beta_0^{\pm} = \beta \pm \beta'$ (we consider a case of a light-axis ferromagnet, so $\beta > \beta'$), $4\pi A_0 = A$, $\boldsymbol{M} = M_0\boldsymbol{m}$, $\boldsymbol{L} = M_0\boldsymbol{l}$. The remaining notation coincides with that of the previous section. Properly normalized, the vectors $\boldsymbol{l}$ and $\boldsymbol{m}$ satisfy the relations

$$\boldsymbol{m}\cdot\boldsymbol{l} = 0\,, \qquad |\boldsymbol{m}|^2 + |\boldsymbol{l}|^2 = 4\,. \tag{2.37}$$

It is not difficult to see that, as in the case of a one-sublattice ferromagnet, monochromatic waves of finite amplitude may propagate along the antiferromagnetic anisotropy axis. Passing to the circular polarization for the transverse component of the vectors $\boldsymbol{m}$, $\boldsymbol{l}$ and $\boldsymbol{h}$ in eqs. (2.36), we arrive at the equations

$$\begin{aligned} &[h_0 + m_{z0} - (1 + A_0 - \beta_0^+)m_z + \mathrm{i}\,\partial/\partial\tau]l_\perp = l_z[h_\perp - (A_0 + \beta_0^-)m_\perp]\,, \\ &[h_0 + m_{z0} - (1 - \beta_0^+)m_z + \mathrm{i}\,\partial/\partial\tau]m_\perp = m_z h_\perp - \beta_0^- l_z l_\perp\,, \\ &\partial l_z/\partial\tau = \mathrm{Im}(h_\perp^* l_\perp - A_0 m_\perp^* l_\perp)\,, \qquad \partial m_z/\partial\tau = \mathrm{Im}(h_\perp^* m_\perp)\,, \\ &(\partial^2/\partial\xi^2 - \partial^2/\partial\tau^2)h_\perp = \partial^2 m_\perp/\partial\tau^2\,, \end{aligned} \tag{2.38}$$

where $\boldsymbol{n} = \boldsymbol{e}_z$, the formula $a_\perp = a_x + \mathrm{i}a_y$ holds for all the quantities, $h_0$ is a constant magnetic field applied along the anisotropy axis. We will seek the solution of system (2.38) in the form

$$a_\perp(\xi, \tau) = a_\omega(\xi)\,\mathrm{e}^{\mathrm{i}\omega\tau}\,, \quad a_z = a_z(\xi)\,. \tag{2.39}$$

It follows from (2.38) that if a wave of the type (2.39) propagates in an antiferromagnet, then the transverse components of the vectors $\boldsymbol{m}$, $\boldsymbol{l}$ and $\boldsymbol{h}$ rotate around the anisotropy axis in the same phase. We will assume that the external field $h_0$ is less than $2[\beta_0^-(A_0+\beta_0^-)]^{1/2} \approx 2(\beta_0^- A_0)^{1/2}$, so that the antiferromagnet in the ground state is in the collinear phase [4]. Here $m_{z0}=0$, so, combining (2.38) and (2.39), we obtain the formulas

$$\begin{aligned} m_\omega &= \kappa_\omega h_\omega = \frac{\beta_0^- l_z^2 + m_z(\omega - h_0 + A_0 m_z)}{\beta_0^- A_0 l_z^2 + (\omega - h_0)(\omega - h_0 + A_0 m_z)}\, h_\omega\,, \\ l_\omega &= \chi_\omega h_\omega = \frac{l_z(\omega - h_0 + A_0 m_z)}{\beta_0^- A_0 l_z^2 - (\omega - h_0)(\omega - h_0 + A_0 m_z)}\, h_\omega\,, \end{aligned} \tag{2.40}$$

that relate the magnetization components with the magnetic field of the propagating wave. It is assumed in eq. (2.40) that $A_0 \gg 1,\ \beta_0^{\pm}$ which is usually true.

Relation (2.37) enables one to express the longitudinal components of the magnetization $m_z$ and $l_z$ that enter in the functions $\kappa_\omega$ and $\chi_\omega$ via the quantity $|h_\omega|^2$. Subsequently the last equation of the set (2.38) is reduced to the closed nonlinear equation

$$\mathrm{d}^2 h_\omega/\mathrm{d}\xi^2 + \omega^2[1 + \kappa_\omega(\omega, |h_\omega|^2)]h_\omega = 0\,, \tag{2.41}$$

which is similar to eq. (2.9) describing nonlinear monochromatic waves in a ferromagnet. The sort of solutions of the above equation is determined by the form of the nonlinear magnetic susceptibility $\kappa_\omega(\omega, |h_\omega|^2)$ which is found from the equations

$$\begin{aligned} &A_0\kappa_\omega(\omega - h_0 + m_z/\kappa_\omega)^2 = \omega_\mathrm{p}^2(1 - \tfrac{1}{4}m_z^2 - \tfrac{1}{4}\kappa_\omega^2|h_\omega|^2)(A_0\kappa_\omega - 1)\,, \\ &A_0 m_z = -\tfrac{1}{2}(\omega - h_0)\{1 - [1 - 4A_0\kappa_\omega(A_0\kappa_\omega - 1)|h_\omega|^2/(\omega - h_0)^2]^{1/2}\}\,, \end{aligned} \tag{2.42}$$

where $\omega_\mathrm{p}^2 = 4\beta_0^- A_0$. One can easily see that in the limit $|h_\omega| \to 0$ relation (2.41) implies the standard expression for the linear magnetic susceptibility of the antiferromagnet

$$\kappa_\omega(\omega, 0) = (1/A_0)\omega_\mathrm{p}^2/[\omega_\mathrm{p}^2 - (\omega - h_0)^2]\,. \tag{2.43}$$

For small but finite values of the wave magnetic field amplitude eqs. (2.42) yield an approximate expression for the nonlinear magnetic permeability,

$$\begin{aligned} A_0\kappa_\omega(\omega, |h_\omega|^2) &\simeq \frac{\omega_\mathrm{p}^2 - (\omega - h_0)^2 - 2|h_\omega|^2}{4|h_\omega|^2}\left[\left(1 + \frac{8|h_\omega|^2\omega_\mathrm{p}^2}{[\omega_\mathrm{p}^2 - (\omega - h_0)^2 - 2|h_\omega|^2]^2}\right)^{1/2} - 1\right] \\ &\to \frac{\omega_\mathrm{p}^2}{\omega_\mathrm{p}^2 - (\omega - h_0^-)^2 - 2|h_\omega|^2}\,, \end{aligned} \tag{2.44}$$

that shows that nonlinear self-action effects of monochromatic EMS waves in antiferromagnets are essential only in the vicinity of the resonance frequencies $\omega = \pm\omega_\mathrm{p} + h_0$.

The expression for the magnetic susceptibility (2.43) enables us two write the solution of eq. (2.41) in quadratures, the general analysis being similar to the one in the above-considered case of a monochromatic wave in a ferrodielectric. As an instructive example, we will examine the structure of standing waves in an antiferromagnet. Taking account of (2.44) the first integral of (2.41) has the form

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\xi}\right)^2 = \omega^2\left[\frac{\omega_\mathrm{p}^2}{2A_0}\ln\left(1+\frac{2R^2}{(\omega-h_0)^2-\omega_\mathrm{p}^2}\right)-R^2\right], \tag{2.45}$$

where $R=|h_\omega|$ and the integration constant is chosen so that the boundary condition $R\to 0$ should be fulfilled at $|\xi|\to\infty$, which corresponds to the autolocalized nonlinear oscillations of the antiferromagnet magnetization. One can easily see that the r.h.s. of (2.45) in the limit $|\xi|\to\infty$ has the form $-\omega^2\mu(\omega,0)R^2$. Thus the solitary nonpropagating waves we look for can be realized in the frequency range corresponding to the opacity region of nonlinear monochromatic waves

$$\omega_\mathrm{p}+h_0<\omega<\omega_\mathrm{p}+h_0+\sqrt{\beta_0^-/A_0}\,,\qquad -\omega_\mathrm{p}+h_0-\sqrt{\beta_0^-/A_0}<\omega<-\omega_\mathrm{p}+h_0\,. \tag{2.46}$$

The analytic solution of eq. (2.45) in the case of small amplitudes has the form

$$R=\frac{\omega_\mathrm{p}}{|\kappa_\omega(\omega,0)|}\sqrt{\frac{|\mu(\omega,0)|}{A_0}}\,\mathrm{sech}[\omega\sqrt{|\mu(\omega,0)|}\,\xi]\,, \tag{2.47}$$

which is typical for weakly nonlinear solitary waves. One can see that the solitary wave amplitude vanishes on the boundaries of the opacity regions of linear waves (2.46). Nevertheless, the condition $R^2\ll|\omega_\mathrm{p}^2-(\omega-h_0)^2|$ used in deriving expression (2.47) from eq. (2.45) permits us to use solution (2.47) only in the vicinity of the values $\omega=\pm\omega_\mathrm{p}+h_0\pm\sqrt{\beta_0^-/A_0}$.

## *2.4. Dynamics of Schrödinger wave packets of electromagnetic–spin waves*

We have thus far studied the propagation of monochromatic EMS waves in magnets. New interesting physical effects appear when EMS waves with a finite spectrum width propagate in a ferromagnet. The analysis of weakly nonlinear quasimonochromatic waves is of special interest, since, on the one hand, such waves are most typical for experiments and applications and, on the other hand, the analysis can be carried out in the general manner by employing the well developed theory of NSE (nonlinear Schrödinger equations) [28–30].

At first, let us examine the simplest problem of the evolution of a quasimonochromatic one-dimensional EMS wave of a finite amplitude propagating in an isotropic ferrodielectric along the d.c. field. The initial equation for the analysis is one that follows from the system (2.9), (2.10); this is a nonlinear dispersion relation,

$$k^2=\omega^2\,\frac{1+h_0^-\omega+\mathrm{i}\eta\omega}{h_0-\omega+\frac{1}{2}\theta^2+\mathrm{i}\eta\omega}\,, \tag{2.48}$$

that relates the wave frequency, wave number and amplitude. In the case at hand of a weakly nonlinear wave, the amplitude, whose role is played by the angle by which the magnetization vector deviates from the direction of wave propagation, $\theta$, is small, $\theta\ll 1$. Let us confine the analysis to the slow branch, assuming that $h_0\ll 1$. Then the solution of eq. (2.48) can be represented as

$$\omega = \omega(k, \theta^2) + \mathrm{i}\gamma(k, \theta^2) . \tag{2.49}$$

In the region of long-wave disturbances where the condition $k^2 \ll 4h_0$ holds, the functions $\omega(k, \theta^2)$ and $\gamma(k, \theta^2)$ are as follows:

$$\omega(k, \theta^2) \simeq \sqrt{h_0}k(1 - k/2\sqrt{h_0} + \theta^2/4h_0) , \qquad \gamma(k, \theta^2) \simeq \tfrac{1}{2}\eta k^2 . \tag{2.50}$$

In the region of short-wave disturbances with $k^2 \gg 4h_0$ the same functions are given by the expressions

$$\omega(k, \theta^2) \simeq h_0(1 - h_0/k^2 + \theta^2/2h_0) , \qquad \gamma(k, \theta^2) \simeq h\eta . \tag{2.51}$$

Following ref. [29], we will seek the solution of the system (2.1), (2.2) giving the transverse magnetization component $m_\perp = m_x + \mathrm{i}m_y$ as

$$m_\perp = U(\xi, \tau)\, \mathrm{e}^{\mathrm{i}\omega(k)\tau - \mathrm{i}k\xi} , \tag{2.52}$$

where $\omega(k) = \omega(k, 0)$ and the envelope $U(\xi, \tau)$ grows slowly as compared to the exponential growth of its arguments.

Relation (2.49) corresponds to the operator equation for the envelope $U(\xi, \tau)$,

$$\left[-\mathrm{i}\,\frac{\partial}{\partial\tau} + \omega(k) - \omega\left(k + \mathrm{i}\,\frac{\partial}{\partial\xi}, |U|^2\right)\right]U = \mathrm{i}\gamma\left(k + \mathrm{i}\,\frac{\partial}{\partial\xi}, |U|^2\right)U . \tag{2.53}$$

Taking into account that the medium is weakly nonlinear and the envelope varies but slowly, the operator $\omega(k + \mathrm{i}\,\partial/\partial\xi, |U|^2)$ in the l.h.s. of eq. (2.53) can be expanded in powers of $\mathrm{i}\,\partial/\partial\xi$ and $|U|^2$ keeping several leading terms (see, e.g., ref. [29]). As a result, we obtain the equation

$$\left(\mathrm{i}\,\frac{\partial}{\partial\tau} + \mathrm{i}\,\frac{\partial\omega}{\partial k}\,\frac{\partial}{\partial\xi} - \frac{1}{2}\,\frac{\partial^2\omega}{\partial k^2}\,\frac{\partial^2}{\partial\xi^2} + \frac{\partial\omega}{\partial|U|^2}\,|U|^2\right)|U| = \mathrm{i}\gamma\left(k + \mathrm{i}\,\frac{\partial}{\partial\xi}, |U|^2\right)U . \tag{2.54}$$

Equation (2.54) with zero on the right is called a nonlinear Schrödinger equation [28]. The form of the solution essentially depends on the sign of the quantity $\alpha_L(\partial^2\omega/\partial k^2, \partial\omega/\partial|U|^2)$. One can easily see that for EMS waves of the slow branch the Lighthill criterion of the modulation instability is satisfied, $\alpha_4 < 0$ [31] [relations (2.50) and (2.51) imply the inequalities $\partial^2\omega/\partial k^2 < 0$ and $\partial\omega/\partial|U|^2 > 0$]. In the assumed conditions the wave in (2.52) is in the evolution process divided into a set of spatially localized wave packets, i.e., solitons of the envelope, for which $U(\xi, \tau) \to 0$ when $|\xi| \to \infty$. Equation (2.54) with $\gamma \neq 0$ describes the magnetization relaxation effect on the soliton evolution. A change in the spectral composition of the wave $m_\perp(\xi, \tau)$ during its propagation can significantly affect the wave absorption conditions (for instance, the spectrum can be shifted towards larger or smaller values of the decrement). So for a correct description of the wave absorption one must consider the behaviour of $\gamma(k, |U|^2)$ in greater detail than the behaviour of $\omega(k, |U|^2)$. To this end, the operator $\gamma(k + \mathrm{i}\,\partial/\partial\xi, |U|^2)$ on the r.h.s. of (2.54) is expressed in a general form.

Let us rewrite (2.54) in the canonical form

$$\mathrm{i}\,\frac{\partial U}{\partial \tau'} + \frac{1}{2}\,\frac{\partial^2 U}{\partial \xi'^2} + |U|^2 U = \delta f\Big(k + \mathrm{i}\tilde{\beta}\,\frac{\partial}{\partial \xi'},\, |U|^2\Big) U = \mathrm{i}\hat{R}[U]\,, \tag{2.55}$$

where

$$\tau' = \frac{\partial \omega}{\partial |U|^2}\,\tau\,, \qquad \xi' = \tilde{\beta}\Big(\xi - \frac{\partial \omega}{\partial k}\,\tau\Big)\,, \qquad \tilde{\beta}^2 = \frac{|\partial\omega/\partial|U|^2|}{|\partial^2\omega/\partial k^2|}\,,$$

$$\delta = \frac{\gamma_0}{|\partial\omega/\partial|U|^2|}\,, \qquad \gamma = \gamma_0 f\Big(k + \mathrm{i}\tilde{\beta}\,\frac{\partial}{\partial \xi'},\, |U|^2\Big)\,, \qquad f \simeq 1\,.$$

For $\delta \ll 1$ the considered problem allows a simple analysis on the basis of the perturbation theory developed for solitons [32] that was created in the framework of the inverse scattering problem. The theory of ref. [32] (see also refs. [33, 34]) allows one to describe a slow variation of the soliton parameters under the action of a structural disturbance $\hat{R}[U]$ and also to study the distortion of the soliton shape in time. Following ref. [32], we will seek the solution of eq. (2.55) in the form

$$U = 2\kappa_0 \exp[\mathrm{i}(\lambda_0/\kappa_0)z + \mathrm{i}\varphi]\,[\operatorname{sech} z + W(z, \tau')]\,, \quad z = 2\kappa_0(\xi' - g')\,, \tag{2.56}$$

where the first term proportional to $\operatorname{sech} z$ describes the envelope soliton with the slowly varying parameters $\kappa_0(\tau')$, $\lambda_0(\tau')$, $\varphi(\tau')$, $g'(\tau')$. The distortion of the soliton shape is described by the term proportional to the function $W(z, \tau')$.

The soliton parameters are found from the set of equations describing the adiabatic approximation [32]. In the case under consideration these equations assume the form,

$$\begin{gathered}
\frac{\mathrm{d}\kappa_0}{\mathrm{d}\tau'} = -\delta\kappa_0\,\mathrm{Re}\Big(\int_{-\infty}^{\infty} \mathrm{d}z\; q^*\hat{f}q\Big)\,, \qquad \frac{\mathrm{d}\lambda_0}{\mathrm{d}\tau'} = -\delta\kappa_0\,\mathrm{Im}\Big(\int_{-\infty}^{\infty} \mathrm{d}z\,\tanh z\; q^* f q\Big)\,, \\
\frac{\mathrm{d}g'}{\mathrm{d}\tau'} = 2\lambda_0 - \frac{\delta}{2\kappa_0}\,\mathrm{Re}\Big(\int_{-\infty}^{\infty} \mathrm{d}z\; z q^* f q\Big)\,, \\
\frac{\mathrm{d}\varphi}{\mathrm{d}\tau'} = 2\lambda_0\,\frac{\mathrm{d}g'}{\mathrm{d}\tau'} - 2(\lambda_0^2 - \kappa_0^2) - \delta\,\mathrm{Im}\Big(\int_{-\infty}^{\infty} \mathrm{d}z\,(1 - z\tanh z) q^* f q\Big)\,, \\
q(z) = \exp[\mathrm{i}(\lambda_0/\kappa_0)z]\,\operatorname{sech} z\,, \qquad \hat{f} = f(\kappa + 2\mathrm{i}\tilde{\beta}\kappa_0\,\mathrm{d}/\mathrm{d}z,\, \kappa_0^2)\,.
\end{gathered} \tag{2.57}$$

Using the Fourier transform of the quantity $q$,

$$q_\nu = \frac{1}{2\pi}\int_{-\infty}^{\infty} q\,\mathrm{e}^{\mathrm{i}\nu z}\,\mathrm{d}z = \tfrac{1}{2}\operatorname{sech}[\tfrac{1}{2}\pi(\nu + \lambda_0/\kappa_0)]\,,$$

we obtain the relation

$$\hat{f}q = \tfrac{1}{2}\,\mathrm{e}^{\mathrm{i}(\lambda/\kappa)z}\int_{-\infty}^{\infty}\mathrm{d}\nu\,\mathrm{e}^{-\mathrm{i}\nu z}f(\kappa - 2\tilde{\beta}\lambda_0 + 2\tilde{\beta}\kappa_0\nu,\,\kappa_0^2)\,\mathrm{sech}(\tfrac{1}{2}\pi\nu) = \tfrac{1}{2}\,\mathrm{e}^{\mathrm{i}(\lambda_0/\kappa_0)z}\tilde{\Phi}\,,$$

that allows substantial simplification of eq. (2.57). Finally, we have such a set of equations,

$$\frac{\mathrm{d}\kappa_0}{\mathrm{d}\tau'} = -\frac{\delta\kappa_0}{\Delta}\int_{-\infty}^{\infty} f(k - 2\tilde{\beta}\lambda + k',\,\kappa_0^2)\,\frac{\mathrm{d}k'}{\cosh^2(k'/\Delta)}\,,\qquad \frac{\mathrm{d}\lambda_0}{\mathrm{d}\tau'} = \frac{2\delta\kappa_0}{\pi\Delta^2}\int_{-\infty}^{\infty} f\,\frac{k'\,\mathrm{d}k'}{\cosh^2(k'/\Delta)}\,,$$
$$\mathrm{d}g'/\mathrm{d}\tau' = 2\lambda_0\,,\qquad \mathrm{d}\varphi/\mathrm{d}\tau' = 2(\lambda_0^2 + \kappa_0^2)\,. \tag{2.58}$$

The set describes the soliton parameter evolution for an arbitrary dependence of the decrement on the wave number and amplitude. Here $\Delta = (4\pi/\beta)\kappa_0$ stands for the envelope spectrum width.

Set (2.57) is obtained under the assumption that the function $\omega(k)$ changes smoothly. A question may arise as to how large the contribution in (2.54) is of the neglected terms of the expansion of the operator $\omega(k + \mathrm{i}\,\partial/\partial\xi)$ in powers of $\mathrm{i}\,\partial/\partial\xi$. Our analysis showed that taking into account of the term proportional to $\partial^2\omega/\partial k^3$ does not change the form of the equation for the parameters $\kappa_0(\tau')$ and $\lambda(\tau')$ in the system (2.57), but the equations expressing the parameter $g'$ and $\varphi$ via $\kappa_0$ and $\lambda_0$ assume the form

$$\mathrm{d}g'/\mathrm{d}\tau' = 2\lambda_0 + 2\tilde{\eta}(\lambda_0^2 + \kappa_0^2/3)\,,\qquad \mathrm{d}\varphi/\mathrm{d}\tau' = 2(\lambda_0^2 + \kappa_0^2) + \tfrac{8}{3}\tilde{\eta}(\lambda_0^2 - \kappa_0^2)\lambda\,,$$
$$\tilde{\eta} = \beta(\partial^3\omega/\partial k^3)(\partial^2\omega/\partial k^2)^{-1}\,. \tag{2.59}$$

Let us pass now to the calculation of the function $W(z, \tau')$ in (2.56) which describes the distortion of the soliton shape in the evolution process. Using the results of ref. [32] and relation (2.58), we arrive at the expression

$$W = -\frac{\delta}{2\pi\mathrm{i}\kappa_0}\int_{-\infty}^{\infty}\mathrm{d}z'\int_{-\infty}^{\infty}\mathrm{d}\mu\Big((\mu - \mathrm{i}\kappa_0 + \tanh z')^2\tilde{\phi}(z') - \frac{\kappa_0^2}{\cosh^2 z'}\,\tilde{\Phi}^*(z')\Big)$$
$$\times\,\frac{(\mu + \mathrm{i}\kappa_0 + \tanh z)^2\exp[\mathrm{i}\,(\mu/\kappa_0)(z - z')]}{(\mu - \mathrm{i}\kappa_0)^3(\mu + \mathrm{i}\kappa_0)^3} + \frac{\delta\kappa_0}{8\pi\mathrm{i}\cosh^2 z}\int_{-\infty}^{\infty}\mathrm{d}z'\int_{-\infty}^{\infty}\mathrm{d}\mu$$
$$\times\Big((\mu + \mathrm{i}\kappa_0 + \tanh z')^2\tilde{\Phi}^*(z') - \frac{\kappa_0^2}{\cosh^2 z'}\,\tilde{\Phi}(z')\Big)\,\frac{\exp[-\mathrm{i}(\mu/\kappa_0)(z - z')]}{(\mu - \mathrm{i}\kappa_0)^3(\mu + \mathrm{i}\kappa_0)^3}\,. \tag{2.60}$$

Equation (2.60) implies that in the reference frame fixed to a moving soliton the explicit dependence of the function $W(z, \tau')$ on the time $\tau'$ is determined by the time-dependent parameters $\lambda_0(\tau')$ and $\kappa_0(\tau')$ of the soliton, the parameter $\lambda_0(\tau')$ entering (2.60) only via the function $\tilde{\Phi}(\kappa, \lambda, z')$ from (2.58) which characterizes the dependence of the decrement on the wave number.

Calculating the integral over $\mathrm{d}\mu$ in (2.60) yields a rather cumbersome expression for $W(z, \tau')$ whose asymptotic behaviour for $|z| \to \infty$ has a simple form,

$$W \to -\frac{\delta}{8\mathrm{i}\kappa_0^2} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x}[\tilde{\Phi}(z+x) + \tilde{\Phi}(z-x)]\,, \tag{2.61}$$

where the function $\tilde{\Phi}(z \pm x)$ has been defined previously.

Of greatest interest is the analysis of the evolution of the wave packet as a whole. By substituting into the adiabatic equations (2.57) the coefficients derived from (2.51) and (2.50) we obtain in the case of the long-wave solitons of the envelope $(k^2 \ll 4h_0)$ the following simple equations

$$\mathrm{d}\Delta/\mathrm{d}\tau' = -2\delta k_c^2 \Delta = -\tfrac{1}{6}\pi^2\delta\Delta^3\,, \qquad \mathrm{d}k_c/\mathrm{d}\tau' = -\tfrac{1}{3}\pi^2\delta k_c \Delta^2\,, \tag{2.62}$$

where $k_c(\tau') = k - 2\tilde{\beta}\lambda_0(\tau')$, in accordance with definition (2.56), is a time-dependent wave number that characterizes the center of the spectral distribution of the wave packet. Here $\delta = 2\sqrt{h_0}\,\eta/k$. The integral of eqs. (2.62)

$$k_c = \tfrac{1}{2}k_{c0}\left\{1 - \tfrac{1}{12}\pi^2 \frac{\Delta_0^2}{k_{c0}^2} + [(1 - \tfrac{1}{12}\pi^2\, \Delta^2/k_{c0}^2)^2 + \tfrac{1}{3}\pi^2\, \Delta^2/k_{c0}^2]^{1/2}\right\}, \tag{2.63}$$

shows that in the evolution process of the soliton its spectral distribution centre is shifted towards smaller $k_c$ $[k \to k_{c0}(1 - \tfrac{1}{12}\pi^2\, \Delta_0^2/k_{c0}^2)$, $\Delta_0$ and $k_{c0}$ being the initial values of the corresponding quantities]. Here, according to the first of eqs. (2.62) the rate of the soliton absorption becomes slower, i.e., here we observe a peculiar effect of a nonlinear self-induced transparency (self-brightening) of the medium. Figure 4 shows the behaviour of $\Delta(\tau')$ calculated by using formulas (2.62) and (2.63) with $\Delta_0^2/k_{c0}^2 = 0$ (curve 1), $\Delta_0^2/k_{c0}^2 = 0.2$ (curve 2) and $\Delta_0^2/k_{c0}^2 = 0.4$ (curve 3).

This peculiarity of the soliton evolution can be simply explained. It is well-known that soliton solutions of the nonlinear Schrödinger equation can be regarded as coupled states of a large number of elementary excitations in the form of plane waves. During absorption in a medium with a decrement which is a growing function of the wave number $k'$, as in the case described by eqs. (2.50), the soliton spectrum components corresponding to the left wing of the spectrum density function $|\psi(k')|^2$,

$$|\psi(k')|^2 = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{\mathrm{i}k'\xi} U(\xi,\tau)\, \mathrm{e}^{\mathrm{i}\omega(k)\tau - \mathrm{i}k\xi}\right|^2 \sim \cosh^{-2}\{[k' - k_c(\tau')]/\Delta(\tau')\}\,,$$

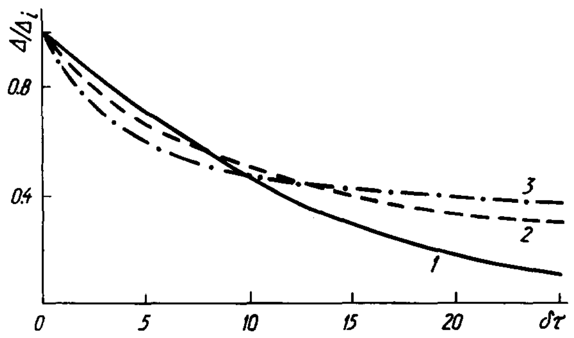


Fig. 4.

are absorbed more slowly than the elementary excitations with larger $k'$ which correspond to the right wing of the spectrum density. That is why the centre of the soliton spectrum distribution is shifted to the left, towards smaller values of the absorption decrement. The nonlinear interaction results in such energy redistribution among the elementary excitations comprising the soliton, so as to make the envelope shape close to the function $U(\xi, \tau)$ (2.56); this is just the shape that ensures equilibrium between dispersion and nonlinearity, which is a necessary condition for the soliton existence.

Short-wave packets of EMS waves of the slow branch [the dispersion and nonlinear characteristics of such waves are described by relations (2.51) which are valid for wave numbers satisfying the inequality $k^2 \gg 4h_0$] decay exponentially owing to the magnetization relaxation. Here the soliton does not move as a whole, since the absorption decrement in the region $k^2 \gg 4h_0$ is independent of the wave number, according to (2.51).

## 2.5. *Quasimonochromatic waves in a ferrite waveguide*

The characteristic spatial scale $gM_0/c$ of EMS waves is of the order of one centimeter, so practically all wave processes involving EMS waves occur in spatially limited structures. In this connection the above analysis of one-dimensional waves gives but a qualitative description of nonlinear effects of the propagation of intense EMS waves in real waveguide and resonator systems. Now we will investigate the weakly linear wave propagation in a strip waveguide filled with an isotropic ferrodielectric and magnetized in the direction normal to the metallic plates that bound the waveguide [35].

Let a wave propagate along the $\boldsymbol{e}_x$ axis, let the external magnetic field $\boldsymbol{h}_0$ (internal field) be directed along the $\boldsymbol{e}_y$ axis. The initial equation set for the problem has the form

$$\nabla \times \boldsymbol{h} = \partial \boldsymbol{\varepsilon}/\partial \tau\,, \qquad \nabla \times \boldsymbol{\varepsilon} = -\partial \boldsymbol{h}/\partial \tau - \partial \boldsymbol{m}/\partial \tau\,, \qquad \partial \boldsymbol{m}/\partial \tau = -\boldsymbol{m} \times (\boldsymbol{h}_0 + \boldsymbol{h})\,, \tag{2.64}$$

where, besides the dimensionless magnetic field $\boldsymbol{h}$, we introduced the dimensionless electric field $\boldsymbol{E}$,

$$\boldsymbol{E} = 4\pi M_0 \boldsymbol{\varepsilon}/\sqrt{\varepsilon_0}\,, \qquad \nabla = \boldsymbol{e}_x\, \partial/\partial x + \boldsymbol{e}_y\, \partial/\partial y\,, \qquad c\xi = 4\pi g M_0 \sqrt{\varepsilon_0} x\,, \qquad c\chi = 4\pi g M_0 \sqrt{\varepsilon_0} y\,,$$

with the remaining rotation coinciding with that explained previously.

We will seek solutions of (2.64) which correspond (without taking account of nonlinearity) to the usual linear waveguide modes. When weak nonlinearity is taken into account, the waveguide harmonics becomes slow functions of the coordinates and time. Our task is to obtain and subsequently to analyze the evolution equations of the amplitudes. The multiple scale technique will be a convenient method for solving this problem [36].

Let us expand the field components in the transverse waveguide modes. We will seek the solution of eqs. (2.64) as asymptotic expansions in powers of the wave amplitude, assuming that the wave TEM components are the principal ones. The quantities $\varepsilon_x$ and $\varepsilon_z$ will be expressed in accordance with the general formula

$$a(\tau, \xi, \chi) = \sum_{n \geq 1} a_n(\tau, \xi) \sin(n k_d \chi)\,, \tag{2.65}$$

while the quantities $m_x$, $m_z$, $h_x$, $h_z$, $\varepsilon_y$ will be obtained in accordance with the formula

$$C(\tau, \xi, \chi) = C_0(\tau, \xi) + \sum_{n \geq 1} C_n(\tau, \xi) \cos(n k_d \chi)\,, \tag{2.66}$$

where $k_d = \pi/d$, $d$ being the ferrodielectric layer thickness. It is convenient to seek the quantity $h_y$ in the form

$$h_y = \tfrac{1}{2}m_\perp^2 + \sum_{n\geq 1} h_{y_n}(\tau, \xi)\sin(nk_d\chi)\,, \tag{2.67}$$

with $m_\perp^2 = m_x^2 + m_z^2$. In the next chapter, representation (2.67) will be derived in analyzing solitary waves in a strip waveguide.

Let us introduce formally the small parameter $\alpha_0 \ll 1$ which characterizes the smallness of the wave amplitude; let us also introduce different temporal and spatial scales $\tau_n = \alpha_0^n \tau$ and $\xi_n = \alpha_0^n \xi$. For the wave TEM components the following formula is valid:

$$C_0 = \sum_{l\geq 1} \alpha_0^l C_0^l(\tau - \xi/u, \tau_1, \tau_2, \ldots, \xi_1, \xi_2, \ldots)\,. \tag{2.68}$$

For the waveguide harmonics having the index $n \geq 1$ the corresponding asymptotic expansions begin with terms proportional to $\alpha_0^2$. The substitution of expressions of the type of (2.68) into set (2.64) and taking into account rules of differentiation, leads to

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\tau_0} + \alpha_0\frac{\partial}{\partial\tau_1} + \alpha_0^2\frac{\partial}{\partial\tau_2} + \cdots\,, \qquad \frac{\partial}{\partial\xi} = \frac{1}{u}\frac{\partial}{\partial\tau_0} + \alpha_0\frac{\partial}{\partial\xi_1} + \alpha_0^2\frac{\partial}{\partial\xi_2} + \cdots\,, \tag{2.69}$$

and related equations of different order in $\alpha_0$. In (2.69) $\tau_0 = \tau - \xi/u$.

The equations of first order in $\alpha_0$ imply the usual formula,

$$m_{z0}^{(1)} = m_0(\tau_1, \tau_2, \ldots, \xi_1, \xi_2, \ldots)\,\mathrm{e}^{\mathrm{i}\omega\tau_0} + \text{h.c.}\,, \tag{2.70}$$

which describes a linear monochromatic wave, with the dispersion law

$$u^2(\omega) = (b_0h_0 - \omega^2)/(b_0^2 - \omega^2)\,, \quad b_0 = 1 + h_0\,. \tag{2.71}$$

Other wave components in the considered approximation have the form

$$h_{x0}^{(2)} = -m_{x0}^{(2)} = (\mathrm{i}\omega/b_0)m_0\,\mathrm{e}^{\mathrm{i}\omega\tau_0} + \text{h.c.}\,, \qquad h_{z0}^{(2)} = u\varepsilon_{y0}^{(2)} = [u^2/(1-u^2)]m_0\,\mathrm{e}^{\mathrm{i}\omega\tau_0} + \text{h.c.}$$

The equations of second order in $\alpha_0$ for the TEM components of the wave are reduced to one equation for the quantity $m_{z0}^{(2)}$,

$$\left(\frac{\partial^2}{\partial\tau_0^2} + \omega^2\right)\frac{\partial}{\partial\tau_0}\, m_{z0}^{(2)} = \frac{2}{b_0}\,[b_0^2h_0 + (b_0h_0 - \omega^2)^2]\left(\frac{\partial m_0}{\partial\tau_1} + v_{\mathrm{gr}}\frac{\partial m_0}{\partial\xi_1}\right)\mathrm{e}^{\mathrm{i}\omega\tau_0} + \text{h.c.}\,, \tag{2.72}$$

whose r.h.s. is a resonance with respect to the operator in the l.h.s. The requirement for the solution to be regular leads to the following equation for the amplitude $m_0$:

$$\frac{\partial m_0}{\partial\tau_1} + v_{\mathrm{gr}}\frac{\partial m_0}{\partial\xi_1} = 0\,, \qquad v_{\mathrm{gr}} = \frac{(c_0h_0 - \omega^2)^{3/2}(b_0^2 - \omega^2)^{1/2}}{b_0^2h_0 + (b_0h_0 - \omega^2)^2}\,, \tag{2.73}$$

that shows that the wave envelope (2.70) in the considered approximation propagates with a group velocity.

Taking into account eq. (2.73), eq. (2.72) implies that the quantity $m_{z0}^{(2)}$ has a structure similar to that of (2.70), so in what follows we will unite $m_{z0}^{(1)}$ and $m_{z0}^{(2)}$ in one expression, (2.70). The second-order approximation equations will lead to the following expressions for the remaining wave components:

$$h_{x0}^{(2)} = -m_{x0}^{(2)} = \frac{1}{b_0}\frac{\partial m_0}{\partial \tau_1}\, e^{i\omega\tau_0} + \text{h.c.}\,,$$

$$h_{z0}^{(2)} = \frac{2u}{1+u^2}\,\varepsilon_{y0}^{(2)} = \frac{2}{i\omega}\frac{u^2}{(1-u^2)^2}\left(\frac{\partial m_0}{\partial \tau_1} + u\,\frac{\partial m_0}{\partial \xi_1}\right) e^{i\omega\tau_0} + \text{h.c.}$$

The system of equations which is of second order in $\alpha_0$ for wave components with the index $n \geq 1$ is represented by the three equations,

$$\left(\frac{\partial^2}{\partial \tau_0^2} - \frac{n^2k_d^2u^2}{1-u^2}\right)\tilde{h}_{yn}^{(2)} + \frac{nk_d u}{1-u^2}\frac{\partial m_{xn}^{(2)}}{\partial \tau_0} = -\frac{(1-\cos \pi n)}{n\pi}\frac{1}{(1-u^2)}\frac{\partial^2}{\partial \tau_0^2}(m_{\perp 0}^{(1)})^2\,,$$

$$b_0\frac{\partial m_{xn}^{(2)}}{\partial \tau_0} + \frac{\partial^2 m_{zn}^{(2)}}{\partial \tau_0^2} = nk_d u\tilde{h}_{yn}^{(2)}, \tag{2.74}$$

$$\left(\frac{\partial^2}{\partial \tau_0^2} - \frac{n^2k_d^2u}{1-u^2}\right)\frac{\partial m_{xn}^{(2)}}{\partial \tau_0} = \left(h_0 - \frac{u^2}{1-u^2}\right)\frac{\partial^2 m_{zn}^{(2)}}{\partial \tau_0^2} - h_0\,\frac{n^2k_d^2u^2}{1-u^2}\,m_{zn}^{(2)}\,,$$

expressing the components of the $n$th waveguide harmonics via the solutions of the first approximation (2.70). The solution of eqs. (2.74) looks rather cumbersome; one can see, however, that the quantity $\tilde{h}_{yn}^{(2)}$ we will need further has the structure

$$\tilde{h}_{yn}^{(2)} = A_n(\omega)m_0^2\, e^{2i\omega t_0} + \text{h.c.} \tag{2.75}$$

The expression for the coefficient $A_n(\omega)$ will later be given for several particular cases.

Let us examine the third-order equations in $\alpha$ for the TEM components of the wave. After some simple algebra the initial rather awkward system is reduced to the three equations,

$$\frac{\partial m_{z0}^{(3)}}{\partial \tau_0} + b_0 m_{x0}^{(3)} = -\frac{\partial m_0}{\partial \tau_2}\, e^{i\omega\tau_0} - \frac{i\omega}{b_0}\sum_n \frac{1-\cos n\pi}{n\pi}\, A_n|m_0|^2 m_0\, e^{i\omega\tau_0} + \text{h.c.}\,,$$

$$\begin{aligned}\frac{\partial m_{x0}^{(3)}}{\partial \tau_0} - h_0 m_{z0}^{(3)} + h_{z0} &= \frac{1}{b_0}\frac{\partial^2 m_0}{\partial \tau_1^2}\, e^{i\omega\tau_0} + \frac{i\omega}{b_0}\frac{\partial m_0}{\partial \tau_2}\, e^{i\omega\tau_0} + \frac{1}{2}\frac{1}{1-u^2}\left(3 + \frac{\omega^2}{b_0^2}\right)|m_0|^2 m_0\, e^{i\omega\tau_0} \\ &\quad + \sum_n \frac{1-\cos n\pi}{\pi n}\, A_n|m_0|^2 m_0\, e^{i\omega\tau_0} + \text{h.c.} + \dots\,,\end{aligned} \tag{2.76}$$

$$\begin{aligned}\frac{\partial h_{z0}^{(3)}}{\partial \tau_0} &= \frac{u^2}{1-u^2}\frac{\partial m_{z0}^{(3)}}{\partial \tau_0} + \frac{2u^2}{(1-u^2)^2}\left(\frac{\partial m_0}{\partial \tau_2} + u\,\frac{\partial m_0}{\partial \xi_2}\right) e^{i\omega\tau_0} \\ &\quad + \frac{1}{i\omega}\frac{u^2}{1-u^2}\left(\frac{\partial}{\partial \tau_1} + u\,\frac{\partial}{\partial \xi_1}\right)\left[(1+3u^2)\frac{\partial}{\partial \tau_1} + u(3+u^2)\frac{\partial}{\partial \xi_1}\right] m_0\, e^{i\omega\tau_0} + \text{h.c.} + \dots\,.\end{aligned}$$

Nonresonance terms have been omitted in the r.h.s. of the equations, because they are inessential for the subsequent analysis.

Equations (2.76) can be easily reduced to one closed equation of the form $(\partial^2/\partial\tau_0^2 + \omega^2)m_{x0}^{(3)} = F\,e^{i\omega\tau_0} + \text{h.c.} + \dots$. The condition that the secular term in the r.h.s. be absent leads to the required evolution equation for the envelope

$$2i\omega\left(\frac{\partial}{\partial\tau_2} + v_{gr}\frac{\partial}{\partial\xi_2}\right)m_0 + \frac{b_0\omega^2(b_0^2-\omega^2)(3b_0h_0+\omega^2u^2)(b_0h_0-\omega^2)}{[b_0^2h_0+(b_0h_0-\omega^2)^2]^3}\frac{\partial^2 m_0}{\partial\xi_1^2}$$
$$+\frac{b_0\omega^2(b_0^2-\omega^2)}{b_0^2h_0+(b_0h_0-\omega^2)^2}\left(\frac{3}{2}+\frac{\omega^2}{2b_0^2}+\frac{1}{b_0}\sum_n\frac{1-\cos n\pi}{\pi n}A_n\right)|m_0|^2m_0 = 0\,. \tag{2.77}$$

In deriving eq. (2.77) we assumed that the TEM component was the principal component of the wave, so, in accordance with the equation just derived, the dispersion properties of the envelope $m_0$ are determined only by the TEM harmonics structure. Waveguide TM harmonics affect the nonlinear wave properties. Since eq. (2.77) is a NSE [28], it is interesting to clarify the question whether any solitary waves of the EMS wave envelope exist in the considered waveguide.

One can easily see that the dispersion coefficient in the term proportional to $\partial^2 m_0/\partial\xi_1^2$ is positive for the slow branch EMS waves ($\omega < \sqrt{b_0h_0}$) and negative for those of the fast branch ($\omega > b_0$). So, to employ the Lighthill criterion (see the preceding section), one must analyze the frequency dependence of the coefficient before the nonlinear term in the evolution equation (2.77).

Assuming the external field $h_0$ to be small, let us examine the slow branch waves. The analysis showed that for $k \gtrsim 1$ in the low frequency region $\omega \ll \sqrt{h_0}$ the expression for $A_n(\omega)$ assumes the form

$$A_n(\omega) \approx -\frac{1-\cos\pi n}{\pi n}\left(1+n^2k_d^2\,\frac{1+4\omega^2/h_0^2}{4\omega^2/h_0^2}\right)^{-1}. \tag{2.78}$$

In the case of $h_0-\omega^2 \ll h_0$ the coefficient $A_n(\omega)$ is given by the formula

$$A_n(\omega) \approx -\frac{1-\cos\pi n}{\pi n}\left(1+n^2k_d^2\,\frac{h_0-\omega^2}{3h_0}\right)^{-1}. \tag{2.79}$$

Substituting expressions (2.78) and (2.79) to (2.77) with the subsequent summing over $n$ shows that the coefficient in the nonlinear term of the NSE is positive. Thus, slow EMS waves are unstable with respect to automodulation.

Let us proceed with the analysis of the fast branch. We will confine ourselves to the case of high frequencies, $\omega \gg 1$; eq. (2.74) implies the following expression for $A_n(\omega)$:

$$A_n(\omega) \approx -4\omega^2(1-\cos\pi n)/n^3k_d^2\pi\,. \tag{2.80}$$

Taking account of (2.80), eq. (2.77) assumes the form

$$2i\omega\left(\frac{\partial m_0}{\partial\tau_2}+v_{gr}\frac{\partial m_0}{\partial\xi_2}\right)-\frac{1}{\omega^2}\frac{\partial^2 m_0}{\partial\xi_1^2}-\frac{\omega^2}{2}\left(1-\frac{\pi^2\omega^2}{3k_d^2}\right)|m_0|^2m_0=0\,. \tag{2.81}$$

Equation (2.81) shows that the fast EMS waves in the considered waveguide are unstable with respect to automodulation in the frequency region $b_0 \approx 1 \ll \omega < \sqrt{3}k_d/\pi$ (here the $k_d$ must be large enough). In the higher frequency region the waves are stable.

The NSE was thoroughly investigated, so there is no necessity to write down the solution of (2.77). Note only that the formal smallness parameter $\alpha$ in this equation should be set to unity. In the solution of eq. (2.77) the small parameter is the soliton amplitude.

## 3. Stationary waves

The above-described monochromatic EMS waves of finite amplitude exist due to the uniformity and azimuthal symmetry of the forces acting on the magnetization vector in the plane normal to the direction of wave propagation. That is why the harmonic oscillations of the magnetization vector are excited by the monochromatic magnetic field even in the case of a finite oscillation amplitude. The magnetization oscillations arising when an EMS propagates at an angle to the magnetic anisotropy axis or to the magnetizing field are not harmonic; as the wave amplitude grows, the wave spectrum becomes enriched with higher harmonics. We will consider the structure of the simplest nonmonochromatic waves stabilized in a ferromagnet, waves depending on the time $t$ and the spatial coordinate $z$ (through the expression $z - vt$) directed along the propagation direction [37–42]. We will pay special attention to analyzing solitary EMS waves which drastically differ from the well-known magnetostatic magnetization solitons [11–13].

### 3.1. Solitary waves in an isotropic ferromagnet

First of all let us examine the structure of stationary EMS waves propagating in the $\boldsymbol{e}_z$ direction in an isotropic ferrodielectric magnetized by the field $H_0$ along the $\boldsymbol{e}_y$ axis. The initial set of equations (2.1), (2.2) can be written in terms of the dimensionless variables,

$$
\begin{gathered}
\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \tau^2}\right) h_y = \frac{\partial^2}{\partial \tau^2} (\cos\theta \cos\varphi)\,, \qquad \cos\theta \,\frac{\partial \theta}{\partial \tau} = h_1 \cos\theta - v_0^2 \frac{\partial}{\partial \xi}\left(\cos^2\theta \,\frac{\partial \varphi}{\partial \xi}\right),\\
\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \tau^2}\right) h_x = \frac{\partial^2}{\partial \tau^2} (\cos\theta \sin\varphi)\,,\\
\cos\theta\, \frac{\partial \varphi}{\partial \tau} = -(\cos\theta + h_2)\sin\theta + v_0^2\left[\sin\theta\cos\theta\left(\frac{\partial\varphi}{\partial\xi}\right)^2 + \frac{\partial^2\theta}{\partial\xi^2}\right],\\
h_1 = h_y \sin\varphi - h_x\cos\varphi\,, \qquad h_2 = h_y\cos\varphi + h_x \sin\varphi\,, \qquad v_0^2 = 4\pi\varepsilon_0 g^2 M_0^2/c^2\,,
\end{gathered}
\tag{3.1}
$$

where $M_x = M_0 \cos\theta \sin\varphi$, $M_y = M_0 \cos\theta\cos\varphi$. The remaining notation coincides with that used previously.

We will be interested in the existence of the solutions of eqs. (3.1) describing the propagation of the stationary solitary waves satisfying the boundary condition $\theta$, $\varphi \ll 1$, $h_y \to h_0 = H_0/4\pi M_0$, $h_x \to 0$ when $|\xi| \to \infty$. Although it is difficult to find exact solutions of the set, one can determine, however, the necessary conditions and the domains of existence for soliton solutions to the magnetization dynamics; for this one needs to examine the solutions of (3.1) in the asymptotic domain $|\xi| \to \infty$, where the homogeneous state of magnetization is established. Here $\theta$, $\varphi \ll 1$, permitting one to linearize set (3.1) and seek solutions in the form of exponentially decaying waves [43, 44]. Setting $\theta$, $\varphi \sim \exp[-\kappa(\xi - u\tau)]$, we get in the asymptotic region the dispersion relation from (3.1),

$$\kappa^2 = -\frac{h_0^2}{4u_m^2}\left[\phi \pm \left(\phi^2 + 4\mu_0 \frac{\mu_0 u^2 - 1}{1 - u^2}\right)^{1/2}\right], \qquad \phi = \frac{u^2}{u_m^2} - \mu_0 + \frac{\mu_0^2 u^2 - 1}{1 - u^2}, \tag{3.2}$$

where $\mu_0 = 1 + 1/h_0$, $u_m^2 = v_0^2 h_0$. It is evident that the solutions of the dispersion equation (3.2), $\kappa = \kappa(u)$, which meet the condition $\mathrm{Re}[\kappa(u)] \neq 0$, correspond to solitary waves.

Analysis of relation (3.2) shows that there exist soliton solutions of the type [43, 44] for which

$$\begin{aligned} &u^2 \leq u_m^2(\mu_0 + 1) \ll 1, \\ &\kappa^2 \simeq \frac{h_0^2}{2u_m^2}\left\{\mu_0 + 1 - \frac{u^2}{u_m^2} \pm \left[\left(\mu_0 + 1 - \frac{u^2}{u_m^2}\right)^2 - 4\mu_0\right]^{1/2}\right\}. \end{aligned} \tag{3.3}$$

No electrodynamic factors practically affect the solutions obtained, due to the small value of the limit velocity $u_m\sqrt{\mu_0 + 1}$ (analogous to the Woker velocity) as compared to the velocity with which the electromagnetic interaction is transferred in the medium.

Relations (3.3) can be obtained directly from the system of equations (3.1), after passing to the magnetostatic limit $h_1 \to h_0 \sin\varphi$, $h_2 \to h_0 \cos\varphi$. It should be noted that when the velocity of the magnetostatic soliton propagation $|u| < u_m(\sqrt{\mu_0} - 1)$, then the angles $\theta(\xi - u\tau)$ and $\varphi(\xi - u\tau)$ monotonically decrease when $|\xi| \to \infty$; when the velocity lies in the range $u_m(\sqrt{\mu_0} - 1) < |u| < u_m(\sqrt{\mu_0 + 1})$ the decrease in $\theta$ and $\varphi$ is accompanied by oscillations (cf. the results of refs. [43, 44]).

Relation (3.2) implies the existence of a new branch of soliton solutions of the magnetization dynamics equations. In the asymptotic region these solutions are characterized by the relations

$$1/\mu_0 \leq u^2 < 1, \qquad \kappa^2 \simeq (\mu_0 h_0^2/u^2)(\mu_0 u^2 - 1)/(1 - u^2). \tag{3.4}$$

The minimum velocity of propagation of the solitary waves of branch (3.4) far exceeds the maximum velocity of the solitons of branch (3.3), the former being of the order of the light velocity. This means that in principle, solutions of (3.4) cannot be obtained within the framework of the magnetostatic approach. A very important feature of the solutions obtained is the independence of the quantity $\kappa$, which describes the spatial localization of a soliton, of the constant $\alpha$; this constitutes evidence of a weak dependence of the spatial structure of the soliton of the considered electromagnetic branch on the effective field of the inhomogeneous exchange interaction. Thanks to that, one can give an exhaustive analytical description of the considered solitary waves in many cases. Setting $v_0^2 = 0$ in eqs. (3.1), we obtain the expressions

$$\begin{aligned} &\sin\theta = \frac{1}{\sigma\cosh[\kappa(\xi - u\tau)]}, \qquad \sin\varphi = \sqrt{\mu_0 \frac{1 - u^2}{\mu_0 u^2 - 1}}\,\frac{-\tanh[\kappa(\xi - u\tau)]}{\sqrt{\sigma^2\cosh^2[\kappa(\xi - u\tau)] - 1}}, \\ &\sigma^2 = 1 + \frac{[\mu_0(1 - u^2) - \mu_0 u^2 + 1]^2}{4\mu_0(1 - u^2)(\mu_0 u^2 - 1)}, \end{aligned} \tag{3.5}$$

where the coefficient $\kappa$ is defined by formula (3.4).

Analyzing (3.5) we see that the dependence of the soliton amplitude $\sin(\theta_{max})$ on its propagation velocity $u$ is nonmonotonic; $\sin(\theta_{max}) = 1$ at $u^2 = (\mu_0 + 1)/2\mu_0$ and vanishes at the end points of the interval of permissible values of the velocity $u$ from (3.4). It is important to bear in mind that at the point $u = [(\mu_0 + 1)/2\mu_0]^{1/2}$ an abrupt change occurs of the topological properties of the solitary wave. It is easy to see that in case of the solitons propagating with the velocities $|u| < \sqrt{(\mu_0 + 1)/2\mu_0}$ the angle $\varphi$ grows monotonically from zero when $\xi \to -\infty$ to $2\pi$ when $\xi \to \infty$, i.e., the magnetization vector makes a complete revolution around the $\boldsymbol{e}_z$ axis in the process of wave propagation. If the soliton velocity exceeds the value $[(\mu_0 + 1)/2\mu_0]^{1/2}$, then the angle $\varphi$ becomes an odd oscillating function, the values of $\varphi$ at $\xi = -\infty$ and $\xi = \infty$ being both equal to zero, while $\sin(\varphi_{max}) = \mu_0^2(1 - u^2)/(\mu_0 u^2 - 1) < 1$. Thus the considered solitary waves are topological solitons in the velocity range $1/\mu_0 < u^2 < (\mu_0 + 1)/2\mu_0$ and are dynamic solitons in the velocity range $(\mu_0 + 1)/2\mu_0 < u^2 < 1$, in analogy with the magnetization magnetostatic solitons [11].

One should notice the relation between the expression $\kappa^2 = \kappa^2(u)$ in formulas (3.4) with the dispersion dependence for linear EMS waves in a transversally magnetized ferromagnet [6],

$$k^2(\omega) = \omega^2(\mu_0^2 h_0^2 - \omega^2)/(\mu_0 h_0^2 - \omega^2)\,, \tag{3.6}$$

which can be easily derived from (3.4) by setting $\kappa \to ik$, $\omega = ku$. Comparing relations (3.4) and (3.6), we see that the interval of the admissible values of the propagation velocities of solitary waves (3.5) coincides with the gap separating the existence domain of the slow-branch linear EMS waves $[u^2 < 1/\mu_0$, according to (3.6)] with that of the fast-branch waves $(u^2 > 1)$.

The interaction of the solitary waves of the form (3.5) is of utter importance, and to elucidate this question one must study the corresponding nonstationary solutions of eqs. (3.1). In the general case the solution of this physical problem requires overcoming considerable mathematical difficulties. We will seize the opportunity to substantially simplify the initial equations in the case where the waves propagate with velocities close to the lower limit (3.4) in a ferromagnet positioned in a field $H_0 \ll 4\pi M_0$, with $H_{x,y} \sim 4\pi M_0 u^2 \ll 4\pi M_0$, according to (3.1). One can easily see that under such conditions the magnetodipole field $H_z = -4\pi M_z$ generating the effective anisotropy of the type of the light magnetization plane will have a dominant effect on the magnetization dynamics. Thanks to this fact the vector $\boldsymbol{M}$ deviates only slightly from the $xy$ plane during the wave propagation process. The last equation in (3.1) implies the approximate relation between the angles $\theta$ and $\varphi$

$$\theta \simeq -\partial\varphi/\partial\tau\,, \tag{3.7}$$

which permits one to reduce two magnetization dynamics equations to one,

$$\partial^2\varphi/\partial\tau^2 = h_y \cos\varphi - h_x \sin\varphi\,, \tag{3.8}$$

not containing the wave function $\theta(\xi, \tau)$.

In the theory of magnetization, magnetostatic waves in a light-plane ferromagnet, relation (3.7) yields (see ref. [11]) a closed sine–Gordon equation of the function $\varphi(\xi, \tau)$. In the case of EMS waves eq. (3.8) must be solved jointly with the equations for the vortex electromagnetic fields from (3.1), $h_x$ and $h_y$, where one must set $\cos\theta \to 1$. We obtain

$$\left(\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\tau^2}\right)h_y = \frac{\partial^2}{\partial\tau^2}\cos\varphi\,, \qquad \left(\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\tau^2}\right)h_x = \frac{\partial^2}{\partial\tau^2}\sin\varphi\,. \tag{3.9}$$

Let us utilize now the fact that the wave velocities are close to the value $1/\sqrt{\mu_0} \approx \sqrt{h_0} \ll 1$ in order to lower the order of the wave equations (3.9). Let us use a reference frame that moves with the velocity $\sqrt{h_0}$. The corresponding Lorentz transformation,

$$\sqrt{1-h_0}\xi' = \xi - \sqrt{h_0}\tau\,, \qquad \sqrt{1-h_0}\tau' = \sqrt{h_0}\tau - h_0\xi\,,$$

virtually coincides with the Galilean transformation. Taking account of the slow variation of the vortex field and magnetization in the new reference frame ($\partial/\partial\tau' \ll \partial/\partial\xi'$), we will get, by combining (3.8) and (3.9), the following equations:

$$\frac{\partial}{\partial\xi'}(h_y - h_0\cos\varphi) = 2h_0\sin\varphi\,\frac{\partial\varphi}{\partial\tau'}\,, \qquad \frac{\partial}{\partial\xi'}(h_x - h_0\sin\varphi) = -2h_0\cos\varphi\,\frac{\partial\varphi}{\partial\tau'}\,,$$
$$h_0\,\frac{\partial^2\varphi}{\partial\xi'^2} = h_x\cos\varphi - h_y\sin\varphi\,. \tag{3.10}$$

Further simplification is achieved by the change of variables,

$$f_1 = h_x\cos\varphi - h_y\sin\varphi\,, \qquad f_2 = h_x\sin\varphi + h_y\cos\varphi - h_0\,. \tag{3.11}$$

By substituting (3.11) into (3.10) and excluding the quantities $f_1$ and $f_2$, we arrive at a closed equation for the angular function $\varphi(\xi', \tau')$,

$$\partial\varphi/\partial\tau' + \tfrac{1}{4}(\partial\varphi/\partial\xi')^3 + \tfrac{1}{2}\partial^3\varphi/\partial\xi'^3 = 0\,. \tag{3.12}$$

First of all, consider a stationary solution of eq. (3.12) which corresponds to a solitary wave. It is not difficult to show that such a solution has the form

$$\varphi = \pi + 2\arctan[\sinh\sqrt{2u'}(\xi' - u'\tau')]\,, \qquad \sin\varphi = \frac{-2\tanh[\sqrt{2u'}(\xi' - u'\tau')]}{\cosh[\sqrt{2u'}(\xi' - u'\tau')]}\,. \tag{3.13}$$

Comparing the obtained solution with the exact solution of (3.5) we see that they coincide, provided that

$$u' = u/\sqrt{h_0} - 1 \ll 1\,, \tag{3.14}$$

which agrees with the assumptions underlying the derivation of eq. (3.12).

The evolution equation (3.12) possesses a remarkable property: it is fully integrable. Differentiating this equation with respect to $\xi'$ and applying the linear exchange of variables $\xi' = 2\xi''$, $\tau' = 16\tau''$ leads to the mKdV equation in the canonical form [45],

$$\frac{\partial\Psi}{\partial\tau''} + \frac{\partial^3\Psi}{\partial\xi''^3} + 6\Psi\,\frac{\partial\Psi}{\partial\xi''} = 0\,, \qquad \varphi = 2\int_{-\infty}^{\xi''}\Psi\,d\xi''\,. \tag{3.15}$$

The general analytic solution of this equation was found in ref. [45] by a method employed in the inverse scattering problem. This $N$-soliton formula enables analytic description of the interaction between the solitary waves (3.15). The interaction of solitary waves in (3.5) can be expected to have the character of elastic collisions in a broader range of velocities than in the case of (3.14). This conjecture is supported by the above-mentioned nonzero topological charge of the solitary waves (3.5) whose velocities lie within the interval $1/\mu_0 < u^2 < (\mu_0 + 1)/2\mu_0$.

Let us dwell on another limit case that also allows a drastic simplification of eqs. (3.1). Expressing the vortex electromagnetic field components $h_y$ and $h_x$ via the angular variables $\theta$ and $\varphi$ by means of the magnetization dynamics equations,

$$h_y + \mathrm{i}h_x = -\cos\theta\, \mathrm{e}^{\mathrm{i}\varphi} + (\mathrm{i}/\sin\theta)\partial(\cos\theta\, \mathrm{e}^{\mathrm{i}\varphi})/\partial\tau ,$$

and substituting the relation obtained into the wave equations (3.1), we obtain the closed equation,

$$\left(\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\tau^2}\right)\left(\frac{\partial}{\partial\tau}\cos\theta\, \mathrm{e}^{\mathrm{i}\varphi}\right)(\sin\theta)^{-1} = -\mathrm{i}\,\frac{\partial^2}{\partial\xi^2}\cos\theta\, \mathrm{e}^{\mathrm{i}\varphi} . \tag{3.16}$$

Let us consider the waves whose propagation velocities are close to the upper limit (3.4), i.e., $u \approx 1$. Equation (3.16) describing such waves can be substantially simplified by using the inequality $\partial/\partial\xi + \partial/\partial\tau \sim (1-u)\partial/\partial\tau \ll \partial/\partial\tau$. It is not difficult to see that the system of equations

$$\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\tau}\right)\frac{\partial\theta}{\partial\tau} - \tfrac{1}{2}\cos\theta\,\frac{\partial\varphi}{\partial\tau} = 0 , \qquad \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\tau}\right)\left(\cot\theta\,\frac{\partial\varphi}{\partial\tau}\right) + \tfrac{1}{2}\sin\theta\,\frac{\partial\theta}{\partial\tau} = 0 , \tag{3.17}$$

with the obvious integral

$$(\partial\theta/\partial\tau)^2 + \cot^2\theta\,(\partial\varphi/\partial\tau)^2 = \text{const.} , \tag{3.18}$$

is valid.

The constant in (3.18) is determined from eq. (3.1) and it is equal to $\mu_0^2 h_0^2$. As a result, we obtain a closed equation for the quantity $\theta(\xi, \tau)$,

$$(\partial/\partial\xi + \partial/\partial\tau)\partial\theta/\partial\tau + \tfrac{1}{2}\theta\sqrt{\mu_0^2 h_0^2 - (\partial\theta/\partial\tau)^2} = 0 . \tag{3.19}$$

In deriving eq. (3.19) we considered the exact solution of (3.5) that shows that the solitary wave propagation with velocities $u \approx 1$ is accompanied with a small excitation of the magnetization ($\theta \ll 1$, $\varphi \ll 1$).

One can easily see that the soliton solution of eq. (3.19),

$$\theta = \sqrt{8\mu_0 h_0(1-u)}\,\cosh^{-1}[\sqrt{\mu_0 h_0/2(1-u)}(\xi - u\tau)] , \tag{3.20}$$

coincides with (3.5), provided that

$$1 - u \ll 1/4\mu_0 h_0 . \tag{3.21}$$

Equation (3.19) is fully integrable and has $N$-soliton solutions. By the exchange of variables $\partial\theta/\partial\tau = \mu_0 h_0 \sin\Psi$, $\xi = \xi' + \frac{1}{2}\mu_0 h_0 \tau'$, $\tau = -\frac{1}{2}\mu_0 h_0 \tau'$, eq. (3.19) is reduced to the well-known sine–Gordon solution,

$$\partial^2\Psi/\partial\xi'\partial\tau' = \sin\Psi\,, \qquad \theta = -2\,\partial\Psi/\partial\xi'\,. \tag{3.22}$$

The soliton solutions of eq. (3.22) have been thoroughly studied; in particular, the $N$-soliton formula describing collisions of $N$ solitons has the form [46]

$$\tfrac{1}{16}\theta^2 = \ln\det|B^2 + I|\,, \qquad B_{ij} = [\sqrt{C_iC_j}/(k_i + k_j)]\,\mathrm{e}^{-(K_i+K_j)\xi'}\,, \quad C_j = \mathrm{e}^{-\tau'/2K_j+\gamma_j}\,, \tag{3.23}$$

where $I$ is an $N \times N$ unit matrix $K_j$, $K_i$ and $\gamma_j$ are constants. Formula (3.23) shows that the collision of the solitary waves under consideration has an elastic character.

### 3.2. Waves in an anisotropic ferromagnet

Let us pass now to analyzing how the magnetic anisotropy affects the propagation of the stationary EMS waves of finite amplitude. First we will consider the solitary wave dynamics in a light axis ferromagnet. We will introduce angular functions $\theta$ and $\varphi$ that describe the magnetization vector in a polar system of coordinates whose axis coincides with the magnetic anisotropy axis $\boldsymbol{n}$. We obtain

$$M_z = M_0(\cos\theta\cos\chi_n - \sin\theta\cos\varphi\sin\chi_n)\,, \tag{3.24}$$

where $\cos\chi_n = \boldsymbol{n}\cdot\boldsymbol{e}_z$, and the axis $\boldsymbol{e}_x$ has been chosen so that the vector $\boldsymbol{n}$ lies in the plane $xz$.

Let us consider the structure of the stationary solitary waves propagating along the anisotropy axis ($\chi_n = 0$). Substituting expressions (3.24) into the general set of equations (2.1), (2.2), we will arrive at a set of two nonlinear equations whose solution is

$$\begin{gathered}\tan^2(\theta/2) = \frac{a-b}{1 + 2a(a+b)^{-1}\sinh^2[\kappa_{\|}(\xi - u\tau)]}\,, \qquad \kappa_{\|}^2 = \beta_0 - \frac{u^2}{4v_0^2} - \frac{u^2}{1-u^2}\,,\\ b = |1 - 2v_0^2/u^2|\,, \qquad a = [(1 - 2v_0^2/u^2)^2 + 4(v_0^2/u^2)\kappa_{\|}^2]^{1/2}\,.\end{gathered} \tag{3.25}$$

The solution differs from the one obtained for the first time in ref. [15] only by the term $u^2/(1-u^2)$ in the expression for $\kappa_{\|}^2(u)$ which is due to the EMS interaction. The domain for the existence of the wave in (3.25) is obtained from the condition $\kappa_{\|}^2(u) > 0$; it is determined by the inequality $u^2 < 4\beta_0 v_0^2 - 16\beta_0 v_0^4 \approx 4\beta_0 v_0^2$, which coincides with the one established in ref. [15] for magnetostatic solitons. Thus only magnetostatic magnetization solitons of the stationary profile can propagate along the anisotropy axis in a light axis ferromagnet.

Another situation occurs for $\chi_n \neq 0$. Let us consider the equation set (2.1), (2.2) in the long wave limit $\kappa_{\|}^2 v_0^2 \ll 1$. Here the equations that determine the variation of the functions $\theta$ and $\varphi$ assume the form

$$\begin{gathered}u\dot\theta = -[\sin\chi_n/(1-u^2)][(1-\cos\theta)\cos\chi_n + \sin\theta\cos\varphi\sin\chi_n]\sin\varphi\,,\\ \sin\theta\left[u\dot\varphi + 1 - \frac{\sin^2\chi_n}{1-u^2} + \left(\beta_0 - \frac{\cos^2\chi_n}{1-u^2}\right)\cos\theta\right]\\ = -\frac{\sin\chi_n\cos\chi_n}{1-u^2}\sin\theta\cos\varphi - \frac{\sin\chi_n}{1-u^2}[(1-\cos\theta)\cos\chi_n + \ln\theta\cos\varphi\sin\chi_n]\cos\theta\cos\varphi\,.\end{gathered} \tag{3.26}$$

The solution of (3.26) is

$$\int \frac{\mathrm{d}y}{yQ(y)} = -\frac{\xi - u\tau}{u(1-u^2)}\,, \quad y = \tan(\theta/2)\,,$$

$$Q^2(y) = [u^2(1+\beta_0+y^2) - \beta_0][\sin^2\chi_n - (y\cos\chi_n \mp \sqrt{u^2(1+\beta_0+y^2)-\beta_0})^2]\,, \tag{3.27}$$

$$\sin^2\chi_n \cos\varphi + (1+\beta_0)[\beta_0/(1+\beta_0) - u^2] + (\cos^2\chi_n - u^2)y^2 + 2\sin\chi_n \cos\chi_n \cos\varphi\ y = 0\,.$$

Examine the structure of the solitary waves with $y \to 0$ when $|\xi| \to \infty$. The condition $Q^2(y) > 0$ in the asymptotic region $|\xi| \to \infty$ implies that the admissible values of the soliton propagation lie in the interval

$$\beta_0/(1+\beta_0) \le u^2 \le (\sin^2\chi_n + \beta_0)/(1+\beta_0)\,. \tag{3.28}$$

The above inequalities, valid for solitons of both branches corresponding to different signs ($\mp$) of the term $Q^2(y)$ in eq. (3.27), show that the minimal admissible velocity of the EMS solitons considered is determined by the value of the magnetic anisotropy constant $\beta = 4\pi\beta_0$, the dependence $u_{\min}(\beta_0)$ being the same as the corresponding dependence of the minimum soliton velocity in an isotropic ferromagnet on the constant transversal magnetic field $H = 4\pi h_0$ [see eq. (3.4)]. This is quite natural, since the quantities $\beta$ and $h_0$ determine the boundary of the opaqueness region for linear monochromatic EMS waves in the corresponding cases [see eq. (3.6)]. According to eq. (3.28), the width of the interval of the admissible values of the solitary wave velocities tends to zero when $\chi_n \to 0$.

A characteristic spatial soliton size is given by the decaying decrement of the function $y(\xi - u\tau)$ in the asymptotic region. Equation (3.27) yields the formula

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \ln[y(\xi - u\tau)] \to \frac{1+\beta_0}{u(1-u^2)} \left(u^2 - \frac{\beta_0}{1+\beta_0}\right)^{1/2} \left(\frac{\sin^2\chi_n + \beta_0}{1+\beta_0} - u^2\right)^{1/2}\,, \tag{3.29}$$

indicating that in the range of velocities given by (3.28) the condition that has been used in deriving (3.27), $\kappa_\parallel^2 v_0^2 \ll 1$, is satisfied. This condition means that the inhomogeneous exchange affects the structure of the considered waves only slightly.

According to eq. (3.29), the solitons under consideration are delocalized on the boundaries of the admissible velocity region. This is true, however, only for the case $u^2 \to (\beta_0 + \sin^2\chi_n)/(1+\beta_0)$. Analysis of eq. (3.27) shows that for $u^2 = \beta_0/(1+\beta_0)$ the solutions become algebraic solitons described by the expression,

$$y = \frac{\sin\chi_n}{|\cos\chi_n \mp \sqrt{\beta_0/(1+\beta_0)}|} \left(1 + \frac{(1+\beta_0)^2 \sin^2\chi_n}{[\cos\chi_n \mp \sqrt{\beta_0/(\beta_0+1)}]^2}\,(\xi - u\tau)\right)^{-1/2}\,. \tag{3.30}$$

At $\chi_n = 0$ the amplitude of the solitary wave (3.30) vanishes which shows that EMS solitons cannot propagate along the anisotropy axis.

Let us analyze the dependence of the soliton amplitude and of the asymptotic values of the magnetization rotation angle $\varphi$ on the velocity $u$ and the orientation angle $\chi_n$.

The soliton amplitude is found by means of solving the equation $Q^2(y^{\max}) = 0$,

$$y_{\mp}^{\max} = \frac{|\sin \chi_n \cos \chi_n \pm \sqrt{(1-u^2)[u^2(1+\beta_0) - \beta_0 \cos^2\chi_n]}|}{|u^2 - \cos^2\chi_n|} \tag{3.31}$$

From formula (3.31) it follows that the amplitude $y_-$ grows infinitely when the soliton velocity tends o $\pm\cos \chi_n$. This situation actually occurs when the orientation angle belongs to the range rctan$(\sqrt{1/(1+\beta_0)}) < \chi_n < \arctan(\sqrt{1/\beta_0})$, the amplitude $y_+$ being finite. The values of $u = \pm\cos \chi_n$ livide the existence domain of the minus-branch solitons into regions with different types of magnetic noment rotation around the anisotropy axis.

Formulas (3.27) permit us to determine the asymptotic values of the angle $\varphi$,

$$\varphi(\xi = -\infty) = -\varphi(\xi = \infty) = -\arccos[\sqrt{u^2(1+\beta_0) - \beta_0}] \sin \chi_n . \tag{3.32}$$

Relations (3.32) hold for solitons of the both branches; they show that the solitary waves under onsideration are magnetization rotation waves, similar to the corresponding magnetostatic waves [1, 15].

As has been shown, the stationary solitary EMS waves cannot propagate along the anisotropy axis. 'hat is why it is interesting to investigate solutions of the more general form

$$\theta = \theta(\xi - u\tau) , \qquad \varphi = \omega\tau + \Psi(\xi - u\tau) , \tag{3.33}$$

hich depend on the two parameters: $\omega$ and $u$. The waves of the type considered which satisfy the oundary conditions $\theta \to 0$ when $|\xi| \to \infty$ are called envelope solitons [47] and are being extensively udied in the magnetic soliton theory [11–13]. By substituting (3.33) into the initial set of equations (2.1), (2.2) and (3.24) at $\chi_n = 0$, we obtain the following equations for the functions $\theta$ and $\Psi$:

$$\begin{aligned} &\left[\frac{d^2}{d\xi^2} + \left(\omega + iu \frac{d}{d\xi}\right)^2\right] R\, e^{i\Psi} + \left(\omega + iu \frac{d}{d\xi}\right)^2 \sin\theta\, e^{i\Psi} = 0 , \\ &\cos\theta\, R = [1 + (\beta_0 + v_0^2 \dot{\Psi}^2 - 1)\cos\theta - \omega + u\dot{\Psi}] \sin\theta \\ &\qquad - v_0^2 \ddot{\theta} - i\cos\theta\, [u\dot{\theta} + v_0^2(2\dot{\theta}\dot{\Psi}\cos\theta + \ddot{\Psi}\sin\theta)] . \end{aligned} \tag{3.34}$$

It is difficult to obtain an analytic solution of eq. (3.34); that is why we will consider the solution in e asymptotic region $|\xi| \to \infty$ where $\theta \to 0$, so that the equation becomes a linear one. By setting $e^{i\Psi} \sim \exp[-i\kappa(\xi - u\tau)]$ we obtain from (3.34) the dispersion relation,

$$(\beta_0 + \kappa^2 v_0^2 - \omega - \kappa u)[\kappa^2 - (\omega + \kappa u)^2] = (\kappa u + \omega)^2 , \tag{3.35}$$

at gives the quantity $\kappa$ as a function of the parameters $\omega$ and $u$.

A necessary condition for the existence of soliton solutions of eq. (3.34) exponentially decreasing nen $|\xi| \to \infty$ is that the dispersion relations (3.35) have complex roots. The region in the $(\omega, u)$ plane nere the function $\kappa(\omega, u)$ assumes complex values determines the existence domain of the envelope litons of EMS waves.

Let us first examine solutions of (3.35) in the long wave limit $\kappa v_0 \to 0$ which separates the EMS anch. A third-order equation following from (3.35) remains rather cumbersome, and the well-known ;ebraic condition that complex roots will appear brings about little information.

In the case of $\beta_0 \ll 1$, realized in practice, the boundaries of the region of existence of EMS solitons can be determined analytically. Taking into account that eq. (3.35) in the regions $|\omega + \kappa u| \ll 1$ and $|\omega + \kappa u| \gg \beta_0$ (these regions overlap owing to the condition $\beta_0 \ll 1$) can be solved with respect to the quantity $\omega + \kappa u$, we will rewrite (3.35) in the form $\omega + \kappa u = \omega_i(\kappa)$,

$$\omega_{1,2}(\kappa) = -\tfrac{1}{2}\kappa^2(1 \mp \sqrt{1 + 4\beta_0/\kappa^2})\,, \qquad \omega_{3,4} = \tfrac{1}{2}(1 \mp \sqrt{1 + 4\kappa^2})\,. \tag{3.36}$$

The number of real roots of eq. (3.35) at $v_0 = 0$ equals the number of intersections of the straight line $\omega + \kappa u$ with the curves $\omega_i(\kappa)$ on the $(\omega, \kappa)$ plane. A simple analysis shows that the boundaries of the domain of soliton existence on the $(\omega, \kappa)$ plane, which corresponds to the absence of three intersections of the straight line $\omega + \kappa u$ with the curves $\omega_i(\kappa)$, are given by the equations of the tangents to the curves $\omega_i(\kappa)$. The domain of existence of the solitons under consideration is shaded in fig. 5, the boundary curves being given by the equations

$$\begin{aligned} u_{1,2}(\omega) &= 2\sqrt{\beta_0}[1 \mp (\omega/2\beta_0)(\sqrt{1 + 8\beta_0/\omega} \mp 1)]^{3/2}[2 \mp (\omega/2\beta_0)(\sqrt{1 + 8\beta_0/\omega} \mp 1)]^{-1}\,, \\ u_3(\omega) &= 2\sqrt{\omega(1-\omega)}\,. \end{aligned} \tag{3.37}$$

The precession frequency of magnetization in the proper reference frame changes within the range $0 < \omega < 1$. One can show that in the case of arbitrary values of the constant magnetic anisotropy $\beta_0$ the EMS soliton velocity is bounded by the condition $|u| < 1$ while the frequency $\omega$ lies within the range $0 < \omega < 1 + \beta_0$.

Let us examine now the domain of existence of magnetostatic envelope solitons. Here we have $\omega/\kappa \sim u \sim v_0 \ll 1$, and we get from (3.35),

$$\beta_0 + \kappa^2 v_0^2 - \omega - \kappa u = \frac{(\omega/\kappa + u)^2}{1 - (\omega/\kappa + u)^2} \approx \left(\frac{\omega}{\kappa} + u\right)^2 \sim v_0^2 \ll 1\,. \tag{3.38}$$

According to (3.38), the domain of existence for the solitons is given by the formula

$$u^2 < 4v_0^2(\beta_0 - \omega)\,, \tag{3.39}$$

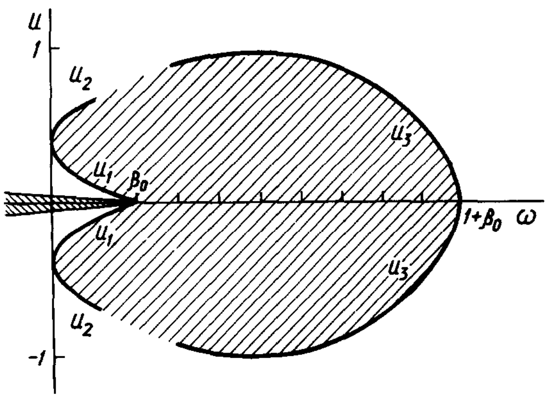


Fig. 5.

which coincides with the result first established in ref. [48]. The parabola in (3.39) that delimits the domain of existence of the magnetostatic solitons is schematically shown in fig. 5.

Consider now the spatial structure of the solitary waves of the EMS envelope, starting the analysis from (3.34) in the long-wave limit $kv_0 \to 0$. According to fig. 5, such an approach is justified, if the soliton parameters $\omega$ and $u$ are not simultaneously close to $\omega = \beta_0$, $u = 0$.

The simplest case to analyze is that of a solitary wave at rest ($u = 0$). Here the waves considered form a particular case of the above considered monochromatic standing EMS waves in a ferromagnet. We will not write down rather awkward general relations, but give only the corresponding weakly linear solution of eqs. (3.34),

$$\theta = (2/\sqrt{\beta_0})\sqrt{(1+\beta_0-\omega)(\omega-\beta_0)}\,\mathrm{sech}[\omega\sqrt{(1+\beta_0-\omega)/(\omega-\beta_0)}\,\xi]\,, \tag{3.40}$$

which is valid for $1 + \beta_0 - \omega \ll 1, \beta_0$. It should be noted that the soliton solutions (3.40) can occur only in an anisotropic ferromagnet and not in an isotropic ferromagnet in the considered case of monochromatic waves.

Finding an analytic solution to the system of equations (3.34) which describes the propagating envelope solitons in the general case is difficult. Examine a particular case with $\beta_0 \ll 1$, $\theta^2 \ll 1$, $\omega \approx 1$. Here the set of equations (3.34) is approximately reduced to the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\,\theta\,\mathrm{e}^{\mathrm{i}\Psi} = \left(\omega + \mathrm{i}u\,\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(1 - \omega - \tfrac{1}{2}\beta_0\theta^2 - \mathrm{i}u\,\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\theta\,\mathrm{e}^{\mathrm{i}\Psi}\,, \tag{3.41}$$

which has the analytic solution

$$\begin{aligned}&\dot{\Psi} \simeq \tfrac{1}{2}u(2\omega - 1 + \tfrac{3}{4}\beta_0\theta^2)\,,\\&\theta \cong \sqrt{[4\omega(1-\omega) - u^2]/\omega\beta_0}\,\mathrm{sech}[\tfrac{1}{2}\sqrt{4\omega(1-\omega)-u^2}(\xi - u\tau)]\,.\end{aligned} \tag{3.42}$$

The soliton propagation velocity (3.42) is bounded by the condition $|u| < 2\sqrt{\omega(1-\omega)}$, coinciding with the equation of the curve $u_3(\omega)$ from (3.37) that delimits the domain of existence of the envelope solitons of EMS waves. It is easy to see that as $u \to 0$, then solution (3.42) becomes (3.40) and accounts for the conditions $\omega \approx 1$, $\beta_0 \ll 1$. According to (3.42), the solitons under consideration are delocalized on the existence domain boundary, the amplitude tending to zero.

The solitary EMS waves under consideration are characterized by a significant (by three to four orders of magnitude) excess of the propagation velocity compared to the maximum propagation velocity of the magnetization magnetostatic solitons. We will analyze below the case of solitary wave propagation in an easy plane ferromagnet along the difficult magnetization axis; here a physically interesting situation arises when magnetostatic solitons continuously become EMS ones.

Let us examine the solutions of the initial system of equations for $\beta_0 < 0$ and $\boldsymbol{n} = \boldsymbol{e}_z$. In this case the magnetization vector lies in the $xy$ plane, the direction of $\boldsymbol{m}$ in this plane being not fixed (an infinite degeneration). We will study the stationary solutions of the system of equations, using the notation $m_x = \sin\theta\cos\varphi$, $m_y = \sin\theta\sin\varphi$, $m_z = \cos\theta$ (in the state of equilibrium the vector $\boldsymbol{m}$ being directed along the $\boldsymbol{e}_x$ axis). The equations for the angular functions $\theta$ and $\varphi$ have the form,

$$u\dot{\theta}\cos\theta = v_0^2 \frac{\mathrm{d}}{\mathrm{d}\xi}(\cos^2\theta\,\dot{\varphi}) + \frac{u^2}{1 = u^2}\cos\theta\sin\varphi\,,$$
$$u\dot{\varphi}\cos\theta = -v_0^2(\ddot{\theta} + \dot{\varphi}^2\sin\theta\cos\theta) + \left(\frac{1}{1-u^2} + |\beta_0|\right)\sin\theta\cos\theta - \frac{u^2}{1-u^2}\sin\theta\cos\varphi\,. \tag{3.43}$$

To delimit the existence domain of the solitary waves, consider solution (3.43) in the asymptotic region $|\xi|\to\infty$ with $\theta \ll 1$ and $\varphi \ll 1$. A dispersion relation similar to (3.2) follows from (3.43); it determines the dependence of the spatial size of the soliton, $k^{-1}$, on its velocity,

$$k^2 = \frac{1}{2v_0^2}\left\{1 + |\beta_0| - \frac{u^2}{v_0^2} - \frac{u^2}{1-u^2} + \left[\left(1 + |\beta_0| - \frac{u^2}{v_0^2} - \frac{u^2}{1-u^2}\right)^2 + \frac{4u^2}{1-u^2}(1+|\beta_0|)\right]^{1/2}\right\}. \tag{3.44}$$

Let us study the limit expression for $k(u)$ in the range of small and large values of the velocity $u$. In the region $u^2 < u_0^2 = (1+|\beta_0|)v_0^2$ eq. (3.44) implies that

$$k^2 \simeq k_{\mathrm{m}}^2 = (\sqrt{1+|\beta_0|}/v_0)\sqrt{(1-u^2)/u_0^2}\,. \tag{3.45}$$

Equation (3.45) describes magnetostatic solitons [15], and is easy to comprehend by considering the solution of (3.43) in the magnetostatic approximation. This solution is

$$\sin\theta = \sqrt{1-u^2/u_0^2}\,\mathrm{sech}[k_{\mathrm{m}}(\xi - u\tau)]\,. \tag{3.46}$$

Expressions (3.45) and (3.46) clearly indicate that there exists a limit velocity of the magnetostatic soliton propagation. When $u \to u_{\mathrm{m}}$ soliton (3.46) is delocalized, the amplitude tending to zero. Taking account of the EMS interaction in the case considered removes the velocity limit for magnetostatic solitons. It follows from the general expression (3.44) that $k(u_0) \simeq \sqrt{1+|\beta_0|}/\sqrt{u_0}$ and $k(u) \simeq [(1+|\beta_0|)/(1-u_0^2/u^2)]^{1/2}$, provided that $u_0^2 < u^2 \ll 1$. In the velocity subrange $u \gtrsim u_0$ magnetization magnetostatic solitons continuously transform into EMS ones, while when $u^2 \gg u_0^2$, $k(u)$ from (3.44) is expressed as

$$k^2 \simeq k_{\mathrm{e}}^2 = \sqrt{1+|\beta_0|}/\sqrt{1-u^2}\,, \tag{3.47}$$

independently of the inhomogeneous exchange constant.

The smooth transformation of the magnetostatic soliton branch to the EMS one is due to the absence, under the conditions considered, of the slow branch of the EMS waves that bound from below the domain of the admissible velocities of the EMS soliton propagation. Indeed, after the substitutions $k \to \mathrm{i}\kappa$, $\omega = \kappa u$ eq. (3.47) yields the dispersion relation for linear waves,

$$\kappa^2(\omega) = \omega^2 - 1 - |\beta_0|\,, \tag{3.48}$$

which allows for the existence of only the fast EMS wave branch $[\omega/\kappa(\omega) > 1]$, in contrast to the similar relation (3.2) describing linear waves in a transversally magnetized ferromagnet.

It is interesting to investigate the influence of the anisotropy in the basic plane which removes the degeneration with respect to azimuthal angle $\varphi$. In order to take account of the magnetic anisotropy in the $xy$ plane, it is sufficient to add in the ground state another term, $\beta_1 m_x$, in the expression for the effective magnetic field affecting the magnetization vector. We will delimit the existence domain of the magnetization stationary solitons in this case by assuming $\beta_1 \ll 1$. It is not difficult to show that a formula similar to (3.44) is valid,

$$k_{\pm}^2 = \frac{1}{2v_0^2}\left\{1 + |\beta_0| - \frac{u^2}{v_0^2} - \frac{u^2}{1-u^2} \pm \left[\left(1 + |\beta_0| - \frac{u^2}{v_0^2} - \frac{u^2}{1-u^2}\right)^2 + 4(1+|\beta_0|)\frac{u^2-\beta_1}{1-u^2}\right]^{1/2}\right\}. \tag{3.49}$$

In analyzing (3.49) one must bear in mind that virtually always $\beta_0 \gg v_0^2 \sim 10^{-9}$.

The most important effect of taking into account the magnetic anisotropy in the basic plane is the appearance of the velocity range $v_0 < |u| < \sqrt{\beta_1}$ which is forbidden for solitons and separates the existence domains of magnetostatic solitons, $0 < u^2 < u_0^2$, and EMS ones. For $\beta_1 \neq 0$ there exist two branches of magnetostatic solitons corresponding to the functions $k_{\pm}(u)$. In the velocity range $u^2 < (1 - 2\sqrt{\beta_1/(1+|\beta_0|)})u_0^2$ the functions $k_{\pm}(u)$ are real, while in the range $(1 - 2\sqrt{\beta_1/(1+|\beta_0|)})u_0^2 < u^2 < u_0^2$ they are complex. In the latter case oscillations are superimposed on the smooth dependence $\theta(\xi - u\tau)$.

In the domain of existence of EMS waves, eq. (3.49) implies that

$$k = (\sqrt{1+|\beta_0|}/u)\sqrt{(u^2-\beta_1)/(1-u^2)}\,, \tag{3.50}$$

which is similar to (3.4) and reduces, as should be expected, to (3.47) when $u^2 \gg \beta_1$.

Let us examine the principal properties of the solitary waves of branch (3.47). Setting $v_0 = 0$ in eq. (3.43), we get

$$\sin\theta = \frac{2u\sqrt{1-u^2}\sqrt{1+|\beta_0|}}{1+|\beta_0|(1-u^2)}\,\operatorname{sech}[k_e(u)(\xi - u\tau)]\,. \tag{3.51}$$

Expression (3.51) shows that when $u \to 1$, then the soliton localization domain and the soliton amplitude tend to zero. The soliton amplitude reaches its maximum value, $\sin(\theta_{\max}) = 1$ when $u^2 = u_k^2 = (1+|\beta_0|)/(2+|\beta_0|)$.

The angular function $\varphi(\xi - u\tau)$ is significantly different in the regions $u^2 < u_k^2$ and $u_k^2 < u^2 < 1$. For $u^2 < u_k^2$ the angle $\varphi$ monotonically grows from zero, when $\xi \to -\infty$, to $2\pi$, when $\xi \to \infty$; thus the magnetization vector makes a complete rotation in the $xy$ plane, as in the case of the soliton propagation in the transversally magnetized isotropic ferromagnet that has been examined in the preceding section. On the other hand, the solitary waves whose velocities lie in the range $u_k < |u| < 1$ are dynamic solitons. Here the angle $\varphi$ is an odd oscillating function with $\varphi(\xi = -\infty) = \varphi(\xi = \infty) = 0$, while the oscillation amplitude is given by the formula

$$\sin(\varphi_{\max}) = \frac{2(1+|\beta_0|)(1-u^2)}{1+|\beta_0|(1-u^2)}\left\{1 + \left[1 - \left(\frac{2u\sqrt{1+|\beta_0|}\sqrt{1-u^2}}{1+|\beta_0|(1-u^2)}\right)^2\right]^{1/2}\right\}^{-1}. \tag{3.52}$$

Here one can see that $\varphi_{\max}$ changes from $\pi/2$ to zero while the soliton velocity grows from $u_k$ to unity.

The analysis given above of the EMS soliton dynamics has not accounted for the energy absorption which always occurs in the process of reversing the ferromagnet's magnetization. The influence of the weak absorption (when the relaxation constant $\eta$ in the equations has the order of $10^{-2}$) can be accounted for within the framework of the adiabatic approximation in the perturbation theory for solitons [49]. The dissipation causes a slow variation in time of the free parameter of the solution of eq. (3.51), the velocity $u$. To find the law governing this variation, $u(\tau)$, let us act as follows. First, it is not difficult to derive from the system of equations (2.1), (2.2) the equation

$$\frac{\partial}{\partial \tau}\left[|\varepsilon_+|^2 + |h_+|^2 + (1+|\beta_0|)\sin\theta\right] = -\frac{2\eta}{\cancel{1+\eta^2\sin^2\theta}}\left|\frac{\partial}{\partial \tau}\cos\theta\, \mathrm{e}^{\mathrm{i}\varphi}\right|^2 + \mathrm{i}\frac{\partial}{\partial \xi}(\varepsilon_+^* h_+ - \varepsilon_+ h_+^*)\,, \qquad (3.53)$$

that expresses the energy conservation law; in eq. (3.53) we have used the notations $4\pi M_0 h_+ = H_x + \mathrm{i}H_y$, $4\pi M_0\varepsilon_+ = \sqrt{\varepsilon_0}(E_x + \mathrm{i}E_y)$. Let us substitute solution (3.51) to eq. (3.53) and integrate the equation with respect to $\mathrm{d}\xi$ from $-\infty$ to $\infty$, as was done in ref. [49]. We shall obtain a simple first-order equation,

$$\frac{\partial}{\partial \tau}W(u) = \frac{\partial}{\partial \tau}16(1+|\beta_0|)^{3/2}\frac{u^2\sqrt{1-u^2}}{[1+|\beta_0|(1-u^2)]^2} = \eta(1+|\beta_0|)(1-u^2)W(u)\,, \qquad (3.54)$$

that describes the absorption of the soliton energy $W(u)$. Note that the soliton energy $W$ as a function of the velocity reaches its maximum at

$$u^2 = u_s^2 = \frac{3+|\beta_0|}{2|\beta_0|}\left(1 + \frac{8|\beta_0|(1+|\beta_0|)}{(3+|\beta_0|)^2}\right) < 1\,.$$

The form of the dependence $u(\tau)$ is substantially determined by the soliton initial velocity. When $u(0) \ll 1$, eq. (3.54) yields

$$u(\tau) = u_0\exp[-\tfrac{1}{2}\eta(1+|\beta_0|)\tau]\,, \quad W \sim u^2\,. \qquad (3.55)$$

Thus, for slow EMS solitons we observe the usual, exponential in time, energy absorption.

In the case of $1-u(0) \ll 1$, $1/|\beta_0|$, eq. (3.54) yields a substantially different result,

$$u(\tau) = 1 - [1-u(0)]\{1 + 4\eta(1+|\beta_0|)[1-u(0)]\tau\}^{-1}\,, \quad W \sim \sqrt{1-u}\,. \qquad (3.56)$$

Formula (3.56) shows that under the influence of the dissipation we shall have here accelerated solitons in contrast to the case of (3.55). Besides we observe here an abrupt suppression of the energy absorption rate of fast EMS solitons that propagate in the given medium at a speed close to that of light.

To explain the first of the above mentioned effects we must note that for $u \sim 1$ the soliton energy $W(u)$ decreases, i.e. $\mathrm{d}W(u)/\mathrm{d}u < 0$. In this case decreasing of the soliton energy due to the dissipation adiabatically transforms the soliton to a state with a greater velocity.

The effect of the abrupt suppression of the fast soliton absorption rate allows for a simple explanation. The soliton total energy $W$ consists of the vortex electromagnetic field energy,

$W_e = \frac{1}{2} \int d\xi\, (|\varepsilon_+|^2 + |h_+|^2)$, and the ferromagnet internal energy $W_m = \frac{1}{2}(1 + |\beta_0|) \int d\xi\, \sin^2\theta$ that consists, in turn, of the magnetic anisotropy energy and the magnet–dipole interaction energy, with $W_e = [(1+u^2)/(1-u^2)]W_m$. According to (3.54), the change in the soliton total energy $W$ is proportional to $-2W_m$; for this reason, when $u^2 \ll 1$ (and $W \approx W_m$) $\partial W/\partial\tau \sim W$, which results in the soliton absorption obeying the exponential law (3.55). When $u \sim 1$, the main part of the soliton energy is contained in the vortex EM field energy, while the ratio $W_e/W_m$ increases during the absorption process, causing the abrupt suppression of the absorption.

### 3.3. Relativistic domain walls in an antiferromagnet

Up to now we have studied the solitary EMS wave properties in a single sublattice ferromagnet. Now we proceed with analyzing solitary waves in a low-sublattice ferromagnet, confining the case to the one-axis antiferromagnet.

Let the wave propagate along the anisotropic axis [4]. Then to analyze the structure, one may use the eqs. (2.36). The wave eqution in (2.36) yields the relation for stationary waves

$$h_\perp = [u^2/(1-u^2)]m_\perp\,. \tag{3.57}$$

Taking into account this equation in the case under consideration of an easy-axis antiferromagnet in the collinear phase of the ground state, we deduce from the initial system of equations the relations $m_z = 0$, $l_\perp \sim \mathrm{i}m_\perp$. Equations (2.36) then become simpler, assuming the form

$$\begin{aligned} &\left(h_0 - \mathrm{i}u\,\frac{\mathrm{d}}{\mathrm{d}\xi}\right)l_\perp = \left(\frac{u^2}{1-u^2} - A_0 - \beta_0^-\right)l_z m_\perp\,, \\ &\left(h_0 - \mathrm{i}u\,\frac{\mathrm{d}}{\mathrm{d}\xi}\right)m_\perp = -\beta_0^- l_z l_\perp\,, \qquad -u\,\frac{\mathrm{d}l_z}{\mathrm{d}\xi} = \left(\frac{u^2}{1-u^2} - A_0\right)\mathrm{Im}[l_\perp m_\perp^*]\,. \end{aligned} \tag{3.58}$$

The system of equations (3.58) can be easily reduced to a closed equation in $l_z$ whose solution,

$$l_z = -2\tanh\left(2\,\frac{\sqrt{\beta_0(1 + A_0 + \beta_0^-)}}{u}\,\sqrt{\frac{u^2 - u_m^2}{1-u^2}}\,(\xi - u\tau)\right), \tag{3.59}$$

has a form typical for propagating magnetostatic domain walls in antiferromagnets [11, 50]. That is why a wave described by (3.59) will be called a relativistic domain wall. The minimum admissible propagation velocity for the waves of (3.59) is determined by the quantity $u_m = [(A_0 + \beta_0^-)/(1 + A_0 + \beta_0^-)]^{1/2}$ which is rather close to the light velocity in an antiferromagnet.

Note that the external magnetic field $h_0$ does not affect solution (3.59). From the system of equations (3.58) one can easily obtain the relation $\varphi = \mathrm{Arg}[l_\perp] = -(h_0/u)(\xi - u\tau)$. Thus, affected by the external magnetic field, transversal components of the antiferromagnet magnetization uniformly rotate in the process of the wave propagation around the anisotropy axis.

We must bear in mind that the description of the solitary waves in question can be substantially simplified. Let us neglect the quantity $\beta_0^-$ in the r.h.s. of the first equation of (2.36), since it is three orders of magnitude less than the exchange constant $A_0$; from here and from the equation describing the time dependence of $l_z$, we will derive the following relation:

$$|l_\perp|^2 + l_z^2 = \text{const.}\,. \tag{3.60}$$

For the solitary waves under consideration the constant in the r.h.s. is obviously equal to 4. According to the second of identities (2.37), eq. (3.60) may be used if $\boldsymbol{m}$ is small. This assumption is similar to the one used in the theory of magnetostatic solitary waves in an antiferromagnet [51].

The system of equations (2.36) implies the estimates $|h_\perp| \sim |m_\perp|$ and $m_z \sim |m_\perp|^2$, so that one may set $m_z \approx 0$ in eqs. (2.36). The first identity from (2.37) yields $m_\perp \sim \mathrm{i}l_\perp$, which allows one to establish that in the case of $h_0 = 0$ considered below the vectors $m_\perp$ and $l_\perp$ vary in mutually orthogonal planes passing through the anisotropy axis.

By setting $l_z = 2\cos\theta$ and $l_\perp = 2\sin\theta$, in accordance with (3.60), let us express $m_\perp$ and $h_\perp$ via the function $\theta$ by the formulas

$$\frac{\partial m_\perp}{\partial \tau} = 4\mathrm{i}\beta_0^- \sin\theta\cos\theta\,, \qquad \frac{\partial h_\perp}{\partial \tau} = \mathrm{i}\,\frac{\partial^2\theta}{\partial\tau^2} + 4\mathrm{i}A_0\beta_0^- \sin\theta\cos\theta\,, \tag{3.61}$$

which follow from (2.36). If one substitutes (3.61) to the wave equation, one obtains a closed equation for the angular function $\theta(\xi, \tau)$,

$$\left(\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\tau^2}\right)\left(\frac{\partial^2\theta}{\partial\tau^2} + 4A_0\beta_0^- \sin\theta\cos\theta\right) = 4\beta_0^- \frac{\partial^2}{\partial\tau^2}\sin\theta\cos\theta\,. \tag{3.62}$$

Equation (3.62) is significantly simpler than the initial system of equations (2.36), so it becomes important to clarify its domain of applicability for studying solitary waves. Let us compare the stationary solution of (3.62),

$$\cos\theta = -\tanh\left(\frac{2\sqrt{\beta_0^- A_0}}{u}\sqrt{\frac{u^2 - u_0^2}{1-u^2}}\,(\xi - u\tau)\right), \tag{3.63}$$

with the exact solution of the initial system of equations (3.59). In formula (3.63) $u_0^2 = A_0/(1 + A_0)$. One can easily see that passing from solution (3.59) to solution (3.63) is achieved by renorming $A_0 + \beta_0^- \to A_0$ in complete accordance with the condition $\beta_0^- \ll A_0$ used in deriving (3.60). Let us consider the formula for the transversal component of the magnetization vector, the one that follows from (2.36) and (3.59),

$$m_\perp = 2\left(1 + \frac{1 + A_0 + \beta_0^-}{\beta_0^-}\,\frac{u^2 - u_\mathrm{m}^2}{1-u^2}\right)^{-1/2} \operatorname{sech}\left(2\,\frac{\sqrt{\beta_0^-(1 + \beta_0 + \beta_0^-)}}{u}\sqrt{\frac{u^2 - u_\mathrm{m}^2}{1-u^2}}\,(\xi - u\tau)\right).$$

As we have noted, a necessary condition of applicability of the suggested approach is the smallness of the quantity $m_\perp$. The relation obtained implies that in the considered case of solitary waves the condition $|m_\perp| \ll 1$ is reduced to the inequality

$$u - u_\mathrm{m} \gg \beta_0^-/2A_0^2\,, \tag{3.64}$$

that delimits the region of admissible values of the solitary wave velocities which can be investigated by using (3.62).

Let us examine now the stability of the domain wall (3.63). Passing to a reference frame that moves with the speed of the domain wall, we will seek the solution of eq. (3.62) in the form

$$\theta(\xi', \tau') = \bar{\theta}(\xi') + e^{p\tau'}\tilde{\theta}(\xi'), \tag{3.65}$$

where $\sqrt{1-u^2}\xi' = \xi - u\tau$, $\sqrt{1-u^2}\tau' = \tau - \xi u$, $\bar{\theta}(\xi')$ is a solution of (3.63), $e^{p\tau'}\tilde{\theta}(\xi')$ is a small added term, $p$ is the spectrum parameter. Substituting (3.65) into eq. (3.62), we get the following linearized equation for the function $\tilde{\theta}(\xi')$,

$$\left(\frac{d^2}{d\nu^2} - \Omega^2\right)\left(\frac{d}{d\nu} - \frac{\Omega}{u}\right)^2\tilde{\theta} = \frac{1}{A_0}\frac{1-u^2}{u^2-u_0^2}\left[\left(\frac{d}{d\nu} - \frac{\Omega}{u}\right)^2 - A_0(1-u^2)\left(\frac{d^2}{d\nu^2} - \Omega^2\right)\right]\cos 2\bar{\theta}(\nu)\,\tilde{\theta}$$

$$p = \kappa\Omega\,, \qquad \nu = \kappa\xi'\,, \qquad \kappa = \frac{2\sqrt{\beta_0 A_0}}{u}\sqrt{\frac{u^2-u_0^2}{1-u^2}}\,. \tag{3.66}$$

We will confine our analysis to the stability of the waves whose velocities $u$ are close to unity.

This case seems to be the most interesting and the reasons for it will become clear somewhat later. It is easy to see that if the condition $1-u \ll 1/2A \ll 1$ is met, then in the r.h.s. of eq. (3.66) the second term in the square brackets may be omitted. How to solve the spectral problem for the resulting equation,

$$d^2\tilde{\theta}/d\nu^2 - \Omega^2\tilde{\theta} - (1-u^2)(1 - 2\,\mathrm{sech}^2\nu)\tilde{\theta} = 0\,, \tag{3.67}$$

is well-known [49]. The only localized solution of eq. (3.67), $\tilde{\theta} \sim \mathrm{sech}(\sqrt{1-u^2}\nu)$, is realized at $\Omega = 0$ and describes a stable translation mode.

The stability of the domain walls is closely related with the complete integrability of the equation that describes them. It is not difficult to see that for the waves whose velocity satisfies the condition $1-u^2 \ll 1/A_0$, eq. (3.62) permits a drastic simplification, assuming the form

$$(\partial^2/\partial\xi^2 - \partial^2/\partial\tau^2)\theta = 4\beta_0^- \sin\theta\cos\theta\,. \tag{3.68}$$

The complete integrability of the obtained equation allows an analytic description of the nonstationary wave process in an antiferromagnet with the participation of the solitary waves under investigation.

We will not dwell on the analysis of solitary waves in an easy-axis antiferromagnet which is in the spin-flop phase in the ground state (see ref. [41]). We will only note that the propagation velocities of such waves are less than those of the solitary waves in an antiferromagnet in the collinear phase. Here the lower boundary of the region of admissible velocities of the considered domain walls depends on the strength of the field that magnetizes the antiferromagnet and tends to zero when the magnetizing field strength approaches the critical value that determines the passage of the antiferromagnet from the spin-flop to the collinear phase.

### 3.4. Solitary waves in a ferrite waveguide

The above given analysis of one-dimensional solitary EMS waves cannot be directly applied for quantitative calculations of actual experiments with solitary waves in spatially limited waveguide structures. Here we will study the propagation of weakly nonlinear solitary EMS waves in a strip waveguide with a normally magnetized isotropic ferrodielectric (the statement of the problem is similar to the one given above in analyzing quasi-monochromatic waves in a ferrodielectric waveguide).

We will solve the system of equations (2.64), assuming, as in the case of quasi-monochromatic waves, that the TEM components of a wave are the principal ones. For the quantities $\varepsilon_x$, $\varepsilon_z$ and $h_y$ we will use expansions (2.65) while for the remaining wave components we will use expansions (2.66). The coefficients in (2.65) and (2.66) will be assumed to be dependent on $\xi$ and $\tau$ in the combination $\xi - u\tau$. Being reformulated in this manner the problem considered of solitary EMS pulses is similar to a very popular problem of the states of two-dimensional nonlinear fields, self-localized along one of the coordinates and periodic along the other (see ref. [52] and the references cited therein).

By substituting expansions (2.65) and (2.66) into the initial set of equations (2.64) we arrive at the equations for the wave components with index $n = 0$,

$$
\begin{aligned}
& h_{z0} = \frac{u^2}{1-u^2}\, m_{z0}\,, \qquad h_{x0} = -m_{x0}\,, \\
& u\,\frac{\mathrm{d}}{\mathrm{d}\xi}\, m_{x0} + \left(h_0 - \frac{u^2}{1-u^2}\right) m_{z0} = -\frac{1}{d}\int_0^d \mathrm{d}\chi\, (h_y m_z + \tfrac{1}{2} m_\perp^2 h_z)\,, \\
& u\,\frac{\mathrm{d}}{\mathrm{d}\xi}\, m_{z0} + (1 + h_0) m_{x0} = \frac{1}{d}\int_0^d \mathrm{d}\chi\, (h_y m_x + \tfrac{1}{2} m_\perp^2 h_x)\,.
\end{aligned}
\tag{3.69}
$$

These equations describe the structure of a TEM-type field. The equations for the remaining coefficients of expansions (2.65) and (2.66) are as follows:

$$
\begin{aligned}
& \hat{L}_n h_{zn} = u^2\,\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\, m_{zn}\,, \qquad \hat{L}_n h_{xn} = -(1-u^2)\,\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\, m_{xn} + \frac{nk_d}{d}\int_0^d \mathrm{d}\chi\, \sin(nk_d\chi)\, m_\perp^2\,, \\
& \hat{L}_n h_{yn} = nk_d\,\frac{\mathrm{d}}{\mathrm{d}\xi}\, m_{xn} - \left(n^2k_d^2 + u^2\,\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\right)\frac{1}{d}\int_0^d \mathrm{d}\chi\, \sin(nk_d\chi)\, m_\perp^2\,, \\
& u\,\frac{\mathrm{d}}{\mathrm{d}\xi}\, m_{zn} - h_0 m_{xn} + h_{xn} = \frac{2}{d}\int_0^d \mathrm{d}\chi\, \cos(nk_d\chi)\,(h_y m_x + \tfrac{1}{2} m_\perp^2 h_x)\,, \\
& \hat{L}_n = (1-u^2)\,\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - n^2k_d^2\,, \qquad m_\perp^2 = m_x^2 + m_z^2\,.
\end{aligned}
\tag{3.70}
$$

First let us establish the existence domain of the solitary waves localized along $\xi$. In the asymptotic region $|\xi| \to \infty$ one can omit nonlinear terms in the equation sets (3.69) and (3.70). The resulting linear equations give the asymptotics of various independent modes. By setting $\boldsymbol{h}_n \sim \mathrm{e}^{-\kappa_n(\xi - u\tau)}$, $\boldsymbol{m}_n \sim \mathrm{e}^{-\kappa_n(\xi - u\tau)}$, $n = 0, 1, \ldots$, we will obtain the dispersion relations $\kappa_n = \kappa_n(u)$ that allow us to clarify necessary conditions and the existence domain for the solitary waves. At $n = 0$ we have the formula

$$\kappa_0^2 = \frac{1 + h_0^2}{u^2} \frac{u^2 - h_0/(1 + h_0)}{1 - u^2} , \tag{3.71}$$

that coincides with similar relations, eqs. (3.4).

Thus, the existence domain of the waves under consideration

$$h_0/b_0 = h_0/(1 + h_0) < u^2 < 1 , \tag{3.72}$$

coincides with that of one-dimensional solitary EMS waves in a transversally magnetized ferromagnet.

For $n \geq 1$ (3.70) implies the dispersion relation

$$\begin{aligned} &n^2 k_d^2 z_n^2 = \frac{b_0^2}{u^2}\left(u^2 - \frac{h_0}{b_0}\right)\left(z_n^2 - \frac{h_0}{b_0}\right)\left(\frac{h_0}{b_0}\frac{1 - u^2}{u^2 - h_0/b_0} + z_n^2\right)(1 - z_n^2)^{-2} , \\ &z_n^2 = (1 - u^2)\kappa_n^2/n^2 k_d^2 . \end{aligned} \tag{3.73}$$

The analysis of eqs. (3.73) shows that in the velocity range delimited by inequalities (3.72), the equation has two solutions of the form $\kappa_n^2 = A_n(u) > 0$ and one solution of the form $\kappa_n^2 = B_n(u) < 0$, the latter corresponding to a wave oscillating and not tending to zero in the asymptotic region.

This solution which oscillates at infinity makes the complete wave localization impossible. But waves whose velocity is close to the lower bound of range (3.72) are delocalized only slightly. Indeed, according to (3.71), the variation scale of the principal wave components with the index $n = 0$ rapidly grows when $u^2 \to h_0/b_0$ (the inequality $u^2 - h_0/b_0 \ll u^2$ appears necessary to warrant the employed condition of weak nonlinearity). At the same time the spatial scale of the oscillating solutions is of the order $(nk_d)^{-1}$ (due to the screening effect of metallic plates), so when one expresses such solutions in terms of the principal components with the use of (3.70), one has to apply the effective averaging which results in the exponential smallness of the amplitude of the wave oscillating components.

Taking into account the weak nonlinearity of the solution of eqs. (3.69) and (3.70), we will seek this solution by the method of asymptotic expansions in powers of the wave amplitude. Under real conditions $h_0$ is usually small, so from now on we will assume that $b_0 \approx 1$. In accordance with the above reasoning we will introduce a small parameter $\delta_0 = \sqrt{1 - h_0/u^2}$. We will attempt to express the quantity $m_{z0}$ in the form,

$$m_{z0} = \sum_{l \geq 1} \delta_0^l m_{z0}^{(l)}(\xi_\delta) , \quad \xi_\delta = \delta_0(\xi - u\tau) . \tag{3.74}$$

It is easy to see that the corresponding series for the quantity $m_{x0}$ begins with a term proportional to $\delta_0^2$. Analyzing the system of equations (3.70), we see that the asymptotic expansion for $h_{yn}$ is given by the formula

$$h_{yn} = \sum_{l \geq 1} \delta_0^{l+1} h_{yn}^{(l)}(\xi - u\tau) . \tag{3.75}$$

Similar expansions for other wave components with index $n \neq 0$ begin with terms proportional to $\delta_0^3$. In the roughest approximation in $\delta_0$ (3.69) and (3.70) yield

$$\frac{\mathrm{d}^2 m_{z0}^{(1)}}{\mathrm{d}\xi_\delta^2} = \left(1 - \frac{1}{2}(m_{z0}^{(1)})^2 - \frac{1}{u^2}\sum_{n\geq 1}\frac{1-\cos n\pi}{n\pi} h_{yn}^{(1)}\right) m_{z0}^{(1)} ,$$
$$\mathrm{d}^2 h_{yn}^{(1)}/\mathrm{d}\xi^2 - n^2k_d^2 h_{yn}^{(1)} = -n^2k_d^2[(1-\cos n\pi)/n\pi](m_{z0}^{(1)})^2 , \tag{3.76}$$

which determine the structure of the principal wave components. The second of eqs. (3.76) yields

$$h_{yn}^{(1)} = \frac{k_d}{2\pi}(1-\cos n\pi)\int_{-\infty}^{\infty}\mathrm{d}\xi'\,\mathrm{e}^{-nk_d|\xi-\xi'|}[m_{z0}^{(1)}(\delta_0\xi')]^2 \approx \frac{1-\cos n\pi}{n\pi}(m_{z0}^{(1)})^2 . \tag{3.77}$$

Substitution of (3.77) to the first of eqs. (3.76), summation over $n$ and subsequent integration lead to the following final expression for the principal term in expression (3.74):

$$m_{z0}^{(1)} = 2u\delta_0\,\mathrm{sech}(\xi_\delta) \tag{3.78}$$

It is easy to see that solution (3.78) coincides in the proper limit with solution (3.5) that describes a one-dimensional solitary EMS wave in a transversally magnetized ferromagnet.

The principal terms of the wave components with an index $n \geq 1$ are determined from the equations

$$(u^2 - h_0)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} m_{zn}^{(1)} + h_0 n^2k_d^2 m_{zn}^{(1)} - u\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - n^2k_d^2\right)\frac{\mathrm{d}}{\mathrm{d}\xi} m_{xn}^{(1)} = 0 ,$$
$$u\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - n^2k_d^2\right)\frac{\mathrm{d}}{\mathrm{d}\xi} m_{zn}^{(1)} - \left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - h_0 n^2k_d^2\right) m_{xn}^{(1)} = \frac{8k_d u^2}{\pi}(1-\cos n\pi)\tanh(\xi_\delta)\,\mathrm{sech}^2(\xi_\delta) . \tag{3.79}$$

The solution of (3.79) can be represented in the form,

$$m_{zn}^{(1)} = -\frac{8\mathrm{i}k_d u^3}{\pi}(1-\cos n\pi)\int_{-\infty}^{\infty}\mathrm{d}\xi'\,\tanh(\delta_0\xi')\,G[nk(\xi - u\tau - \xi')] ,$$
$$G = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}z\,z(1+z^2)\,\mathrm{e}^{\mathrm{i}znk_d(\xi-u\tau-\xi')}}{(h_0+z^2)(h_0-u^2\delta_0^2z^2) - n^2k_d^2u^2(1+z^2)^2z^2} . \tag{3.80}$$

The structure of the Green function in (3.80) is determined by the roots of the expression $\varphi(z^2)$ in the denominator of the integral for $G(\xi - u\tau - \xi')$; the roots coincide of course with those given by the dispersion relation (3.73), after a change in notation: $z_n^2 \to -z^2$. As we have noticed above, the two roots correspond to localized components similar to (3.77). The delocalized part of the function $m_{zn}^{(1)}$ corresponding to the third root $z_3^2 > 0$ is described in the asymptotic region $|\xi - u\tau| \to \infty$ by the formula

$$m_{zn}^{(1)} \approx +8\pi\frac{n^3k_d^3u^3}{\delta_0^3}\,\mathrm{e}^{-\pi z_3 nk_d/\delta_0}\,\frac{1-\cos\pi n}{\pi n}\,\frac{z_3^2(1+z_3^2)}{\mathrm{d}\varphi(z_3^2)/\mathrm{d}z_3^2}\sin[z_3nk_d(\xi - u\tau)] . \tag{3.81}$$

This formula is similar to the one established in ref. [52] and shows that for a wave of a sufficiently small amplitude in (3.78) the delocalization effect due to a nonlinear relation of the localized TEM pulse with the propagating waveguide modes is insignificant. In practice the delocalization effect under discussion causes the finite life time of the solitary wave.

Omitting the calculation of the subsequent terms of expansion (3.74), let us pass to the analysis of the nonstationary process of the localized TEM pulse propagation.

Bearing in mind (3.77), we will expand $h_y$ as $h_y = \frac{1}{2}m_\perp^2 + \Sigma_{n\geq 1} h_{yn} \sin(nk_d\chi)$. Neglecting the intramodal interaction in agreement with the above analysis, we obtain from (2.64) the equations,

$$\begin{aligned} &\frac{\partial m_{z0}}{\partial \tau} = -b_0 m_{x0}\,, \qquad \frac{\partial^2 h_{z0}}{\partial \xi^2} - \frac{\partial^2 h_{z0}}{\partial \tau^2} = \frac{\partial^2 m_{z0}}{\partial \tau^2}\,, \\ &\frac{\partial m_{x0}}{\partial \tau} - h_0 m_{z0} + h_{z0} = \tfrac{1}{2} m_{\perp 0}^2 (m_{z0} + h_{z0})\,, \end{aligned} \tag{3.82}$$

that describe the nonstationary dynamics of the TEM pulse.

In investigating the eqs. (3.82) one must bear in mind that (3.82) is capable of the correct description of the solitary pulse evolution only for waves which propagate with the velocity that satisfies the inequality $1 - h_0/u^2 \ll 1$, thus satisfying the conditions of weak nonlinearity and a weak connection with the modes oscillating over the waveguide cross-section. This inequality gives a lower bound of the duration of the pulse exciting a soliton at the waveguide input. In reality, according to (3.78), the solitary pulse duration is $T \approx (\delta_0 u)^{-1}$. Expressing the velocity $u$ via the duration $T$, one gets from the inequality the following condition:

$$T(\mathrm{s}) \gg 1/g\sqrt{4\pi M_0 H_0}\,. \tag{3.83}$$

Thus, in the experiment one must direct to the system input electromagnetic pulses which are similar in shape to function (3.78).

Let us simplify the system of equations (3.82) by using transformations similar to those which have been employed in deriving the evolution equation (3.12) that describes one-dimensional waves in a homogeneous transversally magnetized ferromagnet. Passing to a moving frame of reference, $\xi' = \xi - \sqrt{h_0}\tau$, $\tau' = \sqrt{h_0}\tau - h_0\xi$, in the transformed equation of set (3.82) we omit the second derivatives with respect to $\tau'$, since the waves in the new coordinate system are slow. We obtain

$$(\partial/\partial\xi')\, h_{z0} = h_0(\partial/\partial\xi' - 2\,\partial/\partial\tau')m_{z0}\,. \tag{3.84}$$

Combining eq. (3.84) with the magnetization dynamics equation (3.82) (setting in the latter $\partial/\partial t \to -\sqrt{h_0}\,\partial/\partial\xi'$), we get a closed mKdV equation for the quantity $m_{z0}$,

$$2\frac{\partial m_{z0}}{\partial \tau'} + \frac{3}{2h_0} m_{z0}^2 \frac{\partial m_{z0}}{\partial \xi'} + \frac{\partial^3 m_{z0}}{\partial \xi'^3} = 0\,. \tag{3.85}$$

It is easy to see that the stationary solution of eq. (3.85),

$$m_{z0} = 2\sqrt{2h_0 u'}\,\mathrm{sech}[\sqrt{2u'}(\xi' - u'\tau')]\,, \tag{3.86}$$

coincides, as should be expected, with expression (3.78) in the case $u' = (u - \sqrt{h_0})/\sqrt{h_0} \ll 1$, in accordance with the assumption used in deriving eq. (3.85).

From the viewpoint of studying opportunities for an experiment aimed at finding solitary EMS waves, the property of the complete integrability of eq. (3.86) is very important. Owing to this property, we do not need the exact coincidence of the shape and amplitude of the pulse excited at the waveguide input by means of an external generator of a solitary pulse. It is known [45] that an initial pulse of sufficient intensity is split during the evolution process described by eq. (3.85) into a number of solitons and a smeared, due to the dispersion, "radiation tail". The scheme for solving the inverse problem developed in ref. [45] enables us to calculate the number and characteristics of the solitons arising from a given input pulse.

The recently developed perturbation theory for the equations bordering on completely integrable ones, enables us to calculate the effect of a small structural disturbance on the soliton evolution. We will consider the magnetization relaxation effect on the single soliton dynamics. Taking into account the relaxation term in the Landau–Lifschitz equation and repeating the line of reasoning used in deriving eq. (3.85), we will obtain an equation for the disturbance in the canonical form,

$$\frac{\partial \Psi}{\partial \tau''} + \frac{\partial^3 \Psi}{\partial \xi''^3} + 6\Psi^2 \frac{\partial \Psi}{\partial \xi''} = \frac{2\eta}{\sqrt{h_0}} \frac{\partial^2 \Psi}{\partial \xi''^2}, \tag{3.87}$$

where $m_{z0} = \sqrt{h_0}\,\Psi$, $\tau' = 16\tau''$, $\xi' = 2\xi''$. Assuming the parameter $\eta/\sqrt{h_0}$ to be small and using perturbation theory [32], let us examine the single soliton evolution.

A small structural disturbance leads to a slow variation of soliton parameters (described within the adiabatic approximation) and to distortion of its shape (described within further orders of the perturbation theory [32]). Following ref. [32], we will seek the solution of eq. (3.87) in the form

$$\Psi = 2g_0(\tau'')[\operatorname{sech}(z) - W(z, \tau'')], \qquad z = 2g_0(\tau'')[\xi'' - \varphi_0(\tau'')], \tag{3.88}$$

where the parameters $g_0(\tau'')$ and $\varphi_0(\tau'')$ are found from the adiabatic approximation equations,

$$\frac{\mathrm{d}g_0}{\mathrm{d}\tau''} = \frac{1}{2}\int_{-\infty}^{\infty} R\,\frac{\mathrm{d}z}{\cosh z}, \qquad \frac{\mathrm{d}\varphi_0}{\mathrm{d}\tau''} = 4g_0^2 + \frac{1}{4g_0^2}\int_{-\infty}^{\infty} R\,\frac{z\,\mathrm{d}z}{\cosh z}, \tag{3.89}$$

while the correction to the soliton, $W(z, \tau'')$, is determined by a cumbersome expression whose asymptotics has the form

$$W = \frac{1}{32g_0^4}\, z^2\, \mathrm{e}^{-z} \int_{-\infty}^{\infty} R\,\frac{\mathrm{d}z}{\cosh z}, \quad z \to \infty, \qquad W = \frac{1}{32g_0^4}\, 2\,\mathrm{e}^{\sigma z} \int_{-\infty}^{\infty} R\,\mathrm{d}z, \quad z \to -\infty. \tag{3.90}$$

In the formulas written above $R$ stands for the r.h.s. of eq. (3.87) where in the capacity of $\Psi$ the function $2g_0 \operatorname{sech}(z)$, $\sigma^{-1} = 8\int g_0^2\,\mathrm{d}\tau''$, must be used. First let us consider the adiabatic approximation equations. By calculating the integrals entering (3.89) we obtain for the soliton velocity,

$$u(\tau'') = \mathrm{d}\varphi_0/\mathrm{d}\tau'' = 4g_0^2(\tau'') = u(0)[1 + \tfrac{4}{3}(\eta/\sqrt{h_0})u^2(0)\tau'']^{-1/2}. \tag{3.91}$$

According to (3.91) the magnetization relaxation effect leads to the soliton deceleration. Thus, in accordance with (3.88), the soliton localization region becomes larger. This fact must be borne in mind in experimental studies of the solitary EMS wave dynamics, for it is possible to interpret this effect of the dissipative smearing of an autolocalized pulse as the usual dispersion smearing of a linear pulse.

Let us consider the variation of the soliton shape. Formula (3.90) imples that

$$2g_0W = (\eta/6\sqrt{h_0})u(\tau'')z^2\,\mathrm{e}^{-z}\,, \quad z \to \infty\,, \qquad 2g_0W = 0\,, \quad z \to -\infty\,. \tag{3.92}$$

In accordance with (3.92), in a reference frame connected with a moving soliton the distortion of the latter's shape does not increase in time, i.e., the soliton displays a certain stability with respect to the structural excitation. Besides, expression (3.92) shows that it is the front part of the pulse that undergoes a distortion.

## 4. Simple (Riemann) and shock waves

### *4.1. Riemann waves in magnetic dielectrics*

The dissipative nature of the magnetization dynamics substantially affects the propagation processes of the above investigated EMS waves, but it is not so important in determining the existence of EMS waves. We will pass now to an analysis of shock EMS waves whose existence is caused by the dissipative nature and nonlinearity of the ferromagnet magnetization motion.

Shock waves appear as a result of an evolution of simple (Riemann) waves which are natural wave excitations of a nonlinear nondispersion medium [53].

Let us examine, for example, the propagation process of low-frequency EMS waves of the slow branch in a longitudinally magnetized ferromagnet. The dispersion equation (2.6) relating the wave number and frequency implies that in the region of sufficiently small frequencies, $\omega \ll \omega_s$, the dispersion dependence becomes linear, i.e., $k(\omega) \sim \omega$. This fact results in a phase synchronization of low-frequency waves with their own harmonics, so during the evolution of low-frequency pulses of EMS waves in a ferromagnet their profiles are distorted as a consequence of the nonlinear generation of higher harmonics.

Passing to the quantitative description, we must first remark that the Landau–Lifshitz equation (2.2) implies in the low-frequency limit that the variables of the magnetic field components $h_\perp = h_x + \mathrm{i}h_y$ are quasistatically related with the magnetization variables $m_\perp = m_x + \mathrm{i}m_y$ (see ref. [53]),

$$h_\perp = (1 + h_0/\sqrt{1 - |m_\perp|^2})m_\perp\,. \tag{4.1}$$

By substituting (4.1) into the Maxwell equations (2.1) we obtain a set of hyperbolic equations in dimensionless variables used in the present paper,

$$\begin{aligned}
&\frac{\partial\varepsilon}{\partial\xi} = b_0\left(\frac{1}{\cos^2\theta}\,\frac{\partial\theta}{\partial\tau}\sin\phi + \tan\theta\,\frac{\partial\varphi_1}{\partial\tau}\cos\phi\right),\\
&\frac{\partial\varphi_2}{\partial\xi}\,\varepsilon = b_0\left(\frac{1}{\cos^2\theta}\,\frac{\partial\theta}{\partial\tau}\cos\phi - \tan\theta\,\frac{\partial\varphi_1}{\partial\tau}\sin\phi\right),\\
&-\frac{\partial\varepsilon}{\partial\tau} = b_0\left(\frac{u_2^2(\theta)}{\cos^2\theta}\,\frac{\partial\theta}{\partial\xi}\sin\varphi + u_1^2(\theta)\tan\theta\,\frac{\partial\varphi_1}{\partial\xi}\cos\phi\right),\\
&-\frac{\partial\varphi_2}{\partial\tau}\,\varepsilon = b_0\left(\frac{u_2^2(\theta)}{\cos^2\theta}\,\frac{\partial\theta}{\partial\xi}\cos\phi - u_1^2(\theta)\tan\theta\,\frac{\partial\varphi_1}{\partial\xi}\sin\phi\right).
\end{aligned} \tag{4.2}$$

The quantities entering the set of equations (4.2) are given by the formulas

$$\varepsilon_x + \varepsilon_y = \varepsilon\,\mathrm{e}^{\mathrm{i}\varphi_2}\,, \qquad m_\perp = \sin\theta\,\mathrm{e}^{\mathrm{i}\varphi_1}\,, \qquad \phi = \varphi_1 - \varphi_2\,,$$
$$u_1^2 = 1 - \cos\theta/b_0\,, \qquad u_2^2 = 1 - \cos^3\theta/b_0\,, \qquad b_0 = 1 + h_0\,. \tag{4.3}$$

We are interested in the process of forming a shock wave which is described by solutions of (4.2) representing simple waves. To find such solutions we will assume for each of the quantities entering (4.2) that

$$\partial y/\partial\tau + u\,\partial y/\partial\xi = 0\,. \tag{4.4}$$

Because of (4.4), differential equations (4.2) become algebraic equations. It follows from the latter that there exist two types of simple EMS waves [53]. The first type is characterized by the property of the magnetization precession angle to be constant, $\theta = \mathrm{const}$. Such waves propagate without changing their shape with the velocity $u = u_1(\theta) = \mathrm{const}$. The corresponding solution of (4.2) is

$$\phi = F(\xi - u_1(\theta)\tau)\,, \qquad \varepsilon = (1 + h_0)\tan\theta\,u_1(\theta)[\sin\phi \pm \sqrt{\sin^2\phi + \mathrm{const.}}]\,, \tag{4.5}$$

where $F$ is an arbitrary function.

The second type waves are linearly polarized ($\varphi_1 = 0$, $\varphi_2 = \pi/2$) and propagate with the velocity $u_2(\theta)$. The variation of the angle during the evolution process is described by the equation

$$\partial\theta/\partial\tau + \sqrt{1 - \cos^3\theta/(1 + h_0)}\;\partial\theta/\partial\xi = 0\,, \tag{4.6}$$

showing that the wave profile points corresponding to large values of the angle $\theta$ move with great velocities. That is why the fragments of the wave profile where the magnetic field $h_\perp$ grows become steeper. The wave shape evolves as

$$\theta = G[\xi - u_2(\theta)\tau]\,, \qquad \varepsilon = -(1 + h_0)\int u_2(\theta)/\cos^2\theta\,\mathrm{d}\theta\,, \tag{4.7}$$

where the function $G$ is determined by the initial conditions.

Solution (4.7) describes an initial stage of forming a shock wave, when a pulse with a sloping front propagates in a ferromagnet. The steeper the front becomes, the greater are the dispersion and dissipation effects [not taken into account in the quasistatic relation (4.1)] which prevent the front from becoming steeper. In this case, if the pulse is semi-infinite, the stationary shock wave is formed.

Before the investigation of the structure of stable shock EMS waves, we will consider several other applications of the Riemann waves in ferromagnetic media. First we will dwell on such waves in antiferromagnets.

In the case of low-frequency waves propagating in an easy-axis ferromagnet along the anisotropy axis one may use the expression for the nonlinear magnetic susceptibility (2.44) but letting the frequency $\omega$ tend to zero in the quasistatic limit. We obtain the equations

$$(\partial^2/\partial\xi^2 - \partial^2/\partial\tau^2)h_\perp = \partial^2/\partial\tau^2\,m_\perp\,, \qquad m_\perp = (1/A_0)[\omega_\mathrm{p}^2/(\omega_\mathrm{p}^2 - h_0^2 - 2|h_\perp|^2)]h_\perp\,. \tag{4.8}$$

Equations (4.8) can be obtained directly from the general equations (2.38) by omitting the operators $\partial/\partial\tau$ in the Landau–Lifshitz equations.

Since the r.h.s. of the first equation of set (4.8) is small ($A_0 \gg 1$), the operator $\partial^2/\partial\xi^2 - \partial^2/\partial\tau^2$ in the l.h.s. of this equation can be simplified by setting

$$\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\tau^2} \equiv \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\tau}\right)^2 - 2\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} \approx -2\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} \tag{4.9}$$

Approximation (4.9) sorts the waves propagating in one direction, taking into account the proximity of the propagation velocity to unity. By setting $h_\perp = h\,e^{i\varphi_n}$ we obtain from (4.8) and (4.9) the hyperbolic equations for the amplitude $h$ and the wave phase $\varphi_n$,

$$\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\tau}\right)h + \frac{1}{2A_0}\frac{\partial}{\partial\tau}\frac{\omega_p^2 h}{\omega_p^2 - h_0^2 - 2h^2} = 0\,, \qquad \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\tau}\right)\varphi_n + \frac{1}{2A_0}\frac{\omega_p^2}{\omega_p^2 - h_0^2 - 2h^2}\frac{\partial}{\partial\tau}\varphi_n = 0\,. \tag{4.10}$$

The first equation of set (4.10) shows an essential difference between Riemann EMS waves in ferromagnets and antiferromagnets. The solution of this equation is given by a function of the type of (4.7) where the nonlinear velocity $u = u_1(h)$ is

$$u_1(h) \cong 1 - \frac{\omega_p^2}{2A_0}\frac{\omega_p^2 - h_0^2 + 2h^2}{(\omega_p^2 - h_0^2 - 2h^2)^2}\,. \tag{4.11}$$

According to (4.11) the parts of the wave profile that correspond to large values of the amplitude $h$ propagate with smaller velocities, in contrast to Riemann waves in ferromagnets. Because of this peculiarity the rear part of the pulse in an antiferromagnet (not the front one as in a ferromagnet) will becomes steeper during the evolution of a time-limited pulse.

Note one peculiarity of the equation set (4.10) from which it follows that the amplitude and the phase of a Riemann wave propagate along different characteristics. This fact enhances further the distortion of the wave profile in the process of its evolution.

Let us examine another example of Riemann waves in ferromagnets, viz. the evolution of such waves in conducting ferromagnets, which have recently been actively studied [54]. We will confine the analysis to nonlinear waves propagating along a magnetic field in an isotropic conducting ferromagnet with frequencies that are small compared to the plasma and cyclotron frequencies of the current carriers (coupled spin–helical and spin–Alfvenian waves [55]). Consider the case of a nondegenerate conductivity plasma (ferromagnetic semiconductors and semimetals). The pulse dependence of the energy for the current carriers will be assumed isotropic and quadratic. The system of equations describing a local mode of the wave evolution is

$$\begin{aligned} &\operatorname{rot}\boldsymbol{H} = \frac{4\pi e}{c}(n_i V_i - n_e V_e) + \frac{\varepsilon_0}{c}\frac{\partial}{\partial t}\boldsymbol{E}\,, \qquad \operatorname{rot}\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}\,, \\ &\nu_p V_p = \delta_p \frac{e}{m_p}\left(\boldsymbol{E} + \frac{1}{c}V_p \times \boldsymbol{B}\right), \qquad \frac{\partial}{\partial t}n_p + V_p(\boldsymbol{n}_p \cdot V_p) = 0\,, \end{aligned} \tag{4.12}$$

where $n_p$, $m_p$, $\nu_p$, $V_p$ stand for the density, effective mass, collision frequency, and the hydrodynamic velocity of the current carrier, respectively; the subscript p assumes two values: p = i for holes and p = e for electrons; $\delta_i = 1$, $\delta_e = -1$.

Analyzing the equation set (4.12) in order to establish the existence domain of Riemann waves, we come, above all, to an important conclusion: under the considered conditions Riemann waves can exist only in a conducting ferromagnet with a compensated electron–hole plasma (in such a plasma equilibrium concentrations of holes and electrons are the same). Indeed, in the case of the conductivity plasma with current carriers of the same sign, or in case of an uncompensated electron–hole plasma, the linearized equation set (4.12) implies a quadratic relation between the frequency $\omega$ and the wave number $k$ of elementary low-frequency wave excitations $\omega = \omega(k) \sim k^2$; this relation is typical for the helical branch of plasma waves [56]. Here the nonlinear process of higher harmonic generation that accompanies nondispersion Riemann wave propagation does not occur due to the absence of the phase synchronization between different exciting agents of elementary waves.

Let us examine the wave evolution in a conducting ferromagnet with a compensated plasma conductivity. In range of frequencies which are small compared to the ferromagnet resonance frequencies, one can use the quasistatic relation (4.1) between the variable components of the magnetic field wave and magnetization. The frequency of a homogeneous ferromagnetic resonance $gH_0$ is usually smaller than plasma and cyclotron frequencies for current carriers in conducting ferromagnets. In view of this condition the initial equation set (4.12) can be substantially simplified to the form (in dimensionless variables),

$$\mathrm{i}u_A^2 \frac{\partial}{\partial \xi}\left(\frac{b_0}{\sqrt{1-|m_\perp|^2}} - 1\right)m_\perp - \frac{\partial}{\partial \tau}\varepsilon_\perp = \nu_0 \varepsilon_\perp , \qquad \mathrm{i}\frac{\partial}{\partial \xi}\varepsilon_\perp = -b_0 \frac{\partial}{\partial \tau}\frac{m_\perp}{\sqrt{1-|m_\perp|^2}} , \tag{4.13}$$

where

$$\nu_0 = \frac{(\nu_i m_i + \nu_e m_e)}{4\pi g(m_i + m_e)} , \qquad u_A = \frac{g(H_0 + 4\pi M_0)}{\omega_{pe}\sqrt{m_i + m_e}}$$

is the normal Alfvenian velocity, $\omega_{pe} = (4\pi e^2 n_0/m_0\varepsilon_0)^{1/2}$, $m_0$ and $n_0$ being the free-electron mass and the equilibrium concentration of the current carriers, respectively.

The equation set (4.13) describes coupled spin–Alfvenian waves [55]. Since the dispersion law for these waves is linear, Riemann waves exist in the considered frequency range. In deriving (4.13) the condition $\nu \ll b_0$ has been assumed to be true, which is necessary for a weak damping of waves. Estimates show that the external field $H_0$ must be of the order of 10 kOe.

Analyzing the equation set (4.13) without accounting for absorption leads to results similar to those obtained for Riemann waves in a ferrodielectric. One can easily see that in the conducting ferromagnet under consideration there exist two types of simple waves. The waves of the first type are linearly polarized. During their propagation their profile is distorted, for some of its points propagate with the velocity $u = u_A(1 - \cos^3\theta/b_0)^{1/2}$ [the magnetization precession angle varies according to the equation $\partial\theta/\partial\tau + u_A(1 - \cos^3\theta/b_0)^{1/2}\partial\theta/\partial\xi = 0$]. The waves of the second type propagate without distorting their shape with the velocity $u = u_A(1 - \cos\theta/b_0)^{1/2}$, $\theta = \text{const}$.

To take account of the collisional absorption on the Riemann wave evolution, let us use the fact that the propagation velocity of the waves under investigation and the Alfvenian velocity $u_A$ are rather close (the weak absorption of waves occurs virtually only if $b_0 \approx h_0 \gg 1$ with $1 - \cos^3\theta/b_0 \approx 1$). Excluding the electric field of the wave, $\varepsilon_\perp$, from eqs. (4.13), we simplify the resulting closed equation for the transversal magnetization $\boldsymbol{m}_\perp$ by means of a relation similar to (4.9). As a result, we obtain simple equations for the wave amplitude and phase ($m_\perp = \sin\theta\, e^{i\varphi}$)

$$\begin{aligned} &\partial\theta/\partial\tau + u_A(1-\cos^3\theta/2b_0)\partial\theta/\partial\xi + \tfrac{1}{2}\nu_0 \sin\theta\cos\theta = 0\,,\\ &\partial\varphi/\partial\tau + u_A(1-\cos\theta/2b_0)\partial\varphi/\partial\xi = 0\,, \end{aligned} \tag{4.14}$$

which can be exactly solved by the method of characteristics. Attention should be paid to a peculiar feature of these equations, which is similar to that of eq. (4.10) that describes Riemann waves in antiferromagnets. According to (4.14), the wave amplitude and phase propagate along different characteristics.

By means of the change of variables $\tau_\nu = \frac{1}{2}\nu_0\tau$ and $\xi_\nu = (\nu_0 b_0/u_A)[\xi - u_A(1-1/2b_0)\tau]$ the first of eqs. (4.14) is reduced to the form

$$\partial\theta/\partial\tau_\nu + (1-\cos^3\theta)\partial\theta/\partial\xi_\nu + \sin\theta\cos\theta = 0\,. \tag{4.15}$$

The Cauchy problem for eq. (4.15) with the initial condition $\theta = \theta_0(\xi_\nu)$ at $\tau_\nu = 0$ has the solution,

$$\begin{aligned} &\tan\theta = e^{-\tau_\nu}\tan\theta_0\,,\\ &\xi_\nu(\tau_\nu) = \xi_0 + \ln\left[\frac{\tan\theta_0}{\tan\theta}\left(\frac{1+\cos\theta_0}{1+\cos\theta}\right)^{1/2}\left(\frac{1-\cos\theta}{1-\cos\theta_0}\right)^{1/2}\right] + \cos\theta - \cos\theta_0\,, \end{aligned} \tag{4.16}$$

where $\theta_0 = \theta(\xi_0)$, $\xi_0 = \xi_\nu(0)$.

According to (4.16), $\theta \to 0$ and $\xi_\nu \to \xi_\infty \neq \infty$ when $\tau_\nu \to \infty$. Thus, the initial broad frequency pulse of spin–Alfvenian waves is completely absorbed at a finite distance from the point of its origin [54].

### 4.2. Shock wave front structure in a one-sublattice ferromagnet

The analysis given above of Riemann wave propagation in ferromagnets does not enable us to investigate the final stage of the wave development when dispersion and absorption result in forming stationary profile shock waves. As we have noted in the introduction, the analysis of the structure of the shock wave front in a ferromagnet was the first study of essentially nonlinear waves in electrodynamics [14]. Following ref. [14], let us consider the structure of the shock wave front of a wave propagating in a longitudinally magnetized ferrodielectric. Instead of eq. (4.1) of the quasistatic relation between the a.c. components of the magnetic field and the magnetization, we will use the exact relations for a stationary wave that are derived from the Maxwell equations (2.1),

$$h_\perp = h_x + ih_y = \frac{u^2}{1-u^2}\sin\theta\,(\xi - u\tau)\,e^{i\varphi(\xi-u\tau)}\,, \qquad h_z = h_0 + 1 - \cos\theta\,(\xi - u\tau)\,. \tag{4.17}$$

Relations (4.17) satisfy the boundary conditions $h_z = h_0$, $h_\perp = 0$, $\theta = 0$ before the shock wave front when $\xi \to \infty$. Substituting (4.17) into the Landau–Lifshitz equations results in

$$-u\,\frac{d\theta}{d\xi} = \eta\,\frac{\cos\theta - b_0(1-u^2)}{1-u^2}\sin\theta\,, \qquad u\,\frac{d\varphi}{d\xi} = \frac{\cos\theta - b_0(1-u^2)}{1-u^2}\,. \tag{4.18}$$

Equations (4.18) shows that the stationary state of the magnetization behind the shock wave front when $\xi \to \infty$ is established when the deviation angle of the magnetization approaches the value $\theta = \theta_\infty$; this is related to the propagation velocity by the expression

$$u^2 = 1 - \cos \theta_\infty / b_0 . \tag{4.19}$$

Note that the velocity $u$ from (4.19) coincides with the phase velocity of nonlinear EMS waves, the latter velocity being given by eq. (1.6) in the case of an isotropic ferromagnet in the limit $\omega \to 0$.

Let us examine the magnetic field value behind the shock wave front $h_\perp = h_\infty$, assuming that $\varphi \to 0$ when $\xi \to -\infty$. The first of eqs. (4.17) yields a formula,

$$h_\infty = (h_0 + 1 - \cos \theta_\infty) \tan \theta_\infty , \tag{4.20}$$

according to which the magnetization deviation angle in the considered shock wave does not exceed $\pi/2$.

Formulas (4.19) and (4.20) show that the propagation velocity of shock EMS waves in a longitudinally magnetized ferromagnet varies, depending on the wave magnetic field, within the limits

$$h_0/b_0 < u^2 < 1 . \tag{4.21}$$

It is easy to see that this range coincides with the opaqueness range for linear right-polarized EMS waves, the range that divides the slow and fast branches.

When $h_\infty$ grows, the shock wave velocity grows too, the variation of the function $u = u(h_\infty)$ being especially great when the d.c. field magnetizing the ferromagnet is small, $h_0 \ll 1$.

The solution of the system of equations (4.18),

$$\begin{gathered}
\frac{1}{\sin^2\theta_\infty} \ln\left( \frac{(\sin \theta)^{1+\cos \theta_\infty}}{(1 + \cos \theta)^{\cos \theta_\infty} (\cos \theta - \cos \theta_\infty)} \right) = \frac{\eta b_0}{u \cos \theta_\infty} (\xi - u\tau) , \\
\eta\varphi = \operatorname{artanh} \cos \theta - \operatorname{artanh} \cos \theta_\infty ,
\end{gathered} \tag{4.22}$$

determines the shock wave front structure. According to (4.22), the duration of the wave front is proportional to the relaxation time of the ferromagnet and essentially depends on the wave amplitude $h_\infty$. For instance, in the case $\theta_\infty \sim h_\infty/h_0 \ll 1$, $\sqrt{2h_0}$ the variation of the precession angle of the magnetization $\theta(\xi - u\tau)$ obeys the relation,

$$\theta(\xi - u\tau) \approx (h_\infty/h)\{1 + \exp[(\eta b_0/u)(h_\infty^2/h_0^2)(\xi - u\tau)]\}^{-1/2} . \tag{4.23}$$

This formula shows that the duration of the front for small intensity shock waves is inversely proportional to the square of the wave amplitude.

The second of eqs. (4.22) shows that during the shock wave propagation the magnetization vector rotates around the d.c. magnetic field $\boldsymbol{h}_0$. The factor $\eta$ is usually of the order of $10^{-2}$, so for the transversal components of the magnetic field and magnetization in a shock wave propagating in a longitudinally magnetized ferromagnet, the profile shows strong oscillations.

Note that in this case the oscillatory structure of the shock wave front is mainly determined by the magnetization relaxation coefficient $\eta$ and depends weakly on the wave magnetic field. Somewhat later we will show that when a shock propagates in a transversally magnetized ferromagnet, the presence or absence of oscillations is determined, to a considerable degree, by the wave amplitude.

Let us consider the structure of an EMS wave propagating along the $\boldsymbol{e}_z$ axis in an isotropic ferrodielectric magnetized in the ground state by the field $h_{\mathrm{i}}$ in the direction $\boldsymbol{e}_y$. For the analysis we will use the system of equations (3.1) which, being completed by the relaxation terms in the dynamic equations, assumes the form,

$$\partial\theta/\partial\tau = h_x \cos\varphi - h_y \sin\varphi - \eta \sin\theta\,(\cos\theta + h_x \sin\varphi + h_y \cos\varphi)\,,$$

$$\cos\theta\,\partial\varphi/\partial\tau = \sin\theta\,(\cos\theta + h_x \sin\varphi + h_y \cos\varphi) + \eta(h_x \cos\varphi - h_y \sin\varphi)\,, \tag{4.24}$$

$$(\partial^2/\partial\xi^2 - \partial^2/\partial\tau^2)h_x = \partial^2/\partial\tau^2 \cos\theta \sin\varphi\,, \qquad (\partial^2/\partial\xi^2 - \partial^2/\partial\tau^2)h_y = \partial^2/\partial\tau^2 \cos\theta\cos\varphi\,.$$

For a stationary wave propagating with velocity $u$, it follows from the Maxwell equations (4.24) that

$$h_y = h_{\mathrm{i}} - [u^2/(1-u^2)](1-\cos\theta\cos\varphi)\,, \qquad h_x = [u^2/(1-u^2)]\cos\theta\sin\varphi\,. \tag{4.25}$$

Behind the front of the shock wave when $\xi \to -\infty$ the ferromagnet is assumed to be reversely magnetized in the direction $-\boldsymbol{e}_y$. The asymptotic value of the wave magnetic field is $\boldsymbol{h} = -\boldsymbol{e}_y h_{\mathrm{f}}$. From (4.25) it indirectly follows that the shock wave velocity is determined by

$$u^2 = h_0/(1+h_0)\,, \qquad 2h_0 = h_{\mathrm{i}} + h_{\mathrm{f}}\,. \tag{4.26}$$

By substituting (4.25) into (4.24) we obtain the equations

$$\begin{aligned} \mathrm{d}\theta/\mathrm{d}\nu &= \Delta h \sin\varphi - \eta[(1+h_0)\cos\theta - \Delta h\cos\varphi]\sin\theta\,, \\ \cos\theta\,\mathrm{d}\varphi/\mathrm{d}\nu &= [(1+h_0)\cos\theta - \Delta h \cos\varphi]\sin\theta + \eta\,\Delta h\sin\varphi\,, \end{aligned} \tag{4.27}$$

where $2\,\Delta h = h_{\mathrm{f}} - h_{\mathrm{i}}$, $\nu = \tau - \xi/u$, and, as we have repeatedly noted, the condition $h_0 \ll 1$ is usually met in practice. By using this condition one can easily reduce the set (4.27) to the nonlinear pendulum equation [57],

$$\mathrm{d}^2\varphi/\mathrm{d}\nu^2 + \eta\,\mathrm{d}\varphi/\mathrm{d}\nu - \Delta h\sin\varphi = 0\,. \tag{4.28}$$

It is easy to see that the solution of eq. (4.28) that is of interest to us, viz., the one that describes how a ferromagnet magnetization is reversed by a shock wave from the state when the vector is directed along $\boldsymbol{e}_y$, exists only for $\Delta h > 0$. This condition has a simple physical meaning: the ferromagnet energy in the final state $-\boldsymbol{M}\cdot\boldsymbol{H}$ must be less than the energy in the initial state ahead of the wave front.

The functions $\varphi(\nu)$ calculated at $\eta = 5\times10^{-2}$ and $\Delta h = 5\times10^{-4}$ (curve 1), $\Delta h = 5\times10^{-3}$ (curve 2) and $\Delta h = 5\times10^{-2}$ (curve 3) are shown in fig. 6. It can be easily seen that with the growth of $\Delta h$ the duration of the shock wave front decreases because the reverse magnetization rate in the ferromagnet increases. For a large enough value of $\Delta h$ ($\Delta h \gtrsim \eta^2$) oscillations appear at the wave front.

It is interesting to examine formally the solutions of the system (4.27) under the condition $h_0 \gg 1$ that ensures a great propagation velocity of a shock wave [see eq. (4.26)]. It is not difficult to show that in

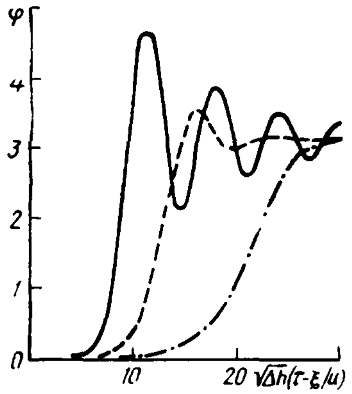


Fig. 6.

the case $h_0 \gg \Delta h$ the system (4.27) is reduced to eq. (4.28) with the substitutions $\nu \to h_0\nu$ and $\Delta h \to \Delta h/h_0$. Thus, if $h \gg 1$, then the propagation velocity of a shock wave increases and its front duration decreases, which may be useful for applications. The condition $h_0 \gg 1$ can be, in principle, realized in a ferrite with two magnetic sublattices with close degree of magnetization. Further we will study the magnetization dynamics in a ferrite in which a shock EMS wave propagates.

### *4.3. A shock wave in a two-sublattice ferromagnet*

Let us examine a model of an isotropic ferrite described by eq. (1.35) with the substitution $\boldsymbol{H}_{\mathrm{eff},1,2} = \boldsymbol{H} - A\boldsymbol{M}_{2,1}$ and with the appropriate relaxation terms added. Since $M_1 - M_2$ is not equal to zero, it is convenient to make dimensionless the temporal and spatial coordinates: $\tau = 4\pi y(M_1 - M_2)t$, $c\xi = 4\pi y(M_1 - M_2)\sqrt{\varepsilon_0}z$. In doing this we obtain for the ferromagnetism vectors, $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2 = (M_1 + M_2)\boldsymbol{m}$, and antiferromagnetism ones, $\boldsymbol{L} = \boldsymbol{M}_1 - \boldsymbol{M}_2 = (M_1 - M_2)\boldsymbol{l}$, the following equations:

$$\begin{aligned}
&\partial \boldsymbol{m}/\partial\tau + \boldsymbol{m}\times\boldsymbol{h} = -\eta^+\{\boldsymbol{m}\times(\boldsymbol{m}\times\boldsymbol{h}) + \boldsymbol{l}\times[\boldsymbol{l}\times(\boldsymbol{h} - A_0\boldsymbol{m})]\\
&\qquad\qquad - \eta^-\{\boldsymbol{l}\times(\boldsymbol{m}\times\boldsymbol{h}) + [\boldsymbol{m}\times(\boldsymbol{l}\times(\boldsymbol{h} - A_0\boldsymbol{m})]\}\,,\\
&\partial \boldsymbol{l}/\partial\tau + \boldsymbol{l}\times(\boldsymbol{h} - A_0\boldsymbol{m}) = -\eta^+\{\boldsymbol{l}\times(\boldsymbol{m}\times\boldsymbol{h}) + \boldsymbol{m}\times[\boldsymbol{l}\times(\boldsymbol{h} - A_0\boldsymbol{m})]\}\\
&\qquad\qquad - \eta^-\{\boldsymbol{m}\times(\boldsymbol{m}\times\boldsymbol{h})] + \boldsymbol{l}\times[\boldsymbol{l}\times(\boldsymbol{h} - A_0\boldsymbol{m})]\}\,,\\
&4\pi A_0 = A\,,
\end{aligned} \tag{4.29}$$

where

$$\boldsymbol{H} = 4\pi(M_1 - M_2)\boldsymbol{h}\,, \qquad 4\eta^{\pm} = (M_1 - M_2)(\eta_1/M_1 \pm \eta_2/M_2)\,.$$

One can easily see that the vectors $\boldsymbol{l}$ and $\boldsymbol{m}$ satisfy the relations

$$\boldsymbol{m}\cdot\boldsymbol{l} = (M_1 + M_2)/(M_1 - M_2) \equiv q\,, \qquad l^2 + m^2 = q^2 + 1\,. \tag{4.30}$$

As in the case of a one-sublattice ferromagnet, we will be interested in the stationary solution of set (4.29) and the corresponding Maxwell equations describing a shock wave that turns a ferrite from the initial state when the vector $\boldsymbol{m}$ is oriented by the field $\boldsymbol{e}_y h_{\mathrm{i}}$ along the $\boldsymbol{e}_y$ axis to the final state when $\boldsymbol{m} = -\boldsymbol{e}_y$ and $\boldsymbol{h} = -\boldsymbol{e}_y h_{\mathrm{f}}$. It is not difficult to see that relations (4.25) and (4.26) are also valid for such a wave. The most interesting waves are the ones propagating with velocity $u \approx 1$ and such that the condition $H_{\mathrm{i}} + H_{\mathrm{f}} \gg 8\pi(M_1 - M_2)$ is satisfied. This condition is satisfiable in the vicinity of the magnetic compensation point of the ferrite when $M_1 \to M_2$ [6].

A specific feature of a ferrite magnetization dynamics is that the moduli of vectors $\boldsymbol{m}$ and $\boldsymbol{l}$ are not conserved. From (4.29) we have

$$\boldsymbol{m} \cdot \dot{\boldsymbol{m}} = -\eta^{+} A_0[q^2(m-1) - m^4] - \eta^{+} q(\boldsymbol{l} \cdot \boldsymbol{h} - q\boldsymbol{m} \cdot \boldsymbol{h}) - \eta^{-}(m^2 \boldsymbol{l} \cdot \boldsymbol{h} - q\boldsymbol{m} \cdot \boldsymbol{h})$$
$$\approx -\eta^{+} A_0 q^2 [m^2 - 1 + (1/A_0 q)(\boldsymbol{l} \cdot \boldsymbol{h} - q\boldsymbol{m} \cdot \boldsymbol{h})] , \tag{4.31}$$

where the coefficient $q$ is defined in (4.30); in our case $q \gg 1$. From (4.31) an interesting result follows: when the modulus of the normed ferromagnetism vector $m(\tau)$ is large enough (when it satisfies the condition $m^2 - 1 \gg h_0/A_0$), the vector satisfies the relation,

$$m^2(\tau) - 1 = [m^2(0) - 1] \exp(-2\eta^{+} A_0 q^2 \tau) , \tag{4.32}$$

independently of external fields. The difference of $m(\tau)$ from unity means that the vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are not collinear. In this situation the effective field of the interlattice exchange creates a sizeable torque under whose action $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ quickly precess around the equilibrium axis, approaching it due to relaxation. In the case $m^2 - 1 \gg h_0/A_0$ the exchange field becomes dominant (the ratio $h_0/A_0$ is usually small since the constant $A_0$ is of the order $10^3$–$10^4$). The characteristic times of the change in the magnetization of the shock wave are much larger than the time when the value $m^2(\tau) - 1$ decreases, so one can assume that $m \approx 1$. It follows from (4.30) that $l \approx q \gg 1$.

Since $|l - q| \ll l$, the modulus of $\boldsymbol{l}$ can be considered as constant, so in the equation for $\boldsymbol{l}$ in set (4.29) one may disregard the relaxation terms. For the subsequent analysis one may use the approach of ref. [51] that has been employed above in studying domain walls in antiferromagnets. Taking the vector product of the equation for $\boldsymbol{l}$ in (4.29) with $\boldsymbol{l}$, we obtain

$$\boldsymbol{m} = \boldsymbol{p} - (1/A_0)[\boldsymbol{p} \times \dot{\boldsymbol{p}} + \boldsymbol{p} \times (\boldsymbol{p} \times \boldsymbol{h})] , \tag{4.33}$$

which expresses the vector $\boldsymbol{m}$ via the vector $\boldsymbol{l} = q\boldsymbol{p}$, $|\boldsymbol{p}| = 1$. Substituting (4.33) into (4.29) results in the equation

$$\dot{\boldsymbol{p}} + \boldsymbol{p} \times \boldsymbol{h} - (1/A_0)[\boldsymbol{p} \times \ddot{\boldsymbol{p}} + \boldsymbol{p} \times (\boldsymbol{p} \times \dot{\boldsymbol{h}}) + 2\boldsymbol{p} \cdot \boldsymbol{h} \dot{\boldsymbol{p}} + \boldsymbol{p} \cdot \boldsymbol{h}\, \boldsymbol{p} \times \boldsymbol{h}] = \eta^{+} q^2 \boldsymbol{p} \times \dot{\boldsymbol{p}} , \tag{4.34}$$

that determines the dynamics of the unit vector $\boldsymbol{p}(\nu)$.

The system of equations (4.26), (4.33) and (4.34) fully determines the front structure and the propagation velocity of a shock EMS wave in a two-sublattice ferrite.

The vector equation (4.34), which involves the angular variables $\theta$ and $\varphi$ that are introduced both for the vector $\boldsymbol{p}$ and for the vector $\boldsymbol{m}$ in the above considered problem of a shock wave in a transversally magnetized one-sublattice ferromagnet, is reduced to the scalar equations,

$$\begin{aligned}
&\dot\theta - h_x \cos\varphi + h_y \sin\varphi + (1/A_0)[\cos\theta\,\ddot\varphi + 2\sin\theta\,\dot\theta\dot\varphi - 2\boldsymbol{p}\cdot\boldsymbol{h}\dot\theta + \boldsymbol{p}\cdot\boldsymbol{h}(h_x\cos\varphi - h_y\sin\varphi) \\
&\quad - \cos\theta\, h_z - \sin\theta\,(\dot h_x \sin\varphi + \dot h_y\cos\varphi)] = -\eta^+ q^2 \cos\theta\,\dot\varphi\,, \\
&\cos\theta\,\dot\varphi - \sin\theta\,(h_x\sin\varphi + h_y\cos\varphi) + \cos\theta\, h_z \\
&\quad + (1/A_0)[-\ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2 + h_x\cos\varphi - h_y\sin\varphi \\
&\quad - 2\boldsymbol{p}\cdot\boldsymbol{h}\cos\theta\,\dot\varphi - \boldsymbol{p}\cdot\boldsymbol{h}\cos\theta\, h_z + \sin\theta\,\boldsymbol{p}\cdot\boldsymbol{h}(h_x\sin\varphi + h_y\cos\varphi)] = \eta + q^2\dot\theta\,.
\end{aligned} \tag{4.35}$$

The magnetic field $\boldsymbol{h}$ entering (4.35) is expressed through the angles $\theta$ and $\varphi$ by means of eqs. (4.26) and (4.33). It is easy to see that in the considered case of $h_0 \gg 1$ the longitudinal field component $h_z$ is of the order $1/h_0 \ll 1$ and it does not greatly affect the magnetization dynamics. So (4.26) and (4.33) yield

$$\begin{aligned}
&(1 - h_0/A_0)(h_x\cos\varphi - h_y\sin\varphi) = \Delta h \sin\varphi - (h_0/A_0)\dot\theta\,, \\
&[1 - (h_0/A_0)\sin^2\theta](h_x\sin\varphi + h_y\cos\varphi) = h_0\cos\theta - \Delta h\cos\varphi - (h_0/A_0)\sin\theta\cos\theta\,\dot\varphi\,.
\end{aligned} \tag{4.36}$$

We will confine our analysis of eqs. (4.35)–(4.36) to the case where $1 \ll h_0 \ll A_0$, which is closer to a realistic experiment. Here the exchange field greatly exceeds the vortex one, so, according to (4.33), $\boldsymbol{m} \approx \boldsymbol{p}$, while the correction to $\boldsymbol{m}$, which should be made because in the magnetization process the vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ become noncollinear, is of the order of $h_0/A_0 \ll 1$. Equations (4.35)–(4.36) take on the form

$$\begin{aligned}
&\dot\theta - \Delta h\sin\varphi + (1/A_0)(\cos\theta\,\ddot\varphi - 2\sin\theta\,\dot\theta\dot\varphi) + \eta^+ q^2\cos\theta\,\dot\varphi = 0\,, \\
&\dot\varphi\cos\theta - \sin\theta(h_0\cos\theta - \Delta h\cos\varphi) - (1/A_0)(\ddot\theta + \sin\theta\cos\theta\,\dot\varphi^2) - \eta^+ q^2\dot\theta = 0\,.
\end{aligned} \tag{4.37}$$

Equations (4.37) differ from eqs. (4.27), which correspond to a one-sublattice ferromagnet, by the terms proportional to $1/A_0$, which account for the precession of the vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ in the exchange field, and also by dissipative terms in the Hilbert form. Taking account of the dissipation in the Hilbert form and in the Landau–Lifshitz form brings about the same result at $\eta^2 q^2 \ll 1$.

A useful piece of information on the shock wave front structure can be obtained by studying the behavior of the solution of eqs. (4.37) near the equilibrium states ahead of and behind the front, i.e., for $\nu \to \mp\infty$ [58]. In these regions eqs. (4.37) can be linearized; besides, one can assume that all the quantities depend on $\nu$ as $\exp(ik\nu)$, with $k$ determined by the dispersion relation

$$k^2 + (h_0 + i\eta^+ q^2 k - k^2/A_0)(\sigma\,\Delta h - i\eta^+ q^2 k + k^2/A_0) = 0\,, \tag{4.38}$$

where $\sigma = 1$ when $\nu \to -\infty$ and $\sigma = -1$ when $\nu \to \infty$.

Approximate solutions of eq. (4.38) have the form

$$\begin{aligned}
&k_{1,2} \approx \pm\{-\sigma\,\Delta h\, h_0 - \tfrac14[\eta^+ q^2(h_0 - \sigma\,\Delta h)]^2\}^{1/2} + \tfrac12 i\eta^+ q^2(h_0 - \sigma\,\Delta h)\,, \\
&k_{3,4} \approx A_0 + i\eta^+ q^2 A_0\,.
\end{aligned} \tag{4.39}$$

According to (4.39), the low-frequency front structure (the front duration and the low-frequency oscillation period) is determined by the vortex magnetic field of the shock wave (roots $R_{1,2}$). The wave profile is a rather smooth function, but high-frequency oscillations caused by the precession of the vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ in the exchange field $AM$ (roots $R_{3,4}$) are superimposed on the profile. Among the solutions described by formulas (4.39) only those which grow at the foot and decrease at the top of the shock wave front are realizable [58]. So the expressions for the roots $R_{3,4}$ imply that the magnetization exchange oscillations reveal themselves mainly at the top of a shock wave front.

To estimate the exchange oscillation amplitude, we will examine the expression

$$\frac{\mathrm{d}E}{\mathrm{d}\nu} = \frac{\mathrm{d}}{\mathrm{d}\nu}\left[\sin^2\theta + \frac{1}{A_0 h_0}(\dot\theta^2 + \cos^2\theta\,\dot\varphi^2) - 2\,\frac{\Delta h}{h_0}\,(1-\cos\theta\cos\varphi)\right] = (\eta^+ q^2/h_0)(\dot\theta^2 + \cos^2\theta\,\dot\varphi^2)\,, \tag{4.40}$$

that follows from eqs. (4.37) and describes the energy dissipation on the shock wave front. The integration constant in (4.40) is selected in such a way that the equality $E=0$ is satisfied before the wave front. According to (4.40), $\mathrm{d}E/\mathrm{d}\nu = 0$ and $E(\nu\to\infty)\to -4\,\Delta h/h_0$. Thus, for the exchange oscillation amplitudes, $\theta_\delta \sim \varphi_\delta$, we obtain the estimate $\theta_\delta < 4\,\Delta h/h_0 A_0 \ll 1$ from (4.40) combined with (4.39).

It follows from (4.40) that for $\Delta h \ll h_0$ the function $\theta(\nu)$ is small. Here eqs. (4.37) are easily reduced to one equation for the angle $\varphi$,

$$\varphi'' + \alpha^+ q^2\varphi' - (\Delta h/h_0)\sin\varphi = -2\alpha^+(h_0/A_0)\varphi''' - (h_0^2/A_0^2)\varphi^{\mathrm{IV}} \approx 0\,, \tag{4.41}$$

where primes denote differentiation with respect to $\eta = h_0\nu$, and the angle $\theta$ proves equal to $\varphi'$. It is easy to see that eq. (4.41) virtually coincides with the appropriate equation that describes the shock wave front structure in a one-sublattice ferromagnet in the case of $h_0 \gg 1$.

### *4.4. Shock waves in an unsaturated ferromagnet*

Thus far we have considered the shock wave structure in a saturated ferromagnet. It should be noted that functioning of practical devices is based on the shock wave propagation in an unsaturated ferromagnet [53, 59].

An unsaturated ferromagnet is a nonlinear dissipative and dispersive medium. To describe a process of magnetization reversal in such a magnet several modal equations are known, the most "physical" of them being the modified Bloch equation,

$$\partial M/\partial t = -(1/\tau_{\mathrm{p}})\{M - M_0\chi(H/H_n)\}\,, \tag{4.42}$$

that seems to have been reported first in ref. [60]. In this equation $\tau_{\mathrm{p}}$ is a typical relaxation time; $M_0$, the magnetic saturation moment; the function $\chi$ is given by the curve of the technical saturation of the given material; $H_n$, the typical saturation field (if $H > H_n$, then $\chi \to 1$).

Equation (4.42) together with the Maxwell equations,

$$\partial E/\partial z = (1/c)\partial B/\partial t\,, \qquad \partial H/\partial z = (\varepsilon/c)\partial E/\partial t\,, \tag{4.43}$$

fully determines the state of a one-dimensional electromagnetic field in a unsaturated ferromagnet.

Let us show that the initial system of equations (4.42)–(4.43) in a broad range of parameters can be reduced to the Burgers equation that has an exact analytical solution.

Note that in the limit $\omega\tau \to 0$ eq. (4.42) implies a static $M = M_0\chi(H/H_n)$; taking it into account permits us to solve (4.43) exactly, the solution corresponding to Riemann waves [14]. Since the relaxation time is finite (usually $\tau_p \sim 10^{-9}$ s), (4.42) yields the relation (in the frequency range $\omega \ll \tau_p^{-1}$),

$$M = M_0\chi(H/H_n) - \tau_p\, \partial M/\partial\tau \approx M_0\chi(H/H_n) - \tau_p(M_0/H_n)\chi'(H/H_n)\partial H/\partial t\,.$$

In case of weakly linear wave processes which can be described within the framework of the quadratic (in the field amplitude) approximation, the above formula can be simplified,

$$M \approx \chi_1(M_0/H_n)H + \chi_2(M_0/H_n^2)H^2 - \tau_p\chi_1(M_0/H_n)\partial H/\partial t\,, \tag{4.44}$$

where $\chi_1$ and $\chi_2$ are the coefficients of the expansion $\chi(H/H_n) = \Sigma_{n\ge 1}\, \chi_n(H/H_n)^n$, usually with $\chi_2 < 0$.

Combining (4.43) and (4.44), we get equations with a short r.h.s.,

$$\begin{gathered}\left(\frac{c}{\sqrt{\varepsilon\mu}}\frac{\partial}{\partial z} \mp \frac{\partial}{\partial t}\right)\left(H \pm \sqrt{\frac{\varepsilon}{\mu}}\,E\right) = \pm\frac{8\pi\chi_2 M_0}{\mu H_n^2} \mp \frac{4\pi\chi_1 M_0\tau_p}{\mu H_n}\frac{\partial^2 H}{\partial t^2}\,,\\ \mu = 1 + 4\pi\chi_1 M_0/H_n\,.\end{gathered} \tag{4.45}$$

Let us examine, for example, waves propagating to the right. For such waves $E \simeq -\sqrt{\mu/\varepsilon}H$, and in the r.h.s. of eq. (4.45) one may set $\partial/\partial t \approx (c/\sqrt{\varepsilon\mu})\partial/\partial z$. As a result, we arrive at the Burgers equation

$$\begin{gathered}\frac{\partial H}{\partial t'} + H\frac{\partial H}{\partial z'} = \nu\frac{\partial^2 H}{\partial z'^2}\,,\\ t' = \frac{4\pi|\chi_2|M_0}{\mu H_n^2}\,t\,, \qquad z' = \frac{\sqrt{\varepsilon\mu}}{c}\,z - t\,, \qquad \nu = \frac{\tau_p\chi_1 H_n}{2|\chi_2|}\,.\end{gathered} \tag{4.46}$$

Reducing the initial system to eq. (4.46) actually solves the general problem, for the Cole–Hopf substitution $H = -2\nu(\partial/\partial z')\ln\varphi$ transforms eq. (4.46) to a linear equation of heat conductivity $\partial\varphi/\partial t' = \nu\,\partial^2\varphi/\partial t'^2$. Thus, the problem of wave propagation in an unsaturated ferromagnet admits a general analytic solution for the frequencies $\omega$ smaller than the relaxation frequency $\tau_p^{-1}$ and for amplitudes $H$ smaller than the saturation field $H_n$ for the given material.

Equation (4.46) opens wide opportunities for the analytic investigation of electromagnetic processes in unsaturated ferromagnetic media, such as turbulent wave states (see ref. [61]) or nonstationary modes of forming shock waves.

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