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Scattering Mechanisms of Conduction Electrons in Transition Metals at Low Temperatures

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References

1. Introduction

The characteristic feature of metallic systems is the presence of conduction electrons; hence the investigation of the effect of those electrons upon various macroscopic properties of metals has received a great deal of interest. Electrical conduction is one of the most important properties essentially depending on the behaviour of the electronic sub-system.

At the same time electrical conductivity is one of the most complex phenomena since it is determined by the properties of all sub-systems of the metal: the electronic, ionic, and (in the case of magnetically ordered metals) magnonic sub-system, and by their interaction. In other words, the conductivity of a given metal depends not only on its electronic structure but also on the mechanisms of relaxation of the conduction electrons which are due to the scattering by structural defects of the lattice, phonons, magnons (in ferro- and antiferromagnetics), and scattering arising from electron–electron interaction. The electrical resistance of simple non-transition metals is mainly established by the first two mechanisms of scattering. For the electric resistance of transition metals, especially at low temperatures, the last mechanism is essential. For magnetically ordered metals, all the above-mentioned mechanisms of scattering make contributions to the resistance.

In the first stage of the investigations of the electrical resistance of metals the main efforts were directed towards the understanding of the scattering of electrons by lattice vibrations and structural defects (impurities, vacancies, dislocations). Several review articles and monographs have been devoted to these mechanisms of scattering (see, for example, [1] to [4]). During the last 10 to 15 years the centre of interest moved to the study of the scattering mechanisms due to interelectronic and electron–magnon collisions. At present a large amount of experimental material has been accumulated; however, there is no review article on these questions so far. With the present paper an attempt is made to fill this gap. However, before passing over to a brief description of the related theoretical concepts and presenting the accumulated experimental material, it seems advisable to make a few remarks concerning some questions of experimental conditions.

In order to isolate the contributions of electron–electron and electron–magnon collisions to electrical resistance, it is necessary to measure the temperature dependence of the resistance of quite pure samples to ensure that the investigated scattering mechanisms are not masked by a large background due to scattering on impurities and other lattice defects (residual resistance). Moreover, according to [5] and [14] at low temperatures the scattering of electrons by vibrating impurity ions leads to an additional contribution to electrical resistance, which is proportional to the concentration of impurities and varies with temperature according to a T^2-law, i.e. in the same way as electron–electron and (for a quadratic dispersion law of spin waves) electron–magnon resistance. In order to estimate the value of this contribution, it is necessary to make measurements on several samples having different concentrations of impurities. Only for samples with a high degree of purity this contribution may be neglected.

However, in low-temperature investigations of pure samples for which the ratio of room temperature to residual resistance $\varrho_{300\,^\circ\mathrm{K}}/\varrho_0$[1]) is about some 10^3 to

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[1]) The value $\varrho_{300\,^\circ\mathrm{K}}/\varrho_0$ the so-called "residual resistance ratio" (RRR), is often used to estimate the purity of the metal.

10^4 it is often necessary to use samples with a small cross-section, or to increase the measuring current through the sample in order to increase the experimental accuracy.

In the first case, when the mean free path of the electrons, l, is comparable with the transverse dimensions of the sample, d, a size effect arises which can distort the real temperature dependence of the bulk resistivity. This is associated with the fact that in thin samples at low temperatures (scattering by small angles) the contribution of normal electron–phonon collisions, the so-called N-processes, grows because in this case each scattering act of an electron by a phonon is followed by a collision with the surface. According to [6] this mechanism of scattering leads to an additional contribution ϱ_{eps} to the temperature-dependent part of electrical resistance²):

$$\varrho_{\text{eps}} = \left(\frac{8\pi}{3}\right)^{1/3} (\varrho_\infty l_\infty)^{2/3} \left(\frac{T}{\theta_{\text{D}}}\right)^{2/3} [\varrho_{\text{ep}}^{\text{N}}(T)]^{1/3}\, d^{-2/3}\,, \tag{1.1}$$

where ϱ_∞ and l_∞ are the resistance and the mean free path of electrons in a bulk sample. θ_{D} is the Debye temperature, and $\varrho_{\text{ep}}^{\text{N}}$ is the resistance due to normal electron–phonon collisions. From the above formula it follows that the thickness of the sample should be chosen according to the condition $d \gg l$.

In the second case, when the magnetic field caused by the measuring current (or external field) has such a magnitude that it is impossible to neglect the radius of curvature of the electronic trajectories in comparison with the mean free path, there arises a galvanomagnetic effect which strongly depends on temperature. The point is that at small fields at which the topological peculiarities of the Fermi surface of the metal are not displayed, the value of the transverse magnetoresistance is $\Delta\varrho(H, T) \sim \varrho(0, T)\, (l(T)/r)^2$. Hence, in measurements of the temperature dependence of electrical resistance at low temperatures which aim at investigating the scattering mechanisms, the condition $l \ll r$ has to be fulfilled.³)

For ferromagnetic metals the experimental situation becomes more complicated due to the fact that even for small currents whose magnetic field may be neglected, there is an internal field due to non-zero magnetic induction which will act on the movement of the electrons. This leads to a galvanomagnetic effect, just as an external magnetic field (or the field of the measuring current) leads to an increase of the electrical resistance of non-ferromagnetic metals. Hence, when discussing experimental results for ferromagnetic metals carried out in zero external magnetic field, it is necessary (see Section 3 of the paper) to take into account the contribution of that internal magnetic field.

2. Scattering Mechanisms Due to Electron–Electron Interaction

2.1 Theoretical concepts concerning the influence of electron–electron interaction on the electrical resistance of metals

The electrostatic interaction between the conduction electrons in a metal is described by a screened Coulomb potential of the type $(e/r)\exp(-q\,r)$; q is

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²) In addition to this temperature-dependent contribution the size effect gives also a temperature-independent contribution which may be evaluated by the well-known formula $\varrho_d = \varrho_\infty\,(1 + \alpha\, l_\infty/d)$ where ϱ_d is the resistance of a sample with thickness d, and α is a parameter characterizing the probability of diffuse reflection from the surface.

³) For evaluations it should be noted that $\boldsymbol{r} = c\,\boldsymbol{P}\times\boldsymbol{H}/(e\,H^2)$, where $\boldsymbol{P}$ and e are the electronic momentum and the charge, and c is the velocity of light.

called the screening constant, and $1/q$ the screening length (or radius). Due to the effect of screening the electrons can interact only at short distances; at distances $r > 1/q$ they may be regarded as independent. Nevertheless, the electron–electron interaction is rather large and is comparable with other types of interaction in a metal, in particular with electron–phonon interaction. This leads to peculiarities in a number of properties of metals, in the first place of kinetic properties. These peculiarities cannot be explained without taking into account electron–electron interaction which results in the occurence of an additional scattering mechanism.

According to the Pauli principle the scattering of two electrons from the states $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ into the states $\boldsymbol{k}_1'$ and $\boldsymbol{k}_2'$ can take place only if both final states are not occupied. Moreover, the initial and final states should lie in an energy range of width kT about the Fermi surface. Due to this fact the probability of interelectronic collisions contains a factor $(kT/\varepsilon_F)^2$. This factor determines the temperature dependence of the electron–electron resistance ϱ_{ee}.

The value ϱ_{ee} depends on the electronic structure of the metal and the probability of electron–electron umklapp processes. Let us discuss two cases which occur in metals:

1. In the metal there is only one group of electrons having the same effective mass, i.e., the metal has spherical (or near-spherical) Fermi surfaces. An example of such a metal is sodium.

2. In the metal there are several groups of current carriers with different effective masses (and different Fermi velocities) corresponding to different sheets of the Fermi surface or different regions of a sheet (strongly anisotropic Fermi surface). The second case is particularly characteristic of transition metals.

1. The influence of interelectronic collisions on the electrical resistance of metals in the first case was first examined in [7]. Later this question was reexamined [4, 8, 12]. It follows from these investigations that the interelectronic collisions lead to a resistance contribution $\varrho_{ee} \sim T^2$ only if in the process of scattering there are electron–electron umklapp processes, as a result of which the total wave vector of the interacting electrons is changed by a reciprocal-lattice vector $\boldsymbol{g}$: $\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_1' - \boldsymbol{k}_2' = \boldsymbol{g}$, i.e. if some momentum is transfered to the lattice as a whole.

According to [4] in this case the electron–electron resistance takes the form

$$\varrho_{ee} = \alpha \frac{e^2}{v_F \, \varepsilon_F} G^2 \frac{g \, k_F}{q^2} \left(\frac{kT}{\varepsilon_F}\right)^2, \qquad (2.1)$$

where α is a constant, e electronic charge, and v_F, ε_F, and k_F are Fermi velocity, Fermi energy, and Fermi wave vector, respectively. G is an interference factor. Experimentally in monovalent non-transition metals (Na, K, Cu, Au, Ag) such a dependence on temperature has never been observed. Since electron–electron umklapp processes are not frozen out even at the lowest temperatures (unlike electron–phonon umklapp processes), this experimental fact apparently indicates a very small value of ϱ_{ee} which is masked by the residual resistance of even very pure monovalent metals.

If there are no umklapp processes, i.e. only normal electron–electron processes ($\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_1' - \boldsymbol{k}_2' = 0$) are essential, interelectronic collisions do not lead to electrical resistance in virtue of the momentum conservation law. Thus normal electron–electron processes of scattering cannot give rise to a relaxation of the electron distribution which is displaced from the equilibrium state by an electric field and has a total momentum different from zero. However, in sufficiently thin samples whose thickness is smaller than the mean free path of the electrons, and in which diffuse scattering by the surface is essential, even normal electron–electron collisions may lead, in principle, to the appearance of resistance. In this case a single electron–electron collision is sufficient to deflect the electrons to the surface, and the diffuse scattering on the surface will lead to an additional contribution to electrical resistance proportional to T^2 and dependent on the thickness of the sample. Hence, in this samples normal electron–electron processes become an effective mechanism of scattering due to surface scattering. The possibility of such a source of resistance was pointed out in [9] and [10], but no experimental evidence of such a mechanism exists.

2. A different situation arises in metals whose electronic sub-system is characterized by several groups of carriers with different velocities (and effective masses). The most characteristic example of such metals are transition metals. If the electronic sub-system of such a metal is not closed (for example, carriers with large effective masses transfer the momentum acquired in interelectronic collisions to the lattice), collisions between carriers with large velocities (and small effective masses) and low-mobility carriers (with large effective masses) will lead to a decrease in the velocities of the "light" carriers and, as a result, the appearance of resistance. It should be noted that in this case the appearance of ϱ_{ee} is possible even in the absence of electron–electron umklapp processes.

Theoretically this case was discussed [4, 11, 12, 13]. It is shown in [11] that the electrical resistance due to interelectronic collisions, i.e. the scattering of "light" carriers with mass m_1 by "heavy" carriers with mass m_2, is proportional to the square of the temperature and is expressed by the formula

$$\varrho_{ee} = \frac{\pi^2 e^2 m_1^2}{16\, n\, h^3} \left(\frac{kT}{\varepsilon'_F}\right)^2 H(\beta, q) = a\, T^2 \tag{2.2}$$

(n number of carriers with mass m_1 per unit volume, h Planck's constant, $\beta = m_2/m_1$, ε'_F Fermi energy of the group of "light" carriers counted from the bottom of the band). In (2.2) the factor $H(\beta, q)$ is essential: as β increases by a factor of 10, the value of $H(\beta, q)$ increases by more than two orders. According to [11] at a given value of the screening constant $H(\beta, q)$ is proportional to $\beta^{2.15}$. With increasing screening constant $H(\beta, q)$ decreases. Hence, the greater the difference between the effective masses of the two groups of carriers and the less the value of q, the greater is the contribution of ϱ_{ee} to the total electrical resistance of the metal.

The evaluation of ϱ_{ee} for a metal with two groups of electrons was also considered in [4]. In this work ϱ_{ee} is expressed not through the ratio of effective masses but through the ratio of Fermi velocities:

$$\varrho_{ee} = \frac{\pi^4}{16} \frac{e^2}{v_1 \varepsilon'_F} \frac{k_F}{q} \left(\frac{v_1 - v_2}{v_2}\right)^2 \left(\frac{kT}{\varepsilon_F}\right)^2. \tag{2.3}$$

Here v_1 and v_2 are the velocities of the carriers with small and great effective mass, respectively; $v_1 > v_2$.

A similar calculation for ϱ_{ee} was carried out in [12]. It is necessary to note that the expressions (2.2) and (2.3) were obtained in the approximation of two spherical Fermi surfaces and quadratic dispersion laws. These assumptions were also made in [12].

Table 1

metal	$\varrho_{273.2\,°K}$ [a] (10^{-6} Ωcm)	θ_D (°K)	$\frac{\varrho_{300\,°K}}{\varrho_{T\to 0\,°K}}$	a (10^{-12} Ωcm deg^{-2})	ref. for ϱ	γ [d] (mJ mol^{-1} deg^{-2})	ref. for γ
Zr	38.6	291	34	≈ 80	[23]	2.80	[41]
Nb	13.5	275	30000	32	[30]	7.79	[36],
			30	140	[2]		[42]
Mo	5.2	425	7700	1.8	[31]	2.01	[37]
			5250	2.6	[23]		
			1600	2.1	[23]		
			800	1.8	[31]		
Ru	$\varrho \perp c$	450	1220	2.7	[22]	2.95	[38]
	$\varrho \perp c$ 7.2	—	90	2.8	[25]		
	$\varrho \parallel c$ 5.4	—	—	2.3	[25]		
Rh	4.4	512	—	—	—	4.65	[39]
Pd	9.7	267	580	33	[2]	9.57	[40]
			400	31	[32]		
Hf	28.1	252	32	≈ 15	[23]	2.16	[41]
Ta	12.1	230	200	≈ 70	[23]	6.27	[37]
W	4.9	379	300000	0.8	[33]	0.84	[43]
			27000	0.6	[34]		
			25000	1.5	[23]		
			9400 to 95000	0.6 to 1.2	[28][b]		
Re	$\varrho \perp c$ 16.8	280	5100	4.3	[20]	2.29	[44]
	$\varrho_{\varphi=82°}$ [c]	—	2240	5	[23]		
Os	$\varrho_{\varphi=60°}$ 7.9	400	310	2.3	[21]	2.35	[45]
	$\varrho_{\varphi=16°}$ 6.0		400	1.8	[21]		
Ir	4.7	310	150	0.9	[24]	3.14	[45]
Pt	9.6	235	5000	12	[10]	6.41	[39]
			2500	14	[2]		
			2300	17	[26]		
			280	19	[46]		

[a] The value $\varrho_{273.2\,°K}$ for Zr, Nb, Rh, Pd, Hf, Ta, Ir, and Pt are taken from [2].
[b] According to [28] there is no correlation between a and $\varrho_{300\,°K}/\varrho_{0\,°K}$.
[c] φ is the angle between the rod axis and the c-axis.
[d] γ is the coefficient of electronic heat capacity.

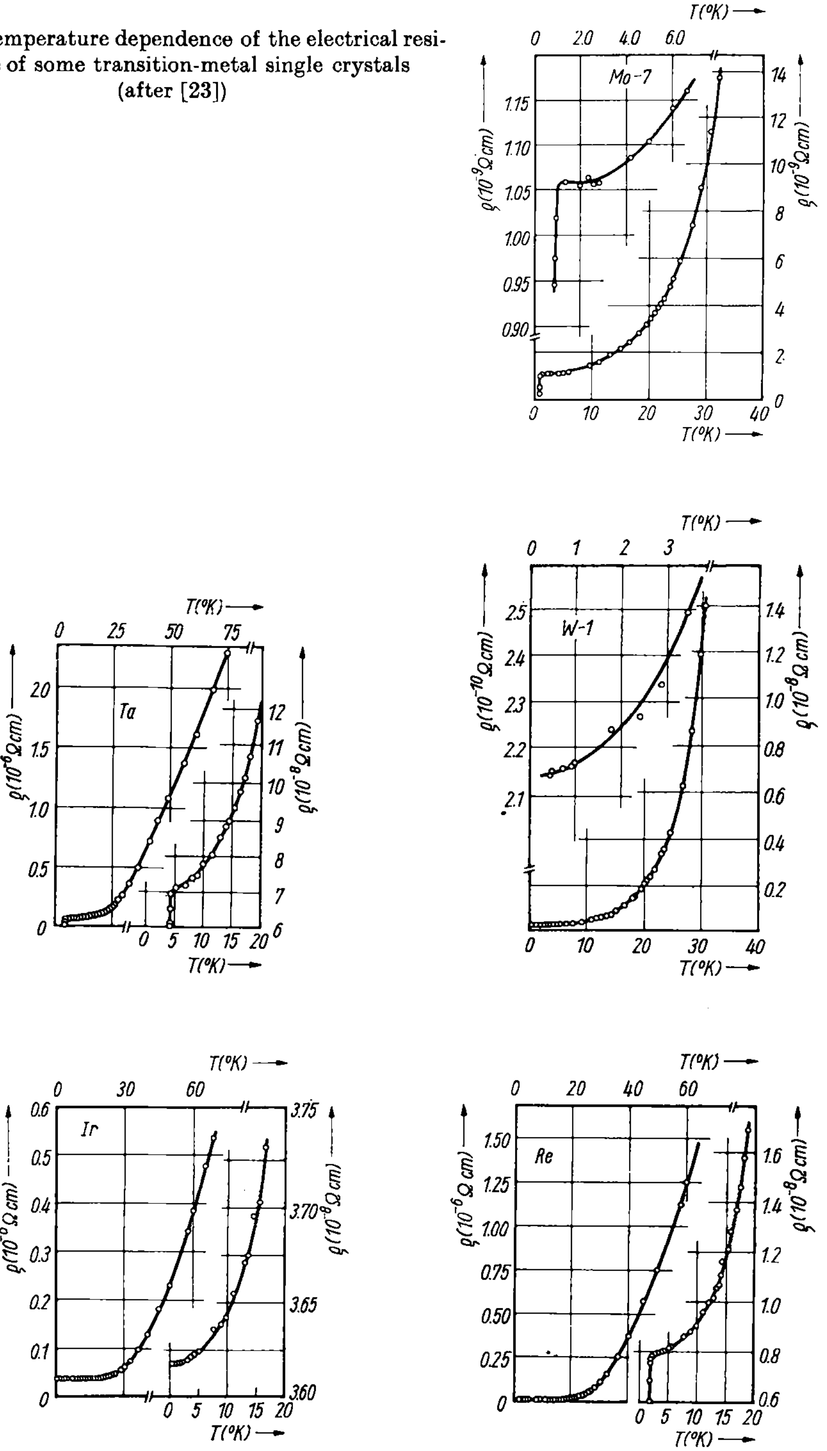

Fig. 1. Temperature dependence of the electrical resistance of some transition-metal single crystals (after [23])

Thus, the results of the theoretical papers [4], [11], and [12] show that at low temperatures ϱ_{ee}[4] may be dominant in the temperature-dependent part of electrical resistance, since with decreasing temperature the electron–phonon resistance decreases much faster ($\varrho_{ep} \sim T^5$ at $T < 0.1\ \theta_D$) than ϱ_{ee}.

2.2 Experimental investigations of the contribution of interelectronic collisions to the electrical resistance of transition metals

For the first time a quadratic dependence of electrical resistance on temperature at $T \ll \theta_D$ was experimentally discovered in Pt in 1933 [16]. Later analogous temperature dependences of electrical resistance were found in Mo [17], W [18], Nb, Pd, Rh, Mn, Fe[5], Co[5], and Ni[5] [2, 19]. In these papers quantitative studies of the contribution $\sim T^2$ were made not for all these metals; however, the main value of the enumerated papers consisted in the experimental evidence of a new scattering mechanism in transition metals due to interelectronic collisions. For many other transition metals the contribution of this mechanism of scattering was detected only recently.

At present the contribution of interelectronic collisions has been observed in most transition metals. The results of these experimental investigations and the references to the papers are given in Table 1.

Typical temperature dependences of the electrical resistance of transition metals are given in Fig. 1. In order to reveal the different scattering mechanisms leading to different temperature dependences of electrical resistance, it is convenient to give the dependence of an "ideal" electric resistance on temperature, $\varrho_i = \varrho(T) - \varrho_0$, in a logarithmic scale. Such dependences for some transition metals are given in Fig. 2. This figure shows the transition from the T^5-dependence, which characterizes electron–phonon resistance, to the T^2-dependence, which occurs at sufficiently low ($\approx$10 °K) temperatures and characterizes electron–electron scattering.

An analysis of the experimental results has shown that the temperature dependence of the electrical resistance of transition metals at $T < 0.1\ \theta_D$ cannot be described by the expression

$$\varrho(T) = \varrho_0 + b\,T^5\,, \tag{2.4}$$

but it follows very well the expression

$$\varrho(T) = \varrho_0 + a\,T^2 + b\,T^5\,. \tag{2.5}$$

The occurrence of such a dependence can be demonstrated by plotting $[\varrho(T) - \varrho_0]/T^2$ versus T^3. The result should be a straight line which intersects the ordinate axis at a (electron–electron scattering); the slope of this line is equal to the coefficient b (electron–phonon scattering). The experimental data treated by this method indeed fit a straight line. Typical plots are shown in Fig. 3. For greater reliability, the measured $\varrho(T)$ dependences were treated in a number of papers (see, for example, [23]) according to the above relation by a least-squares method, using an electronic computer. Both methods give identical results. Moreover, in [26] the temperature dependence of the electrical re-

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[4] In general, the above mechanism of scattering due to interelectronic collisions can apparently manifest itself even in non-transition multivalent metals, which have rather complex Fermi surfaces. However, experimental confirmations of this effect are not numerous at the present time; therefore the discussion of this question would now be premature.

[5] In ferromagnetic metals a T^2-dependence of electrical resistance occurs due to interelectronic scattering as well as due to scattering by spin waves.

sistance of platinum measured on several samples of high purity in the region of liquid helium temperature was discussed on the basis of a power series:

$$\varrho(T) = \varrho_0 + \alpha\, T + a\, T^2 + \beta\, T^3 + \mu\, T^4 + b\, T^5\,, \tag{2.6}$$

instead of using (2.5). It was found that the coefficients α, β, and μ are close to zero, and ϱ_0, a, and b are positive and practically completely determine $\varrho(T)$ for the investigated samples.

As seen from Table 1 there is a considerable scattering of the values of a measured on different samples of the same metal (see Mo, W, Ru, Pt in Table 1). For example, the values of a, which are found in [28] for nine tungsten samples with residual resistance ratios (RRR) from 9400 to 95000, lie in the range

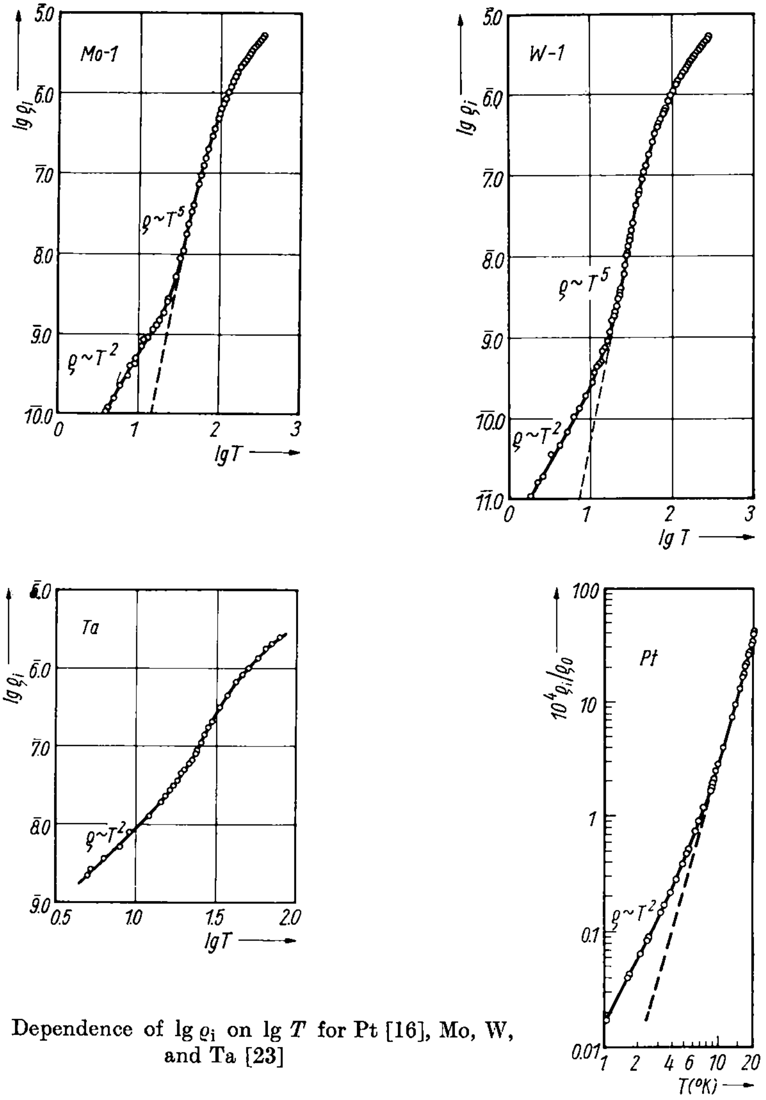

Fig. 2. Dependence of lg ϱ_i on lg T for Pt [16], Mo, W, and Ta [23]

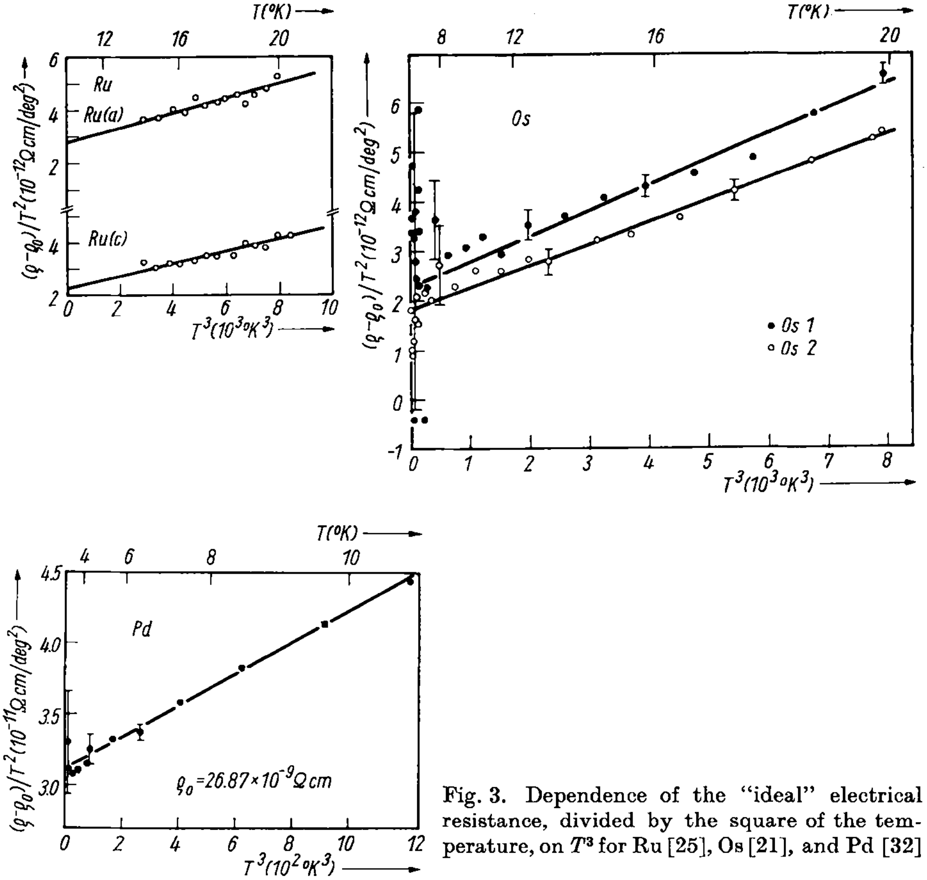

Fig. 3. Dependence of the "ideal" electrical resistance, divided by the square of the temperature, on T^3 for Ru [25], Os [21], and Pd [32]

0.6×10^{-12} to 1.2×10^{-12} Ωcmdeg^{-2}, but there is no correlation between a and RRR, characterizing the sample purity.

With regard to the quantitative correlation between a and γ noted in [27] for Os, Re, Co, Fe, Ni, Pt, and Pd ($a \sim \gamma^2$), it should be noted that the value of a is determined not only by the values of the effective masses (or γ), but also depends on the screening constant, the Fermi energy, and also on the probability of electron–electron umklapp processes. The latter characteristics of the electronic sub-system also differ for different metals, as well as the effective masses and the coefficient of the electronic heat capacity. Electron–electron scattering affects not only the electric resistance, but also the temperature dependence of the thermal resistance. In recent papers (for example, [20 to 22, 28]) this contribution was discovered in rhenium, osmium, ruthenium, tungsten, and other metals. This is additional evidence of the strong influence of electron–electron interaction on the kinetic properties of metals. The temperature dependence of the mean free path characteristic of interelectronic collisions ($l \sim T^{-2}$) was discovered recently in measurements of the radio-frequency size effect in Mo and W [29].

Thus, the available experimental data show that the temperature-dependent part of the electrical resistance of transition metals at sufficiently low temperatures ($T \lesseqgtr 10$ °K) is determined practically completely by the scattering of electrons due to interelectronic collisions. At liquid helium temperature the electron–electron resistance of transition metals is higher than the electron–phonon resistance by several tens of magnitude. So, for example, at $T = 4.2$ °K the ratio $\varrho_{ee}/\varrho_{ep}$ is equal to 30 for tungsten, 60 for molybdenum, 50 for osmium, and 130 for ruthenium and platinum.

It is difficult to draw conclusions on the exact origin of ϱ_{ee} in non-ferromagnetic transition metals: There may be contributions to ϱ_{ee} from collisions of groups of carriers with different effective masses and different Fermi velocities as well as from interelectronic collisions accompanied by electron–electron umklapp processes.

The scattering mechanisms of electrons which determine the temperature dependence of the electrical resistance of magnetically ordered transition metals, will be discussed in Section 3 of this review.

3. Scattering of Conduction Electrons by Spin Waves in Ferromagnetic (3d and 4f) Metals

3.1 Scattering mechanism leading to a quadratic temperature dependence of electrical resistance

Transition metals with magnetic order possess some peculiarities in their kinetic properties. The thermal excitations in the system of atomic magnetic moments (spin waves[6]) lead to the appearance of additional scattering mechanisms for the current carriers. Let us discuss the main results of theoretical papers in which the temperature dependence of the electrical resistance due to the scattering of conduction electrons by spin waves is investigated.

The collision processes of s-electrons and magnons at $T \ll \theta_C$ (θ_C Curie temperature) were first investigated by Vonsovskii on the basis of the s–d model. The interaction between s-electrons and magnons occurs via creation and destruction of spin waves. In these processes the conservation law of quasi-momentum is fulfilled:

$$\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{q} \, ,$$

where $\boldsymbol{k}'$ and $\boldsymbol{k}$ are the quasi-momentum of the s-electron before and after the collision, $\boldsymbol{q}$ is the quasi-momentum of the magnon [47].

The calculation of the energy spectrum of electrons in ferromagnetic crystals and of the transition probabilities in such systems performed in [48] allowed to develop a theory of the electrical conductivity of ferromagnetic metals in the region of low temperatures.

The qualitative behaviour of $\varrho_{em}(T)$ due to any electron–magnon scattering mechanism can be obtained from general physical considerations without solving the kinetic equation [49]. If in the collision the electron would fully lose the momentum obtained in the electric field, the electrical resistance would be proportional to the total number of collisions,

$$\int \overline{N}_q \, w(q) \, \delta \, (E_{k,\sigma} - E_{k-q,\sigma'} - \varepsilon_q) \, \mathrm{d}q \, , \tag{3.1}$$

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[6] The quasi-particles corresponding to spin waves are ferromagnons (or magnons) characterized by the quasi-momentum $\boldsymbol{q}$.

where $\overline{N}_q$ is the average number of magnons with momentum q (the Bose distribution function). The product $w(q)\ \delta\ (E_{k,\sigma} - E_{k-q,\sigma'} - \varepsilon_q)$ gives the probability of collisions in which the electron loses the momentum q and the energy $(E_{k,\sigma} - E_{k-q,\sigma'})$, $w(q)$ is the squared absolute value of the matrix element corresponding to the given transition, and the δ-function ensures energy and quasi-momentum conservation. However, at low temperatures in each collision the component of the momentum along the field changes by a value directly proportional to q^2. Hence, the number of collisions during which the electron will fully lose its momentum is $\sim q^{-2}$, i.e.,

$$\varrho_{\mathrm{em}}(T) \sim \int \overline{N}_q\ w(q)\ \delta(E_{k,\sigma} - E_{k-q,\sigma'} - \varepsilon_q)\ q^2\ \mathrm{d}q\ , \tag{3.2}$$

where the integral is taken over the whole momentum space. Assuming that the energy of the electrons depends only on the absolute value of the quasi-momentum and not on spin, and integrating the expression (3.2) over the angles it follows that

$$\varrho_{\mathrm{em}}(T) = \mathrm{const} \int \overline{N}_q\ w(q)\ q^3\ \mathrm{d}q\ . \tag{3.3}$$

Hence, for the qualitative determination of the temperature dependence $\varrho_{\mathrm{em}}(T)$, according to (3.3) it is sufficient to know $w(q)$. For processes due to exchange electrostatic interaction $w(q)$ is constant, and from (3.3) it follows that

$$\varrho_{\mathrm{em}}(T) = a_1\ T^2\ . \tag{3.4}$$

Analogous results are obtained by more rigorous calculations of the electrical resistance of ferromagnetics [50 to 55] (solution of the kinetic equation and use of the statistic perturbation theory developed by Kubo and applied to the calculation of electrical resistance by Nakano). In [55] it was taken into account that the energy of the conduction electron depends not only on the absolute value of quasi-momentum, but also on the direction of spin:

$$E_{k,\sigma} = E(k) + 2\ \sigma\ I(k)\ ,$$

where $E(k)$ and $I(k)$ are functions of the absolute value of quasi-momentum; $\sigma = \pm 1/2$. As a consequence in (3.4) an exponential multiplier of the type $\exp\ (-T_0/T)$ appears, where the parameter $T_0 = \theta_{\mathrm{C}}(a\ k_0)^2\ (J/E_{\mathrm{F}})^2$ is some critical temperature below which the scattering processes of the type discussed become non-effective (due to the impossibility of the simultaneous conservation of energy and quasi-momentum in electron–magnon collisions). The value of T_0 is found [55, 56] to lie between 0.1 and 10 °K. The coefficient a_1 in (3.4) depends on the functions $E(k)$ and $I(k)$ and their derivatives at the Fermi level [55].

For a quadratic dispersion law $\varrho_{\mathrm{em}}(T)$ takes the form

$$\varrho_{\mathrm{em}}(T) \sim \exp\left(-\frac{T_0}{T}\right) \quad \text{at} \quad T \ll T_0\ , \tag{3.5}$$

$$\varrho_{\mathrm{em}}(T) \sim T^2\ \varphi\left(\frac{T_0}{T}\right) \quad \text{at} \quad T \approx T_0\ , \tag{3.6}$$

where

$$\varphi(t) = \frac{3}{2\ \pi^2} \int_t^{\infty} \frac{x\ \mathrm{e}^x}{(\mathrm{e}^x - 1)}\ (x - t)\ \mathrm{d}x\ ; \quad t = \frac{T_0}{T}$$

and

$$\varrho_{\mathrm{em}}(T) \sim T^2 \quad \text{at} \quad T \gg T_0 \,. \tag{3.7}$$

Since $\varphi(t)$ at $T \approx T_0$ rapidly changes with temperature, $\varrho_{\mathrm{em}}(T)$ falls more rapidly than quadratically as the temperature decreases.

Hence it may be said that the scattering of electrons on spin waves due to s–d exchange interaction results in a quadratic temperature dependence of the electrical resistance of ferromagnetics in the region of applicability of the spin-wave approximation ($T < 0.1\ \theta_{\mathrm{C}}$ but $T > T_0$).

3.2 Scattering mechanisms leading to a linear temperature dependence of electrical resistance

In some theoretical papers temperature dependences of the magnetic component of resistance different from (3.4) were obtained for low temperatures. Turov [49] showed theoretically that the electrical resistance of 3d ferromagnetic metals in the region 1 to 10 °K has a term which is proportional to temperature. Recently in [52] and [57] calculations were made for various scattering mechanisms which qualitatively explain the linear dependence of the electrical resistance of Fe, Co, and Ni in the region of low temperatures.

One of the possible mechanisms is the scattering of conduction electrons by spin waves taking into account the polarization of these electrons. In this case

$$\varrho_{\mathrm{em}}(T) = a_0\, T + a_1\, T^2 \;; \quad T \gg T_0 \,, \tag{3.8}$$

where the second term is due to the scattering of electrons by spin waves (see (3.4) and (3.6)), and the first one arises from the polarization of s-electrons by d-electrons. If the coefficients a_0 and a_1 have values of the same order, $\varrho_{\mathrm{em}}(T)$ is $\sim T^2$ at $T \gg T_0$. The linear term is comparable with the quadratic one only at a sufficiently large value of the ratio a_0/a_1. As is seen from the calculation [57], which was performed for the case of an arbitrary isotropic dispersion law of the electrons, the coefficient a_0 is determined by an expression of the type

$$a_0 \sim k \left(\frac{\mathrm{d}^2 E}{\mathrm{d}k^2} \, \frac{\mathrm{d}k}{\mathrm{d}E} \right)_{E_F} . \tag{3.9}$$

Another reason for the appearance of a linear term in $\varrho_{\mathrm{em}}(T)$ may be the spin–orbit interaction between the spin of the "magnetic" electron and the orbital motion of the conduction electron. Of all the kinds of spin–orbit interaction which are possible in the s–d exchange model, just this interaction plays the main role in the kinetic effects. The probabilities of collisions corresponding to this type of spin–orbit interaction were calculated by Vonsovskii [48], and Turov [49] took into consideration their influence on the electrical resistance. Qualitatively this dependence may be obtained from (3.3) if we substitute

$$w(q) \sim \frac{k^2}{q^2} \sin^2 \vartheta \,, \tag{3.10}$$

where ϑ is the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{q}$.

If the above type of spin–orbit interaction makes the main contribution to the linear term, an anisotropy of a_0 should be observed in a magnetic field. The relative anisotropy originating from the anisotropy of the matrix elements of spin–orbit interaction was calculated in [52] and [57] and found to be $(a_{0\parallel} - a_{0\perp}) \times \times\, a_{0\parallel}^{-1} \approx 0.1$.

The third mechanism which, in principle, can explain the appearance of a linear term in the electrical resistance is associated with the scattering of electrons by excitations in domain boundaries. It was shown by Winter [58] that in a Bloch domain boundary there are some special excitations, which are similar to spin waves with a wave vector having two degrees of freedom. The amplitude of such waves has its maximum at the centre of the boundary and vanishes in the volume of the domain. Theoretical investigations of the scattering by the domain boundaries themselves and of the scattering by two-dimensional spin waves [52, 57] show that for a random distribution of domain boundaries

$$\varrho_{\mathrm{em}}(T) = \nu\,(\alpha\,\mathrm{e}^{-\pi k_{\mathrm{F}}\delta} + \beta\,T^{3/2}) \tag{3.11}$$

(δ thickness of the domain boundaries, ν relative volume of the domain boundaries, k_{F} Fermi wave vector; α, β constants).

The first term in (3.11) corresponds to the scattering by static domain boundaries and only makes a contribution to the residual resistance ϱ_0. Since $\pi\,k_{\mathrm{F}}\,\delta \approx \approx 10^2$ to 10^3, this term may be neglected. Physically this is due to the slow change of the scattering potential of the domain boundary.

The second term in (3.11) arises from the scattering of electrons by two-dimensional spin waves. Since the magnitude of this term depends on the number of domain boundaries, it should strongly decrease with increasing magnetic field in the region of technical magnetization, and be anisotropic, i.e. different for orientation of the current parallel and perpendicular to magnetization. This term can always be roughly approximated by an expression of the type (3.8).

Hence the spin–orbit interaction between the d-electron spin and the orbital motion of the s-electron leads to a linear dependence of $\varrho_{\mathrm{em}}(T)$. The linear term in the electrical resistance will manifest itself only at very low temperatures $T \lessapprox 1$ °K, due to the smallness of spin–orbit interaction compared with the exchange interaction discussed in Section 3.1.

3.3 Contribution of two-magnon processes to the electrical resistance at low temperatures

In Sections 3.1 and 3.2 we discussed scattering processes of s-electrons which involve the creation or annihilation of a spin wave. Such processes are called single-magnon (EM) processes. These processes are inelastic and are associated with the fluctuation of the "transverse" components of magnetization ($S_n^{\pm}$). The dependence of energy on spin has the consequence that single-magnon processes are frozen-in at temperatures $T < T_0$ (see (3.5)), i.e., there is a minimum wave vector of magnon, $g_{\min}$, such that processes with $|\boldsymbol{g}| < g_{\min}$ do not take place.

For two-magnon processes of scattering (ZM) (associated with the fluctuation of the longitudinal component of magnetization (S_n^z)) which, unlike EM processes take place without a change of the spin of the s-electrons, there is no such limitation, i.e., they take place over the whole temperature region where the spin-wave approximation is valid. The contribution of two-magnon processes to the resistance of ferromagnetic metals $\varrho_{\mathrm{ZM}}(T)$ at low temperatures was examined by Rösler [51] using the transition probabilities for ZM processes calculated in [59]. The temperature dependence of electrical resistance due to ZM processes (without umklapp processes) is expressed by

$$\varrho_{\mathrm{ZM}}(T) = A\,T^{7/2}\,, \tag{3.12}$$

whereas that due to EM processes with an accuracy to logarithmic terms is given by the expression (3.5). A comparison of the coefficients in (3.12) and (3.5) shows that ϱ_{ZM} is larger than ϱ_{EM} only for $T < 0.1\ T_0$, i.e. where the resistance is already determined by impurities. So the ZM processes may be neglected practically over the whole region of temperatures.

We should note that the theoretical calculations discussed in Sections 3.1 to 3.3 were carried out for a quadratic dispersion law of spin waves and, in most cases, for an arbitrary isotropic dispersion law of conduction electrons.

3.4 Experimental situation in the determination of the temperature dependence of magnetic resistance — Difficulties in interpretation

The temperature dependence of the electrical resistance of the ferromagnetic metals Fe, Co, and Ni was studied experimentally by many investigators. White and Woods [2] found that below 10 °K the electrical resistance of these metals quadratically depends on temperature: $a\,T^2$, and the coefficient a (see Table 2)

Table 2

metal	chemical purity	RRR ($H = 0$)	a_0 (10^{-12} Ωcm deg^{-1})	a (10^{-12} Ωcm deg^{-2})	temperature region (°K)	ref.
Fe	99.98%	25.6	39.4 to 49.3	9.85 to 11.8	1.23 to 4.2	[61]
	99.98%	25	30.6	10.8	0.38 to 4.2	[61]
	99.98%	40 to 104	—	12.8	<4.2	[2]
	—	180	826	1	<4.2	[72]
	"whisker" $\langle 100\rangle$	10^3 to 4×10^3	49.3	20.7	1 to 4.2 <1	[73]
	"whisker" $\langle 111\rangle$	700 to 2000*)	—	32.5	0.28 to 20	[68]**)
	99.98%	10 to 240	11 to 30	15 to 17	4.2 to 20	[69]
Co	99.9984%	26.3	19 to 32	8.7 to 9.9	1.23 to 4.2	[61]
	99.999%	65 to 91	—	12.8	<10	[2]
	—	66.6	3	57.5	1.2 to 6	[62]
	—	400	15	5 to 24.4	1 to 4.2	[74]
Ni	99.94%	100	5.8 to 16	19.8	1.23 to 4.2	[61]
	99.99%	21 to 31	—	16	<10	[2]
	99.9%	5	8	41	<20	[60]
	—	1400	—	26	5 to 20	[67]
	—	1400	—	14	16 to 20	[63]
	99.99%	—	—	15.5	2 to 50	[64]

*) The RRR value is given for $B = 0$.

**) The electrical resistance was measured in a longitudinal magnetic field with $H = 1.2$ kOe.

for Fe and Co is equal to 13×10^{-12} Ωcmdeg^{-2} for Fe and Co, and to 16×10^{-12} Ωcmdeg^{-2} for Ni. This dependence was confirmed for Fe and Ni below 18 °K by Kondorskii et al. [60], and for Co in the temperature range 1.2 to 6 °K by Radhakrishna and Nielsen [62]. For Ni with RRR ($H = 0$) $= 1400$[7]) the value $a = 14 \times 10^{-12}$ Ωcmdeg^{-2} was obtained by Schwerer and Silcox [63], and $a = 15.5 \times 10^{-12}$ Ωcm deg^{-2} by Greig and Harrison [64]. In a series of papers by Semenenko and Sudovtsov [61] a quadratic as well as a linear term in $\varrho(T)$ are reported for Fe, Co, and Ni at $T < 4.2$ °K. The values of the coefficient a from [61] and [62] are close to the values obtained by White and Woods.

We should note that the experimental values of this coefficient for ferro- and non-ferromagnetic metals are of comparable magnitude. Thus a amounts to 18×10^{-12} Ωcmdeg^{-2} for Pt and 9×10^{-12} Ωcmdeg^{-2} for Ir.

This means that the quadratic temperature dependence of the electrical resistance of ferromagnetic 3d metals at $T \ll \theta_C$ should be essentially due to interelectronic collisions[8]): It should be noted that these collisions are quasi-elastic, while the scattering of electrons by magnons and phonons at these temperatures is essentially inelastic. A quite reliable method af establishing the predominance of elastic scattering in the region of low temperatures is the simultaneous study of the "ideal" parts of the electrical and thermal resistance, ϱ_i and W_i (obtained by subtracting the residual resistances). In the case of elastic scattering their ratio should give a constant value, called the Lorentz number:

$$L_i = \frac{\varrho_i}{W_i}. \tag{3.13}$$

A calculation by Herring [65] considering only electron–electron collisions, for metals with a complex Fermi surface gives the value $L_i = 1 \times 10^{-8}$ WΩdeg^{-2}. This has been confirmed experimentally for non-magnetic transition metals [66], though the accurate value of L_i to some extent depends on the substance. For ferromagnetic metals this problem was experimentally investigated only in two papers. For pure Ni (RRR ($H = 0$) $= 1400$) White and Tainsh [67] obtained $L_i = 1 \times 10^{-8}$ WΩdeg^{-2} in the range 2 to 50 °K (Fig. 4). In [68] the electrical and thermal resistances of Fe single-crystal whiskers with the orientation $\langle 111 \rangle$ and RRR ($B = 0$) $\approx 10^3$ were measured in a longitudinal magnetic field of 1.2 kOe. It was found that below 20 °K L_i has a constant value equal to 1.09×10^{-8} WΩdeg^{-2} (see Fig. 4).

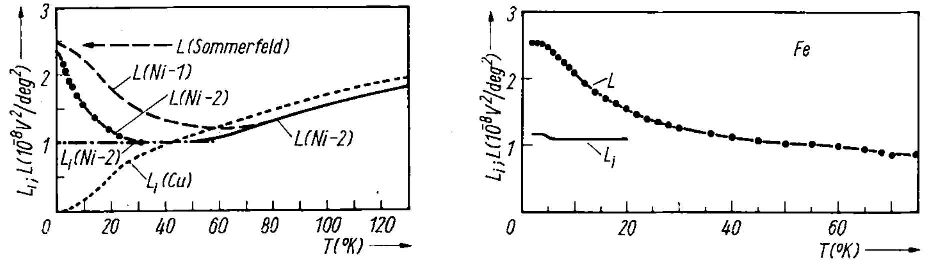

Fig. 4. Temperature dependence of the total Lorentz number and the "ideal" Lorentz number for Ni (Ni-1 by [2], Ni-2 by [67]) and Fe (by [68]). $L = \varrho / W\,T$ [15]

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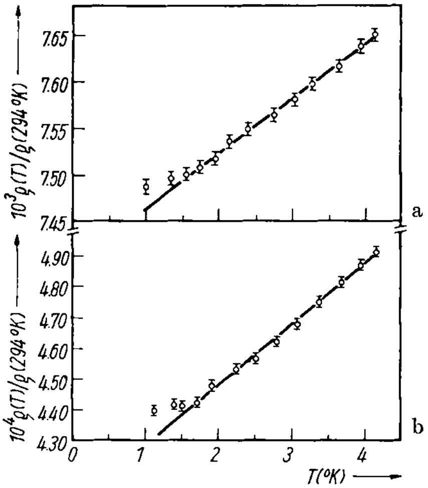
[7]) For ferromagnetics it is necessary to distinguish RRR ($H = 0$) and RRR ($B = 0$) (see Section 3.6).

[8]) Only the temperature-dependent part of the electrical resistance is discussed.

Fig. 5. Temperature dependence of the electrical resistance of Fe whiskers in the temperature region 1.1 to 4.2 °K, at $H = 0$ [72]. a) Fe ⟨100⟩; FC state; ϱ (294 °K) = 0.1038 Ω; b) Fe ⟨100⟩; PS state; ϱ (294 °K) = 0.1038 Ω



Hence, on the basis of [67] and [68] it may be concluded that the contribution of s–d exchange interaction to the T^2-part of electrical resistance is quite small in comparison with the contribution of interelectronic collisions.

Let us analyse late experimental papers devoted to the separation of the term proportional to T in the electrical resistance of Fe. The most characteristic of them are the papers by Coleman et al. [72, 73] performed on maximum-purity Fe single-crystal samples $\left(\text{RRR}\ (H = 0) \approx 130 \text{ to } 250,\ \text{RRR}\ (B = 0) \approx 10^3 \text{ to } 2 \times 10^3\right)$. The electrical resistance of Fe was measured in two magnetic states: in the demagnetized (FC) and the single-domain (PS) state. (The results obtained for one of the samples are shown in Fig. 5.)

The total resistance $\varrho(T)$ in the region 1 to 4.2 °K was treated by means of the formula $\varrho(T) = \varrho_0 + a_0 T$. The average value of the coefficient a_0 for different samples in the FC and PS state is equal to 8.3×10^{-10} Ωcmdeg^{-1} and 2.2×10^{-10} Ωcmdeg^{-1}, respectively. The values of these coefficients are by two orders higher than the values obtained in previous papers. The results of [72] and [73] seem to be consistent with the dominance of the scattering of electrons on spin waves expected theoretically at these temperatures. However, it should be noted that if the term $a\,T^2$ is included in $\varrho(T)$ the coefficient a ($a < 10^{-12}$ Ωcmdeg^{-2}) obtained in [72] is by an order smaller than the value usually observed before (see Table 2).

So far, in analysing experimental data, the formula (3.8) was usually applied, or only one linear term was considered.

However, in ferromagnetics it is necessary to separate the temperature dependence of the electrical resistance into two parts:

1. a part due to electron–magnon scattering: $\varrho_{\text{em}}(T)$;
2. a part due to the dependence of magnetoresistance on relaxation time τ: $\varrho_B(T)$.

The magnetoresistance part ϱ_B appears to be due to the influence of the internal magnetic field, equal to $4\,\pi\,M_s$ (M_s saturation magnetization), on the movement of conduction electrons. This effect is analogous to the behaviour of a non-magnetic metal in an external magnetic field, since the magnetoresistance of the metal depends on the ratio l/r (see Introduction). So the total resistance of a ferromagnetic metal has the form

$$\varrho(\tau) = c_1 \frac{1}{\tau} + c_2(\omega_c\,\tau)^n\,,$$

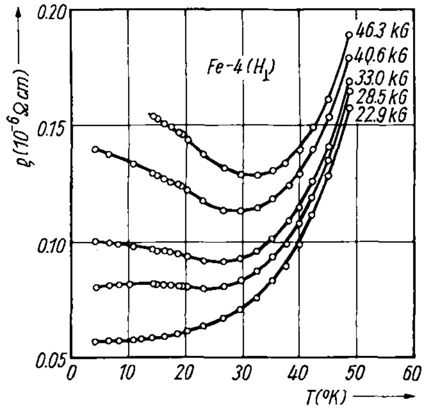


Fig. 6. Temperature dependence of the electrical resistance of Fe in a transverse magnetic field, at some values of magnetic induction [69]

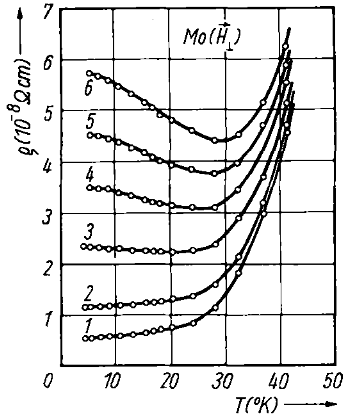


Fig. 7. Temperature dependence of the electrical resistance of Mo in a transverse magnetic field. (1) $H = 0$; (2) 2.5 kOe; (3) 5 kOe; (4) 7 kOe; (5) 8.5 kOe; (6) 9.9 kOe

where

$$\omega_c = \frac{e\,(H_{\text{ex}} + 4\,\pi\,M_s)}{m\,c}. \tag{3.14}$$

Here we have neglected the demagnetization factor. The first term is due to the spin-wave scattering due to spin–orbit interaction. Its value decreases with decreasing temperature due to the growth of τ. The second term is the contribution of magnetoresistance which increases with decreasing temperature. Above a certain value of the magnetic field the second term should be dominant. This will lead to the appearance of a minimum in $\varrho(T)$.

In [69] investigations were carried out on polycrystalline Fe $\big(\text{RRR}\ (H = 0) = 240\big)$ in a transverse magnetic field, which distinctly showed the influence of ϱ_B on the temperature dependence of the electrical resistance (see Fig. 6). At inductions up to 22.9 kG the contribution of the magnetoresistance ϱ_B does not affect the temperature dependence of electrical resistance. However, on slightly increasing the ratio l/r (increasing B to 28.5 kG) a minimum appears in the curve $\varrho(T)$ which shifts towards higher temperatures with increasing magnetic field. A similar behaviour is shown by the electrical resistance of a non-ferromagnetic transition metal, molybdenum, which was measured at the same H_{eff}9) as for iron. In Fig. 7 $\varrho(T)$ curves are shown for a molybdenum single crystal (RRR =

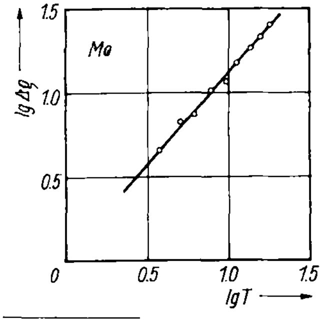


Fig. 8. Contribution of magnetoresistance to $\varrho(T)$ for Mo in a transverse magnetic field; $H = 2.5$ kOe ($\Delta\varrho = \varrho_H - \varrho_{H=0}$)

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9) The value of H_{eff} is determined by the relation $H_{\text{eff}}(T) = H\,\big(\varrho_{300\,°\text{K}}/\varrho(T)\big)$.

$= 900$) at several values of the external magnetic field. From comparison of $\varrho(T)$ at $H = 0$ and $\varrho(T)$ at $H = 2.5$ kOe (see Fig. 8) it may be seen that in the region 4.2 to 20 °K the increase of resistance at the expense of ϱ_B is nearly proportional to temperature: $\Delta\varrho(T) \sim T^{1.1\pm0.1}$. If this additional part of resistance is approximated as $a_0\, T$, one obtains $a_0 = 30 \times 10^{-12}$ Ωcmdeg^{-1}, i.e. the same order as for Fe (see Table 2). Therefore it may be expected that the larger part of the linear term in ferromagnetics (in particular in Fe) is due to the contribution arising from the effect of spontaneous magnetization on the movement of the electrons. Hence the contribution ϱ_B to the resistance of pure Fe samples (essential even in the absence of an external field) gives considerable complications in studies of spin-wave scattering.

We note that the value of ϱ_B depends on whether the magnetoresistance is in the low-field region (impure samples),

$$\omega_c\,\tau \ll 1\ ,$$

in the high-field region (pure samples),

$$\omega_c\,\tau \gg 1\ ,$$

or in the region of intermediate fields,

$$\omega_c\,\tau \approx 1\ .$$

In the second and third case the topological peculiarities of the Fermi surface and the state of compensation of the metal become significant.

So, the above analysis of theoretical and experimental results concerning the temperature dependence of the electrical resistance of ferromagnetic metals at low temperatures leads to the conclusion that the influence of the internal magnetic field (spontaneous magnetization) on the movement of conduction electrons has to be taken into account. Only if the influence of this factor on $\varrho(T)$ can be eliminated, it will be possible to obtain reliable quantitative data on the separation of different scattering mechanisms in ferromagnetic metals.

3.5 Use of the Kohler rule for the determination of electrical resistance of ferromagnetic metals at zero magnetic induction

One of the methods of eliminating the magnetoresistance contribution to $\varrho(T)$ is based on the application of the Kohler rule. According to this rule the influence of the magnetic field on the electrical resistance of a metal can be described by the Kohler function [4]

$$\frac{\Delta\varrho}{\varrho(0)} = \frac{\varrho(H) - \varrho(0)}{\varrho(0)} = F\left(\frac{H}{\varrho(0)}\right), \tag{3.15}$$

where $\varrho(0)$ is the resistance at zero magnetic field, and $F\,(H/\varrho(0))$, which depends on the relative orientation of the magnetic field and the measuring current, is a universal function independent of temperature and sample purity. The temperature independence of $F(H/\varrho(0))$ follows from the assumption of an isotropic relaxation time of the electrons. As follows from (3.15) the parameter determining the magnetoresistance obeys the relation $H/\varrho(0) \sim l/r$, where l is the mean free path and r is the radius of the cyclotron orbit, of the electron. The Kohler rule is fulfilled for many metals [4], but deviations from it may be observed. As noted in [70] in the case of anisotropic Fermi surfaces the occurrence of small-angle scattering of electrons leads to a deviation from the Kohler rule if the background of isotropic scattering on impurities is not too high. In ferromagnetic metals deviations may also be observed due to the fact that $\varrho(H = 0, T)$ contains the magnetoresistance contribution ϱ_B.

Numerous experiments indicate that the Kohler rule is applicable only in the case of large-angle scattering of electrons. This circumstance limits the range of sample purity for which $\varrho_{\text{em}}(T)$ can be separated from $\varrho(T)$ by means of the Kohler rule: samples with very small impurity concentration are inadequate, since on passing from helium to hydrogen temperature large-angle scattering of electrons (mainly due to impurity scattering) should remain dominant over small-angle scattering (collisions with long-wavelength phonons). In the reverse case there may be deviations from the Kohler rule associated with topological peculiarities of the Fermi surfaces of the investigated metals.

This approach to the problem of separating the magnetoresistance contribution to $\varrho(T)$ was used for polycrystalline Fe $\big(\text{RRR}\ (H = 0) = 240\big)$ [69], Ni $\big(\text{RRR}\ (H = 0) = 1400\big)$ [63], and Co samples $\big(\text{RRR}\ (H = 0) = 150\big)$.

If the Kohler function would be known, it would be possible to determine the values $\varrho(B = 0, T)$. But usually this function is unknown; therefore we present a method of estimating it for the example of Fe. The relation (3.15) may be written as

$$\frac{\varrho(B = 0, T)}{K_T\,\varrho(H = 0, T)} = 1 + F\left(\frac{B}{K_T\,\varrho(H = 0, T)}\right) = \Phi\left(\frac{B}{K_T\,\varrho(H = 0, T)}\right). \tag{3.16}$$

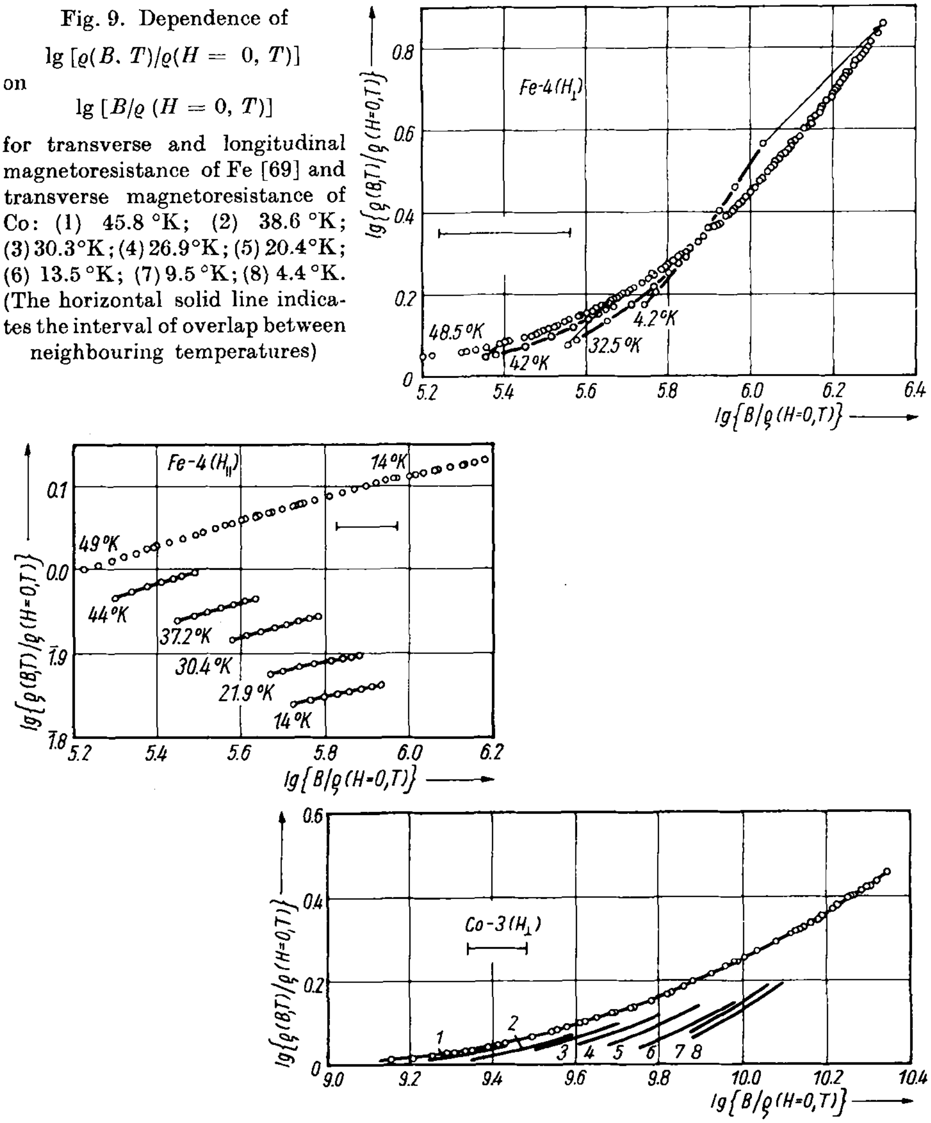
It can be shown experimentally [69] that the magnetoresistance at $T > 40$ °K (due to the small value of l/r) can be neglected, and it may be assumed that $\varrho(H = 0, T) \approx \varrho(B = 0, T)$. Therefore we may put $K_T = 1$ at $T > 40$ °K, and for $T < 40$ °K the values of K_T can be determined from (3.16). For this purpose, using the isotherms of transverse magnetoresistance, it is necessary to construct the dependences

$$\lg\left(\frac{\varrho(B = 0, T)}{\varrho(H = 0, T)}\right) \quad \text{vs.} \quad \lg\left(\frac{B}{\varrho(H = 0, T)}\right). \tag{3.17}$$

Displacing all the curves (3.17) for $T < 40$ °K so that they form one continuous curve, and ensuring a sufficient interval of overlap between neighbouring temperatures we obtain a continuous smooth curve from 48.5 to 4.2 °K. The advantage of the logarithmic scale lies in the fact that the fit of the magnetoresistance isotherms to one monotonous function involves a simple shift of the curves along two axes by the value $\lg K_T$. The maximum error in the determination of $\lg K_T$ is ± 0.005, which allows one to determine $\varrho(B = 0, T)$ with an accuracy of 2.5 to 5%. From Fig. 9 it can be seen that the transition from phonon scattering prevailing at $T > 20$ °K to impurity scattering does not lead to appreciable deviations from the Kohler rule in Fe of given purity. Using the obtained values of K_T it is possible to determine $\varrho(B = 0, T)$, which (Fig. 10) essentially differs from $\varrho(H = 0, T)$.

A similar analysis of the longitudinal magnetoresistance in the temperature region 14 to 48 °K, which is illustrated in Fig. 9, gives values of $\varrho\ (B = 0, T)$ coinciding with those calculated from transverse magnetoresistance. If we use the isotherms of longitudinal magnetoresistance measured in the region 22.9 to 36 kG to construct $\varrho(T)$ at constant field, a minimum in these curves does not appear. This is associated with the fact that the longitudinal magnetoresistance

Fig. 9. Dependence of $\lg[\varrho(B, T)/\varrho(H = 0, T)]$ on $\lg[B/\varrho(H = 0, T)]$ for transverse and longitudinal magnetoresistance of Fe [69] and transverse magnetoresistance of Co: (1) 45.8 °K; (2) 38.6 °K; (3) 30.3°K; (4) 26.9°K; (5) 20.4°K; (6) 13.5 °K; (7) 9.5 °K; (8) 4.4 °K. (The horizontal solid line indicates the interval of overlap between neighbouring temperatures)



saturates in the region $l/r \gg 1$ and is proportional to B^n $(0 < n < 1)$ in the transition region $l/r \approx 1$.

In a similar way the magnetoresistance contribution to $\varrho(T)$ was taken into account in experiments performed on cobalt $(\text{RRR}\ (H = 0) = 150)$ and nickel $(\text{RRR}\ (H = 0) = 1400)$ [63] in a transverse magnetic field. The results for Co are shown in Fig. 9 and 10, and those for Ni in Fig. 11. In [63] it was noted that if in constructing the Kohler function for Ni of a given purity, the magnetic induction B is used instead of the magnetic field H, and $\varrho(0)$ is taken corresponding to $\varrho(B = 0, T)$ but not to $\varrho(H = 0, T)$, the Kohler rule is fulfilled in the temperature range where the scattering of electrons on phonons is dominant, i.e. at

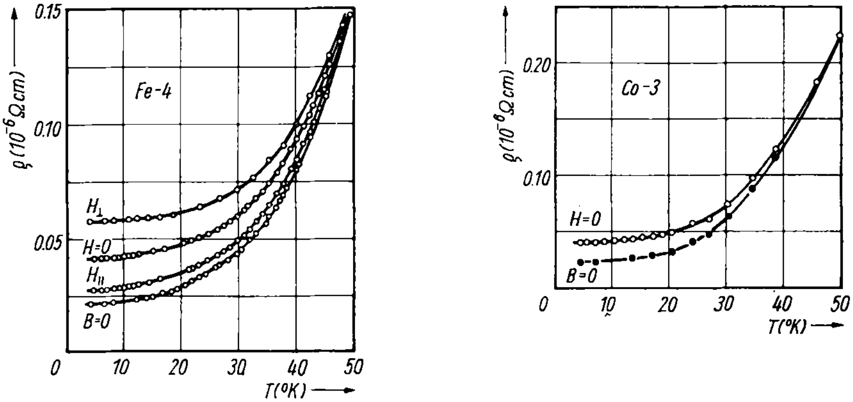


Fig. 10. Temperature dependence of the electrical resistance of Fe at $B = 0$, $H = 0$, and $H_{\parallel} = H_{\perp} = 1.1$ kOe [69], and of Co at $B = 0$ and $H = 0$

17 °K $< T <$ 64 °K. Below 17 °K deviations from the Kohler rule were observed, and these were associated with the influence of small-angle scattering because of the high purity of the samples; therefore it was impossible to find $\varrho(B = 0, T)$. The problem of impurity resistance may be solved using the same rule, but a set of samples is required containing impurities of the same type (see Section 3.6).

For Fe and Co in the temperature range 4.2 to 20 °K the temperature-dependent part of the electrical resistance was described by the expression (3.8) in which each term is responsible for a definite scattering mechanism. (In this region the electron–phonon part of the resistance can be excluded.) The corresponding values of coefficients are shown in Table 3. In this table $\varrho(T)$ data at $H = 0$ for Fe and Co samples of different purity are also given for comparison.

Hence, in order to obtain the temperature dependence of electrical resistance at zero magnetic induction, $\varrho(B = 0, T)$, in ferromagnetics it is possible to apply the Kohler rule for longitudinal and transverse magnetoresistance. The corresponding samples should contain enough impurities so that deviations from the Kohler rule associated with the transition from large- to small-angle scattering (by long-wavelength phonons) do not occur. The investigation of the tem-

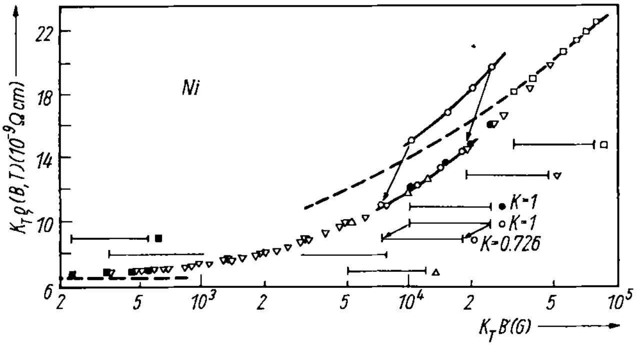


Fig. 11. Transverse magnetoresistance of Ni represented as the dependence of $K_T\,\varrho$ on $K_T\,B$ (data for fourteen temperatures from 1.4 to 64 °K) [63]

Table 3

samples	RRR ($H = 0$)	RRR ($B = 0$)	a_0 (10^{-12} Ωcm deg^{-1})			a (10^{-12} Ωcm deg^{-2})		
			$H = 0$	$H_{\parallel}$	$B = 0$	$H = 0$	$H_{\parallel}$	$B = 0$
Fe-1	10	10	11	9	—	16	17	—
Fe-2	50	57	16	52	—	17	17	—
Fe-3	71	80	21	—	—	16	—	—
Fe-4	240	465	30	93	2	15	15	15
Co-1	20	—	20	—	—	16	—	—
Co-2	60	—	33	—	—	15	—	—
Co-3	150	277	55	—	8	15	—	16

perature dependence of electrical resistance in the demagnetized state ($H = 0$) shows that with decreasing purity the linear term decreases considerably while the quadratic term remains practically unchanged. In the temperature dependence $\varrho(B = 0, T)$ the linear term is considerably smaller than the values corresponding to the resistance in the demagnetized state (at $H = 0$).

3.6 Residual electrical resistance of ferromagnetic metals

In order to clarify the character of the temperature dependence $\varrho(T)$, it is necessary to known how to determine correctly the residual resistance ϱ_0. In ferromagnetic metals $\varrho_{0,\,H=0}$ is not proportional to the concentrations of impurities and other lattice defects, i.e. is not determined only by the scattering on inhomogeneities of the electrostatic potential. Moreover, Berger and de Vroomen [71] showed that for Fe $\varrho_{0,\,H=0}$ as a function of impurity concentration has a minimum which corresponds to RRR ($H = 0$) ≈ 300. The anomalous behaviour of $\varrho_{0,\,H=0}$ for ferromagnetics arises from the fact that there is a contribution to $\varrho_{0,\,H=0}$ which comes from the magnetoresistance $\varrho_B(0)$ due to the presence of spontaneous magnetization (see Section 3.4). Therefore the residual resistance of ferromagnetic metals at $H = 0$ may be written as

$$\varrho_{0,\,H=0} = k\,c + \varrho_B(0)\,, \tag{3.18}$$

where k is a constant, c the concentration of impurities, and ϱ_B the contribution to resistance arising from the internal magnetic field $B = 4\,\pi\,M_s$.

Thus, to obtain the residual resistance it is necessary to know the value $\varrho_B(0)$. Fujii and Morimoto [75] determined ϱ_B for a zone-refined Fe polycrystal at 4.2 °K from measurements of $\varrho(T)$ in a longitudinal field up to 5.5 kOe. An analysis of the correlation between the resistance ϱ ($B = 0$; 4.2 °K) obtained by extrapolation and $\varrho(H = 0$; 4.2 °K) shows that the electrical resistance measured at $H = 0$ contains a contribution from magneto-

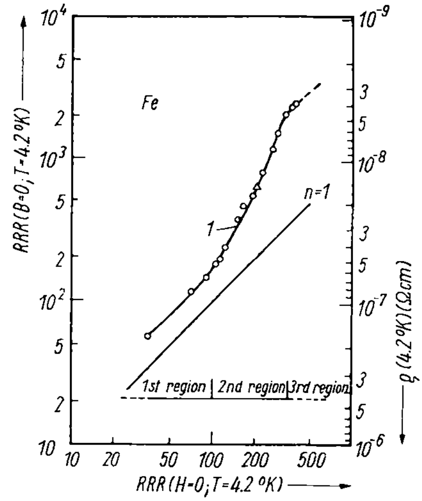


Fig. 12. Connection between RRR ($B = 0$, $T = 4.2$ °K) and RRR ($H = 0$, $T = 4.2$ °K) for Fe: ○ by [75] and △ by [35]

resistance lying between 38 and 84% for samples with RRR ($H = 0$) $= 35$ to 400, and depends on the concentration of impurities. The maximum value of ϱ_B (4.2 °K) is 0.02×10^{-6} Ω cm. Fig. 12 shows the correlation between the resistance-ratios at $B = 0$ and $H = 0$. The difference between the straight line $n = 1$ and the RRR ($B = 0$; 4.2 °K) curve gives the value of the magnetoresistance. As seen from Fig. 12 it increases with increasing RRR. Roughly speaking three regions may be distinguished:

1. RRR ($H = 0$) $=$ 35 to 100: small ϱ_B.
2. RRR ($H = 0$) $=$ 100 to 300: RRR ($B = 0$; 4.2 °K) is two to six times larger than RRR ($H = 0$; 4.2 °K).
3. RRR ($H = 0$) $=$ 330 to 400: RRR ($H = 0$; 4.2 °K) contains the constant value $\varrho_{B,\,4.2\,°\mathrm{K}} = 0.02 \times 10^{-6}$ Ωcm.

Hence, the above-mentioned nomogram may be used to determine the purity of polycrystalline Fe.

Such an effect was observed by Fujii in Ni [76]. He found that for Ni samples with RRR ($H = 0$) $= 300$ to 1600 the contribution of magnetoresistance to $\varrho(H = 0; 4.2$ °K) lies between 15 and 23.5% and depends on the concentration of impurities (see Fig. 13). A similar effect should be observed in Co; however, its influence on the residual resistance has not been studied so far.

It should be noted that the longitudinal magnetoresistance tends to saturation when the magnetic field increases (see, for example, [77]), and therefore the determination of $\varrho(B = 0; 4.2$ °K) performed in [75] and [76] by means of linear extrapolation is not always correct, especially for pure samples whose magnetoresistance reaches the region of saturation.

A consistent analysis of residual resistance (not making use of linear extrapolation) has been carried out by Schwerer and Silcox [78] on the basis of the Kohler rule.

For ferromagnetic metals two additional scattering mechanisms should be mentioned which make a contribution to residual resistance:

1. scattering by an inhomogenous distribution of d-electron spins caused by the presence of internal stresses (magnetostriction residual resistance);
2. scattering by the inhomogeneous spin distribution in domain boundaries (magnetocrystal residual resistance).

The first mechanism has been investigated in [79] where it is shown that at large magnetic field strengths this part of resistance decreases as H^{-2}. Therefore, if we determine ϱ_0 from measurements in a magnetic field, this part of the resistance may be neglected.

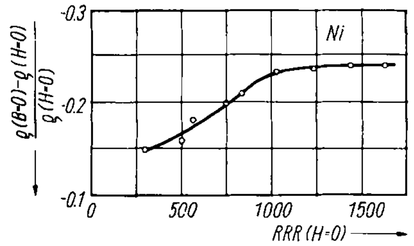


Fig. 13. Values of the magnetoresistance of Ni at $T = 4.2$ °K for samples of different purity [78]

Hence, the residual electrical resistance of a ferromagnetic metal is determined not only by the number of static lattice imperfections, but also by inhomogeneous distributions of d-electron spins and the contribution of the magnetoresistance due to the internal magnetic field. This should be taken into account when the contributions of different scattering mechanisms to the electrical resistance are to be separated.

3.7 Influence of the spin-wave dispersion law on $\varrho_{\mathrm{em}}(T)$ in ferromagnetically ordered rare-earth metals

The temperature dependence of the electrical resistance due to the scattering of electrons by spin waves is determined essentially by the energy spectrum of the spin waves. In Section 3.1 a discussion was given for a quadratic spin-wave dispersion law which occurs in ferromagnetics with small anisotropy. However, the ferromagnetics with hexagonal close-packed structure including heavy rare-earth metals usually possess considerable magnetic anisotropy between the hexagonal axis and the basal plane, as well as in the basal plane. As shown by Niira [80] in such crystals the lowest branch of the spectrum of spin waves has the form (for sufficiently small q)

$$\hbar\omega \sim \Delta \left\{1 + \alpha_1 \left[(q_1 + q_2)^2 + \frac{1}{3}(q_1 - q_2)^2\right] + \alpha_2 q_3^2\right\}, \tag{3.19}$$

where the constants Δ, α_1, and α_2 are determined by the exchange integral and anisotropy energy; q_1, q_2, q_3 are the components of the vector $\boldsymbol{q}$ along the basis vectors of the reciprocal lattice. The energy gap Δ which is the minimum energy required for the excitation of a spin wave, is determined by the values of the magnetic anisotropy constants for the hexagonal axis k_2 and the basal axis k_6. Δ may be evaluated by means of the formula

$$\Delta = \frac{(k_2\, k_6)^{1/2}}{k_{\mathrm{B}}},$$

where k_{B} is the Boltzmann constant.

It was shown by Turov and Shavrov [81] that in the spin-wave spectrum of ferro- and antiferromagnetics, in addition to the "pure magnetic" gap (which is due to the magnetic anisotropy energy and the energy of demagnetizing fields), there is an energy gap associated with spontaneous deformations caused by magnetoelastic energy. A real possibility for the appearance of a magnetoelastic gap occurs in h.c.p. crystals with a small anisotropy constant in the basal plane and with the easy magnetization axis in the basal plane, but with a sufficiently large anisotropy constant in the direction of the hexagonal axis. Hence, in the spin-wave dispersion law an energy gap associated with the magnetic anisotropy energy or magnetoelastic energy may appear.

As mentioned in Section 3.1, in the case of a quadratic dispersion law of spin waves the magnetic part of the resistance of ferromagnetic metals at low temperatures is proportional to T^2 (see (3.4)). In the presence of magnetic anisotropy an exponential factor appears in the expression for $\varrho_{\mathrm{em}}(T)$ which is connected with the occurrence of an energy gap in the spectrum of spin waves:

$$\varrho_{\mathrm{em}}(T) = B\, T^2 \exp\left(-\frac{\Delta}{T}\right), \tag{3.20}$$

where B is a coefficient independent of temperature and Δ is the value of the energy gap expressed in °K.

Yoshimori [82] and Kaplan [83] found that for a magnetic structure which has the form of a ferromagnetic spiral the spin-wave dispersion law is linear. For a linear dispersion law

$$\varrho_{\mathrm{em}}(T) \sim T^4 \,. \tag{3.21}$$

In this case anisotropy can also cause the appearance of an energy gap; it follows

$$\varrho_{\mathrm{em}}(T) = C\, T^4 \exp\left(-\frac{\Delta}{T}\right). \tag{3.22}$$

Accordingly the character of the spin-wave spectrum is reflected in the temperature dependence of the kinetic coefficients, but also in that of all quantities at thermodynamic equilibrium, in particular spontaneous magnetization, $\sigma_{\mathrm{s}}(T)$, and magnetic contribution to heat capacity, $C_{\mathrm{m}}(T)$ (see Table 4).

Table 4

spin-wave dispersion law	$\varrho_{\mathrm{em}}(T)$	$\sigma_{\mathrm{s}}(T) - \sigma_{\mathrm{s}}$ (0 °K)	$C_{\mathrm{m}}(T)$
quadratic, $\Delta = 0$	T^2	$T^{3/2}$	$T^{3/2}$
quadratic, $\Delta \neq 0$	$T^2 \exp(-\Delta/T)$	$T^{3/2} \exp(-\Delta/T)$	—
linear, $\Delta = 0$	T^4	T^2	T^3
linear, $\Delta \neq 0$	$T^4 \exp(-\Delta/T)$	$T^2 \exp(-\Delta/T)$	—

3.8 Experimental investigation of the temperature dependence of the magnetic part of the resistance of rare-earth metals (polycrystals)

The temperature dependences of the total resistance (curve 1) and the magnetic part of the resistance (curve 3) of polycrystalline rare-earth metals (Gd, Tb, Dy, Er, and Tm) are presented in Fig. 14. The comparatively low purity of the investigated rare-earth samples allows one to neglect the contribution ϱ_B which is due to the influence of the internal field on the movement of conduction electrons (see Section 3.4). Indeed, in effective fields where $l/r \ll 1$, $\Delta\varrho/\varrho$ is proportional to $(l/r)^2$; for metals with RRR $(H = 0) \approx 10$ in a magnetic field $H = 3 \times 10^4$ Oe the ratio l/r is about 0.1, i.e., the contribution of ϱ_B does not exceed 1 per cent of the total resistance $\varrho(T)$. It follows that ϱ_0 can be obtained by extrapolation of $\varrho(T)$ to 0 °K and that the contribution of $\varrho_B(T)$ to $\varrho_{\mathrm{em}}(T)$ is negligible.

For separating the temperature-dependent parts of resistance it is assumed that the various scattering mechanisms give additive contributions to the total resistance $\varrho(T)$:

$$\varrho(T) = \varrho_0 + \varrho_{\mathrm{ee}}(T) + \varrho_{\mathrm{em}}(T) + \varrho_{\mathrm{ep}}(T) \tag{3.23}$$

(ϱ_0 residual resistance, $\varrho_{\mathrm{ee}}(T)$ resistance associated with electron–electron collisions, $\varrho_{\mathrm{em}}(T)$ and $\varrho_{\mathrm{ep}}(T)$ resistance due to the scattering of electrons by magnons and phonons, respectively).

In order to determine the dependence $\varrho_{\mathrm{em}}(T)$ without postulating the form of this function, it is necessary, using definite assumptions, to separate $\varrho_{\mathrm{ee}}(T)$ and $\varrho_{\mathrm{ep}}(T)$ from $\varrho_{\mathrm{i}}(T) = \varrho(T) - \varrho_0$. One of the possible ways of separating $\varrho_{\mathrm{ee}}(T)$ is to use data on the temperature dependence of the resistance of a metal, which

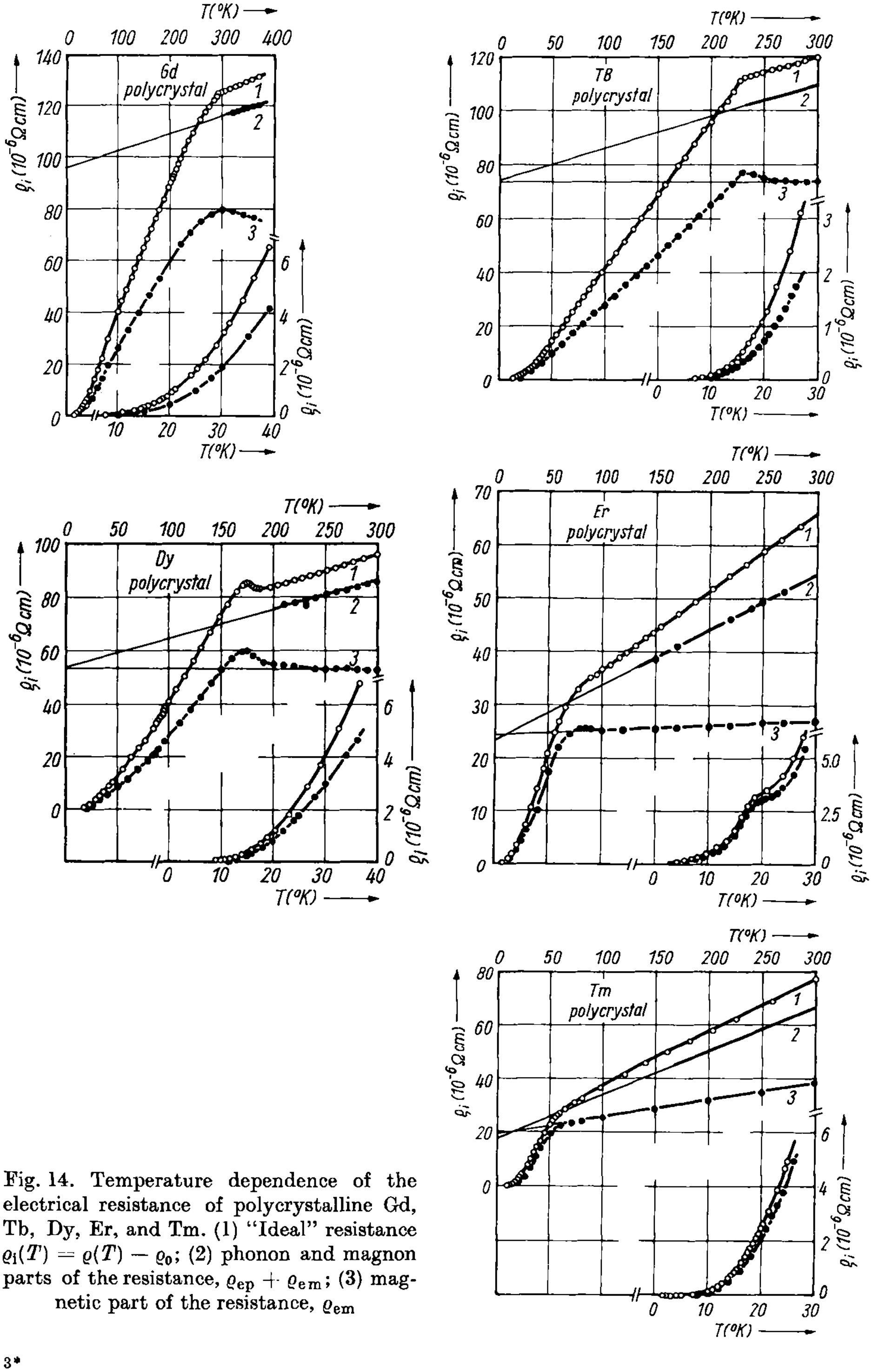


Fig. 14. Temperature dependence of the electrical resistance of polycrystalline Gd, Tb, Dy, Er, and Tm. (1) "Ideal" resistance $\varrho_i(T) = \varrho(T) - \varrho_0$; (2) phonon and magnon parts of the resistance, $\varrho_{ep} + \varrho_{em}$; (3) magnetic part of the resistance, ϱ_{em}

has a similar electronic structure, but does not possess a magnetically ordered state. In the rare-earth series such a metal is lutetium.

Indeed, calculations by the method of augmented plane waves considering relativistic effects [84] show that the Fermi surfaces and the densities of states at the Fermi surfaces of the heavy rare-earth metals Tb, Dy, Er, and Lu are similar. This is also confirmed experimentally: The coefficient γ characterizing the electronic heat capacity is practically the same for all these metals, and is equal to ≈ 10 mJ deg^{-1} mol^{-1} [85]. Thus it may be concluded that the contributions $\varrho_{ee}(T)$ should not be very different for the heavy rare-earth metals.

In Fig. 14, curve 2 corresponds to the expression

$$\varrho_i(T) - \varrho_{ee}(T) = \varrho_{ep}(T) + \varrho_{em}(T)\,, \tag{3.24}$$

i.e. contains contributions from only two scattering mechanisms: phonon and magnon scattering.

The phonon part $\varrho_{ep}(T)$ may be separated with the aid of the Bloch-Grüneisen function. For this purpose it is sufficient to know the Debye temperature and the value of the phonon resistance at this temperature, $\varrho_{ep}(\theta_D)$. The possibility of using the Bloch-Grüneisen law to describe the temperature dependence of the electrical resistance of heavy rare-earth metals was also checked for lutetium [86].

For magnetically ordered rare-earth metals θ_D can be determined from measurements of the elastic constants, since an evaluation of θ_D from the heat capacity or the electrical resistance of magnetically ordered rare-earth metals is not possible due to the uncertainty of the magnetic contributions $C_m(T)$ and $\varrho_{em}(T)$, and the neglect of these contributions will lead to an underestimated value of θ_D. In determining $\varrho_{ep}(\theta_D)$ corrections should be made for the mass number M and the Debye temperature of the given metal. (For $T > \theta_D$, $\varrho \sim r\; T/(M\; \theta_D^2)$, where r is a characteristic constant proportional to the radius of the Debye sphere.[10]))

By such a treatment of the experimental data it was found that in the paramagnetic region ϱ_{em} does not depend on temperature and coincides with the value obtained by linear extrapolation of curve 2 to 0 °K. This coincidence confirms the correctness of the applied method of separating the magnetic part of the resistance of rare-earth metals.[11])

Taking into account the energy gap in the spin-wave spectra of Tb and Dy, good agreement is obtained between the calculated curves and experiment in the low-temperature region. The electrical resistance [86, 87, 93, 94], spontaneous magnetization [80], and the magnetic part of the heat capacity [88, 89, 94] of these metals vary with temperature according to Table 4 with $\Delta \approx 20$ °K for Tb and $\Delta \approx 30$ °K for Dy, which corresponds to the known magnetic anisotropy of these metals. Thus, the experimental results confirm the presence of energy gaps in the spin-wave spectra of these metals.

The magnetic structures of holmium and erbium in the low-temperature region are ferromagnetic spirals for which the spin-wave dispersion law is linear. Therefore, it may be expected that for the electrical resistance of these metals the relation (3.21) will be fulfilled. The behaviour of the electrical resistance of

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[10]) For heavy rare-earth metals the radii of the Debye spheres are similar, and the correction for the coefficient r practically does not change the final result.

[11]) An exceptional behaviour is shown by gadolinium in which ϱ_{em} depends on temperature at $T > \theta_C$.

holmium and erbium at low temperatures shows that the energy gap in the spin-wave spectrum of these metals is comparatively small. In the temperature region 4.2 °K $< T <$ 20 °K $\varrho_{\mathrm{em}}(T)$ of holmium and erbium varies as T^4. The magnetic part of the resistance of thulium exhibits a similar variation.

Thus, from the temperature dependence of the electrical resistance, $\varrho_{\mathrm{em}}(T)$, as well as of the spontaneous magnetization, $\sigma_s(T)$, and heat capacity, $C_{\mathrm{m}}(T)$, some conclusions can be drawn regarding the energy spectrum of spin waves. It is also interesting to study the anisotropy of the energy gap for different directions in Tb and Dy single crystals and to elucidate the question of its origin.

At present it would be of great interest to study the temperature dependence of the electrical resistance of polycrystals with the aim of explaining the law $\varrho_{\mathrm{em}}(T)$, as well as to investigate the anisotropy of $\varrho_{\mathrm{em}}(T)$ in single crystals of rare-earth metals (explanation of the main part of the anisotropy of $\varrho_{\mathrm{em}}(T)$ not only at low temperatures but in a wide temperature range including temperatures above the Curie point). This question is discussed in Section 3.9.

3.9 Anisotropy of the scattering by spin waves in single crystals of rare-earth metals

In a number of papers the temperature dependence of the electrical resistance of rare-earth single crystals was investigated for different crystallographic directions [90]. However, the magnetic part of the resistance, $\varrho_{\mathrm{em}}(T)$, was separated only for Tb, Dy, and Er [91, 92]. The method of separation of $\varrho_{\mathrm{em}}(T)$ proposed in [91] allows to determine the type of the temperature dependence as well as its value, without postulating the form of $\varrho_{\mathrm{em}}(T)$. For separating the contributions of different scattering mechanisms the assumption (3.23) was used. In Fig. 15, curves 1 and 2 give $\varrho_{\mathrm{i}}(T) = \varrho(T) - \varrho_0$ for Tb, Dy, and Er in the main crystallographic directions [0001] (c-axis) and $[1\bar{2}10]$ (a-axis) in the range 4.2 to 370 °K. Curves 3 in these figures represent the electrical resistance of polycrystalline samples, and curves 4 are similar to curves 2 in Fig. 14.

In order to find $\varrho_{\mathrm{em}}(T)$ for different crystallographic directions it is necessary to know the value $(\varrho_{\mathrm{ee}} + \varrho_{\mathrm{ep}})$ for these directions at each temperature. In the case of Lu it is simple to determine these values as well as the coefficients A_1 and A_2 connecting $(\varrho_{\mathrm{ee}} + \varrho_{\mathrm{ep}})$ of polycrystals with the resistance of single crystals:

$$(\varrho_{\mathrm{ee}} + \varrho_{\mathrm{ep}})^{c}_{\mathrm{scr}} = A_1\,(\varrho_{\mathrm{ee}} + \varrho_{\mathrm{ep}})_{\mathrm{pcr}}\,, \tag{3.25}$$

$$(\varrho_{\mathrm{ee}} + \varrho_{\mathrm{ep}})^{a}_{\mathrm{scr}} = A_2\,(\varrho_{\mathrm{ee}} + \varrho_{\mathrm{ep}})_{\mathrm{pcr}}\,. \tag{3.26}$$

The coefficients A_1 and A_2 for Lu practically do not depend on temperature. Because of the similarity of the crystal structure and the Fermi surfaces of the investigated metals, we may assume that the relations (3.25) and (3.26) are valid for other heavy rare-earth metals as well. Subtracting the components (3.25) and (3.26) from $\varrho_{\mathrm{i}}(T)$ for magnetically ordered rare-earth metals we obtain the desired values of magnetic resistance. In Fig. 15 curves 6 and 7 represent the magnetic part of the resistance along the a- and c-axis, respectively.

The magnetic part of the electrical resistance of rare-earth metals is strongly anisotropic in the whole investigated region of temperatures: In the ferro- and antiferromagnetic regions $\varrho^{c}_{\mathrm{em}}(T)$ is greater than $\varrho^{a}_{\mathrm{em}}(T)$. At low temperatures from 20 to 60 °K the dependence $\varrho_{\mathrm{em}}(T)$ for Tb and Dy is close to quadratic. Below 20 °K, for describing $\varrho_{\mathrm{em}}(T)$ it is necessary to introduce an exponential

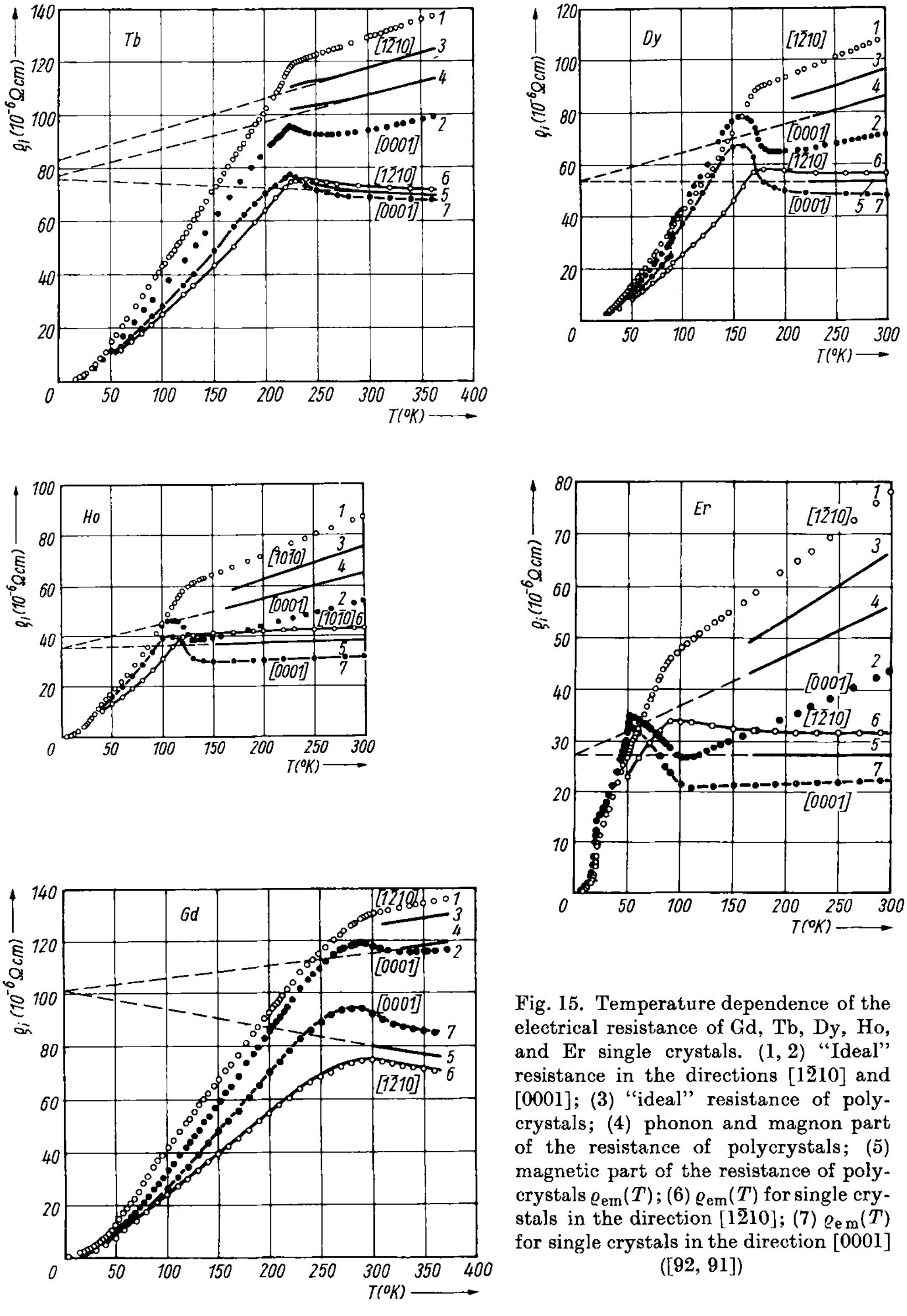

Fig. 15. Temperature dependence of the electrical resistance of Gd, Tb, Dy, Ho, and Er single crystals. (1, 2) "Ideal" resistance in the directions [1$\bar{2}$10] and [0001]; (3) "ideal" resistance of polycrystals; (4) phonon and magnon part of the resistance of polycrystals; (5) magnetic part of the resistance of polycrystals $\varrho_{\mathrm{em}}(T)$; (6) $\varrho_{\mathrm{em}}(T)$ for single crystals in the direction [1$\bar{2}$10]; (7) $\varrho_{\mathrm{em}}(T)$ for single crystals in the direction [0001] ([92, 91])

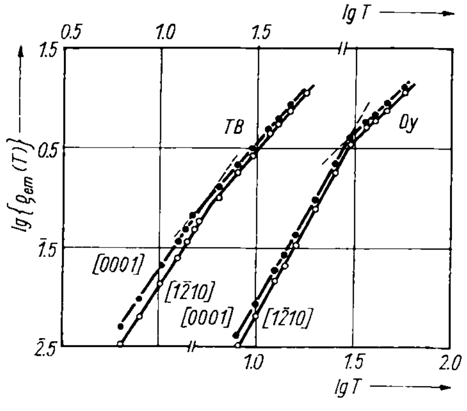

Fig. 16

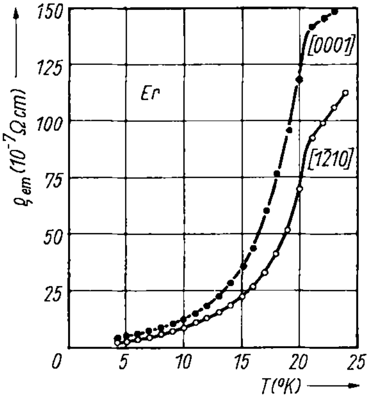

Fig. 17

Fig. 16. Dependence of lg $[\varrho_{\mathrm{em}}(T)]$ on lg T for Tb and Dy single crystals

Fig. 17. Temperature dependence of the magnetic part of the resistance for an Er single crystal in the ferromagnetically ordered state [92]

factor, which agrees with the measurements of $\varrho_{\mathrm{em}}(T)$ on polycrystals (see Section 3.8).

The temperature corresponding to the value of the energy gap in the spin-wave spectrum is observed directly in the dependence $\varrho_{\mathrm{em}}(T)$ plotted on a logarithmic scale (see Fig. 16). From measurements of the electrical resistance of Tb single crystals along the c-axis the values $\Delta \approx 21.5$ °K (for a non-annealed sample) and 17 °K (for an annealed sample) were determined [94]. The differences in the values of Δ found before and after annealing can be associated with different values of the magnetic elastic energy in these two states (see Section 3.7). For Dy the value of Δ is between 29 and 31 °K (the measurements were carried out on several samples).

$\varrho_{\mathrm{em}}(T)$ curves of an Er single crystal in the low-temperature range are shown in Fig. 17. In the region of ferromagnetic ordering (4.2 °K $< T <$ 20 °K) it is found that

$$\varrho^{a}_{\mathrm{em}}(T) \sim T^{3.7\pm0.3} \quad \text{and} \quad \varrho^{c}_{\mathrm{em}}(T) \sim T^{4.0\pm0.2}\,.$$

The obtained result agrees well with the calculations for ferromagnetics having a linear spin-wave dispersion law, with the theoretical investigations of the electrical resistance of polycrystals (see Section 3.8), and with measurements of magnetization, $\sigma_{\mathrm{s}}(T)$ [96]. In the paramagnetic region the magnetic resistance does not depend on temperature, but it is anisotropic, and the character of anisotropy is different from that in the magnetically ordered state: $\varrho^{c}_{\mathrm{em}} < \varrho^{a}_{\mathrm{em}}$. The anisotropy of the scattering by magnetic disorder may be understood from the following considerations: In the general case the electrical conductivity σ_{ij} depends on the processes of scattering (through the relaxation time τ) and on the characteristics of the electronic structure:

$$\sigma_{ij} = \frac{e^2\,\tau}{4\,\pi^3\,\hbar} \int v_i \,\mathrm{d}S_j$$

(v_i electronic velocity in the direction i, $\mathrm{d}S_j$ surface element on the Fermi surface). If it is assumed that the relaxation times of electrons for scattering by magnons, τ_{m}, and by phonons, τ_{p}, are isotropic, we have, as shown in [95], in the paramagnetic region

$$\frac{\varrho_{\text{em}}^{a}}{\varrho_{\text{em}}^{c}} = \frac{\alpha_a}{\alpha_c} = \frac{\int v_c \, \mathrm{d}S_c}{\int v_a \, \mathrm{d}S_a} > 1 \,,$$

where α_a and α_c are the coefficients in the electron–phonon term of resistance. For Tb, Dy, Ho, and Er these values are given in Table 5.

Table 5

rare-earth metal	$\varrho_{\text{em}}^{a}/\varrho_{\text{em}}^{c}$	α_a/α_c
Tb	1.1	1.5
Dy	1.2	2.0
Ho	1.3	1.6
Er	1.4	1.5

Thus, for magnetically ordered rare-earth metals the anisotropy of ϱ_{em} in the paramagnetic region is due not only to the anisotropy of the Fermi surfaces, but probably also to the anisotropy of the relaxation times.

We should note that an analogous separation of $\varrho_{\text{em}}(T)$ performed for Gd single crystals (see Fig. 15) without considering the differences in the Fermi surfaces of Gd and Lu leads to results which are basically different from those obtained for other rare-earth metals. This indicates that the present method of determining $\varrho_{\text{em}}(T)$, as should be expected, is not applicable to Gd due to considerable differences in the electronic structures of Gd and Lu [84].

4. Conclusions

Thus, the available experimental data on the temperature dependence of the electrical resistance of transition metals allows one to conclude that the temperature-dependent part of the resistance of non-ferromagnetic metals at sufficiently low temperatures ($T \lesseqgtr 10$ °K) is mainly due to the scattering of conduction electrons by interelectronic collisions. The contribution of electron–phonon scattering to the electrical resistance of transition metals at $T \lesseqgtr 4.2$ °K is by one order smaller than ϱ_{ee}. It should be noted that in the available theoretical studies for transition metals, which qualitatively correctly describe this scattering mechanism, considerable simplifications are made. For a correct description of the electrical resistance due to interelectronic collisions, it is necessary to take into account the peculiarities of the electronic structure and the topology of the Fermi surface of the given metal as well as the possibility of electron–electron umklapp processes.

In the electrical resistance of magnetically ordered transition metals at low temperatures an additional contribution is introduced by the scattering of conduction electrons by spin waves. The value of this contribution is extraordinarily great in the case of magnetically ordered rare-earth metals, exceeding the value of electron–electron resistance by several times and that of electron–phonon resistance by more than one order. The character of the temperature dependence $\varrho_{\text{em}}(T)$ is determined by the peculiarities of the energy spectrum of spin waves. Therefore experimental investigations of the temperature dependence of the contribution due to spin-wave scattering enables one to get information on the type of the spin-wave dispersion law, and to estimate the value of the energy gap.

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