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The One-Dimensional Classical Electron Gas

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Equilibrium states of the one-dimensional classical electron gas are considered in detail. It is proved that this system is in a crystalline state, at all temperatures and densities.

INTRODUCTION

The electron gas which consists of charged particles moving in an uniform neutralizing background has been the subject of many investigations with different motivations depending on the suspected domain of applications: equilibrium properties of strong electrolyte solutions, physics of high temperature plasma, electron correlation in solids [1]. We would like to recall however, that it is not yet known whether such a system has a correct thermodynamic behavior [2]. In fact approximate treatments of the classical version of the model, as well as numerical studies, indicate an instability at moderately low temperatures or high densities [3].

This paper is concerned with the equilibrium properties of this model in one dimension, with neglect of quantum effects. Under such circumstances the system has indeed a normal thermodynamic behavior and some of his thermodynamic properties have been obtained ten years ago by Baxter [4]. He was however especially interested in the possible differences with a system of equal number of positive and negative charged particles, analyzed by Lenard and Prager [5]. He found that although the thermodynamics of the two systems does differ, the discrepancy is small. Both systems are correctly described by Debye–Hückel theory in the small coupling limit. The main result of this paper is that, despite the thermodynamic validity of Debye–Hückel theory, our system is never in a "plasma" state, but in a crystalline one. More precisely, we show that if we choose appropriate boundary conditions, the correlation functions obtained by going to the infinite volume limit are periodic, of periodic ρ^{-1}, and this for all densities ρ and temperatures. Thus we see that this model provides us an example of a classical Wigner lattice [6]. It is amusing to note that in our case as in the usual quantum Wigner lattice, the particles are well localized at low densities. But this is a mere consequence of the dimensionality.

A starting point in Debye–Hückel theory (and its improvements) is that the equilibrium state is translation invariant. Since it is not so in one dimension and since dimensionality does not play any special role in this theory, it appears to us that some additional argument (unknown to us) is required to justify this theory. This point seems to deserve some further study.

Another interesting feature of our result is that we have an example of a "bona fide" crystal, obtained without the usual assumptions of lattice dynamics (which still wait a justification from "first principles"). A traditional lattice dynamical treatment of the model, describes it as a set of independent harmonic oscillators, each having the plasma frequency. A comparison with the exact result, shows that this approximation is valid only in the very strong coupling limit. A closer look shows that this discrepancy comes from the neglect in lattice dynamics of what could be called kinematical restrictions by analogy with spin-wave theory: in order to develop the potential in powers of the relative displacements of the nuclei, one should satisfy the conditions

$$(\mathbf{u}_i - \mathbf{u}_j)^2 + 2(\mathbf{u}_i - \mathbf{u}_j)(\mathbf{r}_i - \mathbf{r}_j) \geqslant -(\mathbf{r}_i - \mathbf{r}_j)^2,$$

$\mathbf{u}_i$ being the displacement of the ith nucleus from its equilibrium position $\mathbf{r}_i$.

This suggests that such restrictions might give contributions competing with the usual anharmonic ones, even in three dimensions, although their effect is certainly more pronounced in one dimension.

We also think that an extension of these results to the quantum case (always in one dimension) might be rewarding for the following reasons: whereas a crystalline phase will certainly survive in the strong coupling regime, quantum effects become important in the weak coupling limit. But in this regime classically the particles already tend to be delocalized, so that it is not excluded that the uncertainty and the exclusion principle will keep the particles sufficiently far apart that they go into a gas phase. But evidently the presence or absence of such a transition remains to be proved.

The content of this paper is the following. We put two kinds of boundary conditions on our system: the so-called free or rigid walls boundary conditions (f.b.c) in which the particles move in a given box and the periodic ones (p.b.c.) in which one repeats indefinitely the box and takes into account the interaction of the particles between themselves in the box and with their images in the other baxes. (Periodic boundary conditions are usually considered in many-body theory because of their computational convenience, whereas f.b.c. appear traditionally in studies of fluids and gases. In each separate case, we study successively the thermodynamics and the correlation functions (in particular their cluster properties).

The thermodynamics is the same for both boundary conditions. The correlation functions are periodic if we use f.b.c. and have the product property. But if we use p.b.c. they are translation invariant and do not satisfy the product property. This is so because p.b.c. correlation functions decompose naturally into f.b.c. correlation functions.

An appendix gives the proofs of some mathematical results used all along the text.

A. Free Boundary Conditions

1. *Thermodynamics*

Our system consists of particles of charge $-\sigma$, enclosed in a box $\Lambda = [0, L]$ and imbedded in an homogeneous background of charge $+\sigma$ and density ρ. The interaction potential between these particles will be the one-dimensional analog of the Coulomb one, namely $-2\pi \mid x \mid$.[1] Therefore, the Hamiltonian of our system will be defined by

$$H = -2\pi\sigma^2 \sum_{1\leqslant i<j\leqslant N} \mid x_i - x_j \mid + 2\pi\sigma^2 \sum_{i=1}^{N} \int_0^L \rho \mid x - x_i \mid dx - \frac{2\pi\sigma^2}{2} \int_0^L \int_0^L \rho^2 \mid x - y \mid dx\, dy. \tag{1}$$

As usual in one dimension, it is convenient to order the particles

$$0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant L, \tag{2}$$

so that the energy of such an ordered configuration reads:

$$E = -2\pi\sigma^2 \sum_{i=1}^{N} (2i - 1 - N)\, x_i + 2\pi\sigma^2\rho \sum_{i=1}^{N} x_i^2 - 2\pi\sigma^2\rho L \sum_{i=1}^{N} x_i + 2\pi\sigma^2\rho \frac{NL^2}{2} - 2\pi\sigma^2\rho^2 \frac{L^3}{6}. \tag{3}$$

Since our particles interact via a long range potential, it can be of interest to study the influence of boundary contributions to the energy on the equilibrium properties of the system. We will therefore impose only asymptotic charge neutrality; that is,

$$\rho = (N + s)/L \qquad \text{where} \quad \lim_{N\to\infty} (s/N) = 0. \tag{4}$$

[1] The Coulomb potential in ν dimension is defined as the solution of the corresponding Poisson's equation: $\Delta\Phi(x) = -4\pi\delta(x)$, $x \in R^\nu$.

Under these conditions, the energy of an ordered configuration is given by

$$E = 2\pi\sigma^2\rho \sum_{i=1}^{N} \left[x_i - \rho^{-1}\left(i - \frac{N+1-L\rho}{2}\right)\right]^2$$

$$+ 2\pi\sigma^2\rho \left[\frac{1}{2} NL^2 - \frac{1}{6}\rho L^3 - \rho^{-2} \sum_{i=1}^{N} \left(i - \frac{N+1-L\rho}{2}\right)^2\right]$$

$$= 2\pi\sigma^2\rho \sum_{i=1}^{N} \left[x_i - \rho^{-1}\left(i + \frac{s-1}{2}\right)\right]^2 + \frac{2\pi\sigma^2}{12}\rho^{-1}(1-3s^2)N + O(1). \tag{5}$$

We can now write the partition function in the following form.

$$Q_\Lambda{}^N = \int_{0\leqslant x_1\leqslant\cdots\leqslant x_N\leqslant L} dx_1 \cdots dx_N$$

$$\times \exp\left\{-\pi\lambda^2 \sum_{i=1}^{N} \left[\rho x_i - \left(i + \frac{s-1}{2}\right)\right]^2 - \frac{\pi\lambda^2}{12}(1-3s^2)N + O(1)\right\} \tag{6}$$

where $\lambda^2 = 2\beta\sigma^2/\rho$ (7) is a dimensionless coupling constant, which measures the ratio of the average potential energy to the average kinetic energy.

Let us make the change of variables

$$y_i = \rho x_i - (i + (s-1)/2), \qquad i = 1,\ldots, N.$$

We get (neglecting terms of order 1)

$$Q_\Lambda{}^N = \rho^{-N} e^{-(\pi\lambda^2/12)(1-3s^2)N} \int_{-\infty}^{+\infty} dy_1 \cdots dy_N\, e^{-\pi\lambda^2\Sigma_{i=1}^N y_i{}^2} K\left(\frac{1-s}{2}, y_1\right)$$

$$\times \prod_{i=1}^{N-1} K(y_i, y_{i+1})\, g_{(s+1)/2}(y_N) \tag{8}$$

where

$$K(x, y) = \begin{cases} 1, & y \geqslant x - 1, \\ 0, & y < x - 1, \end{cases} \tag{9}$$

and

$$g_a(x) = \begin{cases} 1 & x \leqslant a, \\ 0, & x > a. \end{cases} \tag{10}$$

We see that if we introduce the operator $\mathbb{K}$ defined by

$$(\mathbb{K}f)(x) = \int_{-\infty}^{+\infty} dy\, e^{-\pi\lambda^2 y^2} K(x, y) f(y), \tag{11}$$

we can write the partition function in the simple form

$$Q_\Lambda{}^N = \rho^{-N} e^{-(\pi\lambda^2/12)(1-3s^2)N} (\mathbb{K}^N g_{(s+1)/2})((1-s)/2). \tag{12}$$

In the appendix (Lemma 1(b), Eqs. (A.6)–(A.7)) we prove the following. If $f \in \mathscr{L}^2(R, \sigma)$, where $\mathscr{L}^2(R, \sigma)$ denotes the Hilbert space of square integrable functions with respect to the measure

$$d\sigma(x) = e^{-\pi\lambda^2 x^2}\, dx, \tag{13}$$

then

$$\lim_{N\to\infty} z_0^{-N} (\mathbb{K}^N f)(x) = \nu(f)\, \Psi_0(x) \tag{14}$$

uniformly in x where

$$\nu(f) = \frac{\int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(-x)\, f(x)}{\int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(-x)\, \Psi_0(x)}. \tag{15}$$

Here z_0 denotes the largest eigenvalue, and $\Psi_0(x)$ the corresponding eigenvector in $\mathscr{L}^2(R, \sigma)$ of the integral equation

$$z\Psi(x) = \int_{x-1}^{+\infty} d\sigma(y)\, \Psi(y). \tag{16}$$

Moreover $z_0 > 0$ and $\Psi_0(x) > 0$ when x is finite.

We can now casily compute the free energy density f, and we get

$$-\beta f(\rho, \beta) = -\beta f_0(\rho, \beta) - (\pi\lambda^2/12)(1 - 3s^2) + \ln(z_0(\lambda)/e) \tag{17}$$

where $f_0(\rho, \beta)$ is the free energy density of the free gas.

Before analyzing this result, we want to discuss the following technical point: why do we have taken the thermodynamic limit, by keeping the background density fixed, instead of the particles one N/L as usual. Mainly for reasons of notational simplicity. In any case, one can easily see that the limiting free energy is the same.

The first thing we note is that the charge defect $s\sigma$ gives a contribution even in the thermodynamic limit to the free energy: $\pi(s^2\sigma^2/2)\, L$. This is nothing else but the electrostatic field energy within the boundaries of the system due to the presence of two opposite charges $s\sigma/2$ fixed at the surface of the system (here $x = 0$ and $x = L$). This phenomenon is characteristic of Coulomb interactions and has been proved to occur in general for a three-dimensional quantum system of discrete charges (at least one of them consisting of fermions) [7].

Second, if we compute the pressure p we find

$$p = kT\rho(1 + \lambda^2(\partial/\partial\lambda^2)\ln z_0(\lambda)) - (\pi\sigma^2/6)(1 - 3s^2). \tag{18}$$

It is then possible to prove that p is an increasing function of ρ as it should be, and that

$$-(\pi\sigma^2/6)(1 - 3s^2) + \tfrac{1}{2}kT\rho \leqslant p \leqslant -(\pi\sigma^2/6)(1 - 3s^2) + kT\rho. \tag{19}$$

This means that

$$\lim_{T\to 0} p(T, \rho) = \lim_{\rho\to 0} p(T, \rho) = -(\pi\sigma^2/6)(1 - 3s^2), \tag{20}$$

and this shows that the pressure will be negative, if the density or the temperature is low enough, when $|s| < 3^{-1/2}$.

Finally and more importantly, does this system exhibit a phase transition? The answer is *no*, because $z_0(\lambda)$ and hence the free energy is an analytic function of λ on $(0, +\infty)$. This is proven in the appendix (lemma 1(a)).

2. *One-Point Correlation Function*

This quantity, which represents the local density is the one which will allow us to understand the nature of the phase in which the system exists.

In the canonical ensemble, it is defined by:

$$\rho_\Lambda{}^N(x) = \sum_{i=1}^{N} \langle \delta(x - x_i)\rangle_{N,\Lambda} \tag{21}$$

or

$$\rho_\Lambda{}^N(x) = \frac{\left\{\begin{array}{l}\int_{0\leqslant x_1\leqslant\cdots\leqslant x_N\leqslant L} dx_1 \cdots dx_N \\ \quad \times \sum_{i=1}^N \delta(x - x_i) \exp\{-\pi\lambda \sum_{i=1}^N [\rho x_i - (i + (s-1)/2)]^2\}\end{array}\right\}}{\int_{0\leqslant x_1\leqslant\cdots\leqslant x_N\leqslant L} dx_1 \cdots dx_N \exp\{-\pi\lambda^2 \sum_{i=1}^N [\rho x_i - (i + (s-1)/2)]^2\}}. \tag{22}$$

Let us first consider the numerator. It is equal to

$$\sum_{j=1}^{N} \int_{0\leqslant x_1\leqslant\cdots\leqslant x_{j-1}\leqslant x\leqslant x_{j+1}\leqslant\cdots\leqslant x_N\leqslant L} dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_N$$
$$\times\, e^{-\pi\lambda^2(\rho x_j - j - (s-1)/2)^2} e^{-\pi\lambda^2 \sum_{i\neq j}[\rho x_i - (i+(s-1)/2)]^2}. \tag{23}$$

If we make the change of variables

$$y_k = \rho x_{j+k} - (j + k) - (s - 1)/2, \qquad k = 1,\ldots, N - j,$$

then, if $j \leqslant N-1$,

$$\int_{x \leqslant x_{j+1} \leqslant \cdots \leqslant x_N \leqslant (N+s)\rho^{-1}} dx_{j+1} \cdots dx_N \, e^{-\pi\lambda^2 \sum_{i=j+1}^{N} (\rho x_i - i - (s-1)/2)^2}$$

$$= \rho^{-(N-j)} \int d\sigma(y_1) \cdots d\sigma(y_{N-j}),$$

$$K\left(\rho x - j + \frac{1-s}{2}, y_1\right) \prod_{l=1}^{N-j-1} K(y_l, y_{l+1}) \, g_{(s+1)/2}(y_{N-j})$$

$$= \rho^{-(N-j)} (\mathbb{K}^{N-j} g_{(s+1)/2}) \left(\rho x - j + \frac{1-s}{2}\right) \tag{24}$$

On the other hand, if $j \geqslant 2$,

$$\int dx_1 \cdots dx_{j-1} \, e^{-\pi\lambda^2 \sum_{i=1}^{j-1} (\rho x_i - i - (s-1)/2)^2} = \rho^{-(j-1)} (\mathbb{K}^{j-1} g_{\rho x - j + (3-s)/2}) \left(\frac{1-s}{2}\right). \tag{25}$$

It is now possible to express the numerator as follows:

$$\rho^{-(N-1)} \sum_{j=2}^{N-1} e^{-\pi\lambda^2 (\rho x - j - (s-1)/2)^2}$$

$$\times \, (\mathbb{K}^{j-1} g_{\rho x - j + (3-s)/2}) \left(\frac{1-s}{2}\right) (\mathbb{K}^{N-j} g_{(s+1)/2}) \left(\rho x - j + \frac{1-s}{2}\right)$$

$$+ \, \rho^{-(N-1)} e^{-\pi\lambda^2 (\rho x - (s+1)/2)^2} (\mathbb{K}^{N-1} g_{(s+1)/2}) \left(\rho x - \frac{s+1}{2}\right)$$

$$+ \, \rho^{-(N-1)} e^{-\pi\lambda^2 (\rho x - (N+s-1)/2)^2} (\mathbb{K}^{N-1} g_{\rho x - N + (3-s)/2}) \left(\frac{1-s}{2}\right). \tag{26}$$

At this point, we need to define precisely the limiting density. In order to avoid surface effects, we will require that $L \to \infty$ and $x \to \infty$ in such a way that

$$L - x \to \infty. \tag{27}$$

A natural way to realize these conditions is to take

$$x = y + \alpha L, \qquad 0 < \alpha < 1, \tag{28}$$

and to keep y fixed, whereas we go to the limit $L \to \infty$. Hence, we are led to study $\lim_{N\to\infty} \rho_N(y)$ where

$$\rho_N(y) = \rho \left[(\mathbb{K}^N g_{(s+1)/2}) \left(\frac{1-s}{2}\right)\right]^{-1} \left\{ \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y + r + N\alpha - j)^2} \right.$$
$$\times (\mathbb{K}^{j-1} g_{\rho y + r + N\alpha - j + 1}) \left(\frac{1-s}{2}\right) (\mathbb{K}^{N-j} g_{(s+1)/2})(\rho y + r + N\alpha - j)$$
$$+ e^{-\pi\lambda^2(\rho y + r + N\alpha - 1)^2} (\mathbb{K}^{N-1} g_{(s+1)/2})(\rho y + r + N\alpha - 1)$$
$$\left. + e^{-\pi\lambda^2(\rho y + r + N\alpha - N)^2} (\mathbb{K}^{N-1} g_{\rho y + r + N\alpha - N + 1}) \left(\frac{1-s}{2}\right) \right\} \tag{31}$$

where

$$r = s(\alpha - \tfrac{1}{2}) + \tfrac{1}{2}. \tag{32}$$

In order to compute this limit we will use the following basic estimates, proved in the appendix (Lemma 1(b), Eqs. (A.8), (A.9), and (A.10)). If $f \in \mathscr{L}^2(R, \sigma)$ then, when $N \geqslant 1$,

$$| z_0^{-N} (\mathbb{K}^N f)(x) - \nu(f)\, \Psi_0(x) | < \bar{m}\, \| f \|\, \epsilon_{N-1}, \tag{33}$$

where

$$\| f \|^2 = \int_{-\infty}^{+\infty} d\sigma(x)\, | f(x) |^2, \tag{34}$$

and the ϵ_N satisfy

$$0 < \epsilon_{N+M} \leqslant \epsilon_N \epsilon_M < \sum_{N=0}^{\infty} \epsilon_N < \infty. \tag{35}$$

In our case

$$g_a(x) \in \mathscr{L}^2(R, \sigma) \qquad \text{and} \qquad \| g_a \|^2 < \lambda^{-1} \tag{36}$$

$$\nu(g_{a+1}) = \frac{z_0 \Psi_0(-a)}{\int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(x)\, \Psi_0(-x)}. \tag{37}$$

We see that $\nu(g_{a+1})$ is uniformly bounded in a, since $\psi_0(x) \leqslant 1$ as proved in the appendix (Lemma 2.2, see Eq. (A.12)). We will call

$$g = \sup_{a\in R} \nu(g_a).$$

Let us first handle the last two terms in the expression for $\rho_N(y)$. They tend to zero because

$$z_0^{-N}(\mathbb{K}^N g_{(s+1)/2})\left(\frac{1-s}{2}\right) \to \nu(g_{(s+1)/2})\, \Psi_0\left(\frac{1-s}{2}\right), \qquad \text{(see (44))}$$

$$z_0^{-(N-1)}(\mathbb{K}^{N-1} g_{(s+1)/2})(\rho y + r + N\alpha - 1) \to \nu(g_{(s+1)/2}) \cdot 0, \qquad \text{(see (33))}$$

$$z_0^{-(N-1)}(\mathbb{K}^{N-1} g_{\rho y+r+N\alpha-N+1})\left(\frac{1-s}{2}\right) \to 0 \cdot \Psi_0\left(\frac{1-s}{2}\right), \qquad \text{(see (33))}$$

and

$$e^{-\pi\lambda^2(\rho y+aN+b)^2} \to 0,$$

when y, a, b are kept fixed.

Retaining from now on only the first sum in (31) we get the following inequality, using (33)

$$\Big| z_0^{-N}(\mathbb{K}^N g_{(s+1)/2})\left(\frac{1-s}{2}\right) \rho_N(y) - z_0^{-1}\rho \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2}$$
$$\times\, \nu(g_{\rho y+r+N\alpha-j+1})\, \Psi_0\left(\frac{1-s}{2}\right) \nu(g_{(s+1)/2})\, \Psi_0(\rho y + r + N\alpha - j)\Big|$$
$$< z_0^{-1}\rho \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2} \nu(g_{\rho y+r+N\alpha-j+1})\, \Psi_0\left(\frac{1-s}{2}\right) \bar{m} \,\| g_{(s+1)/2} \|\, \epsilon_{N-1-j}$$
$$+ z_0^{-1}\rho \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2} \bar{m} \,\| g_{\rho y+r+N\alpha-j+1} \|\, \epsilon_{j-2}\nu(g_{(s+1)/2})$$
$$\times\, \Psi_0(\rho y + r + N\alpha - j) + z_0^{-1}\rho \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2} \bar{m}^2$$
$$\times\, \| g_{\rho y-j+N\alpha+r+1} \| \,\| g_{(s+1)/2} \|\, \epsilon_{j-2}\epsilon_{N-1-j}\,. \qquad (38)$$

Each of the sums appearing at the right-hand side of this inequality tends to zero. Let us show it for the first one, since the proof for the two others is similar. We will use (33), (35), and (36).

$$\sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2} \nu(g_{\rho y+r+N\alpha-j+1})\, \epsilon_{N-1-j}$$
$$\leqslant g \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2} \epsilon_{N-1-j}$$
$$\leqslant g \sum_{j=2}^{[N\alpha^{1/2}]} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2} \epsilon_{N-1-j} + g \sum_{j=[N\alpha^{1/2}]+1}^{N-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2} \epsilon_{N-1-j}$$

$$\leqslant g\epsilon_{N-[N\alpha^{1/2}]-1} \sum_{j=2}^{[N\alpha^{1/2}]} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2} \epsilon_{[N\alpha^{1/2}]-j}$$

$$+ g\left(\sum_{k=0}^{\infty} \epsilon_k\right) \sum_{j=[N\alpha^{1/2}]+1}^{N-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2}$$

$$\leqslant g\left(\sum_{k=0}^{\infty} \epsilon_k\right) \epsilon_{N-[N\alpha^{1/2}]-1} \sum_{j=-\infty}^{+\infty} e^{-\pi\lambda^2 j^2}$$

$$+ g\left(\sum_{k=0}^{\infty} \epsilon_k\right) \sum_{l=-[N(\alpha^{1/2}-\alpha)]-1}^{-[N(1-\alpha)]-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-[N\alpha]+l)^2}$$

$$\leqslant g\left(\sum_{k=0}^{\infty} \epsilon_k\right) \epsilon_{N-[N\alpha^{1/2}]-1} \sum_{j=-\infty}^{+\infty} e^{-\pi\lambda^2 j^2}$$

$$+ g\left(\sum_{k=0}^{\infty} \epsilon_k\right) \sum_{l=-[N(\alpha^{1/2}-\alpha)]-1}^{-[N(1-\alpha)]-1} e^{-2\pi\lambda^2(\rho y+r+1)l} e^{-\pi\lambda^2 l^2}$$

and we note that the last term of this inequality tends to zero when y belongs to a compact subset of R.

Hence we have

$$\lim_{N\to\infty} \rho_N(y)$$

$$= \lim_{N\to\infty} z_0^{-1}\rho \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+r+N\alpha-j)^2} \nu(g_{\rho y+r+N\alpha-j+1})\, \Psi_0(\rho y + r + N\alpha - j). \quad (39)$$

In general this limit does not exist. We can however always extract a subsequence such that a limit exists. (This result is in agreement with a general statement about the limiting correlation functions of a superstable system [8]). Indeed, if we make the change of variables $[N\alpha] - j = k$ in the sum and take a sequence of N such that $N\alpha - [N\alpha] = m$ with m given, then

$$\lim_{\substack{N\to\infty \\ N\alpha-[N\alpha]=m}} \rho_N(y) = \rho n(\rho y + s(\alpha - 1/2) + m + 1/2) \quad (40)$$

where

$$n(x) = \frac{\sum_{k=-\infty}^{+\infty} e^{-\pi\lambda^2(x+k)^2}\Psi_0(-x-k)\, \Psi_0(x+k)}{\int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(x)\, \Psi_0(-x)}. \quad (41)$$

This is so, because the series (41) converges, since

$$0 \leqslant \Psi_0(x+k) \leqslant 1.$$

The most important conclusion to draw from this result is that, in any case, the functions of y defined by (40) are *periodic*, of period ρ^{-1}, that is the average distance between the particles. However a constant is also a periodic function so that strictly speaking we need to show that $n(x)$ as given by (41) does not define a constant.[2] This will be done later on.

But if it is so, then (40) tells us that the *equilibrium state is not unique*. since we get different states by choosing different subsequences (different m's here), or by changing the origin of the coordinates (i.e., varying α). These states of the infinite system break the translation invariance of the equilibrium states of the finite one. Note however that if $\alpha = \frac{1}{2}$, then

$$\lim_{\substack{N\to\infty \\ N\text{even}}} \rho_N(y) = \rho n(\rho y + 1/2) = \rho n(-\rho y + 1/2), \tag{42}$$

$$\lim_{\substack{N\to\infty \\ N\text{odd}}} \rho_N(y) = \rho n(\rho y) = \rho n(-\rho y), \tag{43}$$

so that the "rotational" invariance ($y \to -y$) is not broken by these symmetric boundary conditions.

But if $\alpha \neq \frac{1}{2}$

$$\lim_{\substack{N\to\infty \\ N\alpha-[\alpha N]=m}} \rho_N(y)$$

is not an even function of y, whatever value s takes, and these states break the "rotational" symmetry. We will see later that it is possible though to construct states which keep the translational and "rotational" invariance even in the thermodynamic limit.

The dependence of the local density $\rho(x)$ upon the subsequence of N chosen, has the following interesting consequence. If instead of the canonical ensemble, we would have used the grand-canonical one, then, owing to the equivalence of ensembles (which could certainly be rigourously proved) the density $\rho_{z,\beta}(x)$ would be given by

$$\rho_{z,\beta}(x) = \rho \frac{1}{q} \sum_{k=0}^{q-1} n\left(\rho x + s\left(\alpha - \frac{1}{2}\right) + \frac{1}{2} + \frac{kp}{q}\right)$$

$$= \rho \frac{1}{q} \sum_{k=0}^{q-1} n\left(\rho x + s\left(\alpha - \frac{1}{2}\right) + \frac{k}{q}\right)$$

when α is rational, $\alpha = p/q$. Here $\rho = \rho(z, \beta)$ is obtained by means of the thermodynamic relation $\rho = z(\partial/\partial z)\, \beta p(z, \beta)$. Thus we see that in this ensemble the density

[2] The reader, who considers this to be fairly obvious should confront his intuition with the following simple example: $1 = \sum_{n=-\infty}^{\infty} f(x+n)$ if $f(x) = \frac{1}{2}\int_{x-1}^{x+1} dy\, e^{-\pi y^2}$.

would have the period $q^{-1}\rho^{-1}$, instead of ρ^{-1}, when α is rational. And when α is irrational the density would probably be an almost periodic function of x.

Let us now look to some analytic properties of $n(x)$, considered as a function of x and λ, which will appear useful later on.

We show in the appendix (Lemma 2.4) that $\Psi_0(x)$ considered as a function of λ can be extended to an analytic function in some neighbourhood D of the positive real axis. Moreover $\Psi_0(x)$ is uniformly bounded in x and λ on all the compact subsets of D. Hence $\sum_{k=-\infty}^{+\infty} e^{-\pi\lambda^2(x+k)^2}\Psi_0(-x-k)\,\Psi_0(x+k)$ converges uniformly on the compact subsets of D, since $\sum_{k=-\infty}^{+\infty} e^{-\pi(\mathrm{Re}\lambda^2)(x+k)^2}$ is uniformly bounded on such subsets. Noting that $\int_{-\infty}^{+\infty} d\sigma(x)\,\Psi_0(x)\,\Psi_0(-x) \neq 0$ and is analytic in some neighborhood of the positive real axis, we conclude that $n(x)$ can be extended to an *analytic function of* λ in some neighborhood of the positive real axis.

We also show in the appendix (Lemma 2.3) that considered as a function of x, $\Psi_0(x)$ can be extended to an entire function. Moreover this function is uniformly bounded on all the strips of the form $a < \mathrm{Im}\, x < b$ of the x-plane. Proceeding as before, we conclude that $n(x)$ can be extended to an *entire function of* x.

Let us show now that $n(x)$ is not constant when λ is large enough.[3] In order to do this, we need to determine the behavior of z_0 and $\Psi_0(x)$ when $\lambda \to \infty$.

The integral equation

$$\lambda z_0 \Psi_0(x) = \int_{x-1}^{+\infty} \lambda\, d\sigma(y)\, \Psi_0(y)$$

becomes, if

$$\lim_{\lambda\to\infty} \lambda z_0 = \mu \qquad \text{and} \qquad \lim_{\lambda\to\infty} \Psi_0(x) = f_0(x)$$

$$\mu f_0(x) = \theta(1-x)\, f_0(0)$$

since

$$\lim_{\lambda\to\infty} \lambda\, d\sigma(y) = \delta(y)\, dy$$

and this equation has the solution $\mu = 1, f_0(x) = \theta(1-x)$. Hence we expect that

$$\lim_{\lambda\to\infty} \lambda z_0(\lambda) = 1, \qquad \lim_{\lambda\to\infty} \Psi_0(x) = \theta(1-x). \tag{44}$$

Suppose that this argument is correct, then since

$$n(0) \geqslant \frac{\lambda \Psi_0^2(0)}{\int_{-\infty}^{+\infty} \lambda\, d\sigma(y)\, \Psi_0(y)\, \Psi_0(-y)}$$

and

$$n(\tfrac{1}{2}) \leqslant \frac{\sum_{k=-\infty}^{+\infty} \lambda e^{-\pi\lambda^2((1/2)+k)^2}}{\int_{-\infty}^{+\infty} \lambda\, d\sigma(y)\, \Psi_0(y)\, \Psi_0(-y)},$$

$$n(0) - n(\tfrac{1}{2}) \geqslant \lambda\Psi_0^2(0) - 2\lambda\left(e^{-\pi\lambda^2/4} + \int_{+1/2}^{+\infty} d\sigma(x)\right), \tag{45}$$

[3] I owe most of this argument to H. J. Brascamp.

and consequently $n(0) - n(\frac{1}{2}) > 0$ if λ is sufficiently large. It remains to prove rigourously (44).

Since

$$\Psi_0(y) \leqslant 1,$$

$$\lambda z_0 = \int_{-\infty}^{+\infty} \lambda \, d\sigma(y) \, \Psi_0(y) \leqslant 1, \tag{46}$$

and since $\Psi_0(x)$ is a positive decreasing function of x (see appendix, Lemma 2.2),

$$\lambda z_0 \Psi_0(\tfrac{1}{2}) \geqslant \int_{-1/2}^{+1/2} \lambda \, d\sigma(y) \, \Psi_0(y) \geqslant \Psi_0(\tfrac{1}{2}) \int_{-1/2}^{+1/2} \lambda \, d\sigma(y)$$

so that

$$\lambda z_0 \geqslant \int_{-1/2}^{+1/2} \lambda \, d\sigma(y) \tag{47}$$

and we see that

$$\lim_{\lambda \to \infty} \lambda z_0 = 1.$$

On the other hand, we prove in the appendix (Lemma 2.2, Eqs. (A.13) and (A.14)) that

$$\begin{aligned} 1 - z_0^{-1} \int_{-\infty}^{x-1} d\sigma(x_1) &\leqslant \Psi_0(x) \\ &\leqslant 1 - z_0^{-1} \int_{-\infty}^{x-1} d\sigma(x_1) + z_0^{-2} \int_{-\infty}^{x-1} d\sigma(x_1) \int_{-\infty}^{x_1-1} d\sigma(x_2) \end{aligned} \tag{48}$$

and since

$$z_0^{-1}\lambda^{-1} \int_{-\infty}^{x-1} \lambda \, d\sigma(x_1) \to \theta(x-1),$$

$$z_0^{-2}\lambda^{-2} \int_{-\infty}^{x-1} \lambda \, d\sigma(x_1) \int_{-\infty}^{x_1-1} d\sigma(x_2) \to 0$$

pointwise, we see that

$$\lim_{\lambda \to \infty} \Psi_0(x) = \theta(1-x).$$

In order to extend the argument to all λ, we use the fact that $n(x)$ is an analytic function of λ on $(0, +\infty)$. This shows indeed that $n(0) - n(\frac{1}{2})$ being positive on a segment $(\lambda_0, +\infty)$ cannot be zero except possibly on a countable number of points of $(0, +\infty)$. If we wanted to strengthen this result, so that such a possibility would be ruled out, we should prove that

$$n(0) \geqslant n(x) \geqslant n(\tfrac{1}{2}) \qquad \text{when} \qquad 0 \leqslant x \leqslant \tfrac{1}{2}$$

and that $n(0) - n(\frac{1}{2})$ is an increasing function of λ, all properties which seem reasonable, but that we were unable to prove.

3. *Pair Correlation Function*

The study of correlation functions between two or more particles will give us more insight into the nature of the equilibrium states, particularly if we look at their clustering properties.

The pair correlation function in a finite system is given by:

$$\rho_\Lambda{}^N(x_1, x_2) = \left\langle \sum_{i \neq j} \delta(x_1 - x_i') \, \delta(x_2 - x_j') \right\rangle_{N,\Lambda}. \tag{49}$$

In one dimension however it is sufficient to know this function when $x_1 < x_2$ and in this case:

$$\begin{aligned} \rho_\Lambda{}^N(x_1, x_2) = \rho^N &\left[(\mathbb{K}^N g_{(s+1)/2}) \left(\frac{1-s}{2} \right) \right]^{-1} \\ &\times \int_{0 \leq x_1' \leq x_2' \leq \cdots \leq (N+s)/\rho} \prod_{l=1}^{N} d\sigma \left(\rho x_l' - l - \frac{s-1}{2} \right) \\ &\times \sum_{1 \leq i < j \leq N} \delta(x_1 - x_i') \, \delta(x_2 - x_j'). \end{aligned} \tag{50}$$

Making the usual change of variables $y_i = \rho x_i' - (i + (s-1)/2)$, we get

$$\begin{aligned} \rho_\Lambda{}^N(x_1, x_2) = \rho^2 &\left[(\mathbb{K}^N g_{(s+1)/2}) \left(\frac{1-s}{2} \right) \right]^{-1} \\ &\times \int \prod_{l=1}^{N} d\sigma(y_l) \, K \left(\frac{1-s}{2}, y_1 \right) \prod_{l=1}^{N-1} K(y_l, y_{l+1}) \, g_{(s+1)/2}(y_N) \\ &\times \sum_{1 \leq i < j \leq N} \delta \left(\rho x_1 - y_i - i - \frac{s-1}{1} \right) \\ &\times \delta \left(\rho x_2 - y_j - j - \frac{s-1}{2} \right). \end{aligned} \tag{51}$$

Calling

$$f_a(y) = K(a, y) \tag{52}$$

and

$$K^*(x, y) = K(y, x) \tag{53}$$

we can write the numerator in the following form:

$$\rho^2 \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} \int \prod_{i=1}^{k} d\sigma(y_i) \prod_{i=1}^{k} K^*(y_{i+1}, y_i) f_{(1-s)/2}(y_1)\, \delta\left(\rho x_1 - k - \frac{s-1}{2} - y_k\right)$$

$$\times \prod_{i=k+1}^{k+m} d\sigma(y_i) \prod_{i=k+1}^{k+m} K(y_i, y_{i+1})\, \delta\left(\rho x_2 - k - m - \frac{s-1}{2} - y_{k+m}\right)$$

$$\times \prod_{i=k+m+1}^{N-1} d\sigma(y_i)\, K(y_i, y_{i+1})\, g_{(s+1)/2}(y_N).$$

And proceeding as for the one point correlation function (see (24)–(25)), this quantity can be written as

$$\rho^2 \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} e^{-\pi\lambda^2(\rho x_2-k-m-(s-1)/2)^2-\pi\lambda^2(\rho x_1-k-(s-1)/2)^2} (\mathbb{K}^{*k-1} f_{(1-s)/2}) \left(\rho x_1 - k - \frac{s-1}{2}\right)$$

$$\times (\mathbb{K}^{m-1} g_{\rho x_2-k-m-(s-1)/2+1}) \left(\rho x_1 - k - \frac{s-1}{2}\right)$$

$$\times (\mathbb{K}^{N-k-m} g_{(s+1)/2}) \left(\rho x_2 - k - m - \frac{s-1}{2}\right).$$

Here $\mathbb{K}^*$ denotes the adjoint of $\mathbb{K}$:

$$(\mathbb{K}^* f)(x) = \int_{-\infty}^{+\infty} d\sigma(y)\, K^*(x, y) f(y).$$

Since we are interested only by the bulk properties, we need to make the same change of variables than in the case of the one point correlation function:

$$\begin{aligned} \rho x_1 &= \rho y_1 + \alpha\rho L, \\ \rho x_2 &= \rho y_2 + \alpha\rho L, \end{aligned} \tag{54}$$

and to keep y_1, y_2, α fixed whereas we go to the thermodynamic limit.

Hence we have to compute $\lim_{N\to\infty} \rho_N(y_1, y_2)$ $(y_1 < y_2)$ where

$$\rho_N(y_1, y_2) = \rho^2 \left[(\mathbb{K}^N g_{(s+1)/2}) \left(\frac{1-s}{2}\right)\right]^{-1} \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} e^{-\pi\lambda^2(\rho y_1+r+N\alpha-k)^2} (\mathbb{K}^{*k-1} f_{(1-s)/2})$$

$$\times (\rho y_1 + r + N\alpha - k)\, e^{-\pi\lambda^2(\rho y_2+r+N\alpha-k-m)^2} (\mathbb{K}^{m-1} g_{\rho y_2+r+N\alpha-k-m+1})$$

$$\times (\rho y_1 + r + N\alpha - k)(\mathbb{K}^{N-k-m} g_{\,s+1\,/2})(\rho y_2 + r + N\alpha - k - m). \tag{55}$$

r has been defined before ((32)). In order to compute this limit, we need the same estimates as in Section 1, that is (33) and the corresponding property of the adjoint $\mathbb{K}^*$.

$$| z_0^{-N}(\mathbb{K}^{*N}f)(x) - \nu^*(f)\, \Psi_0(-x)| < \bar{m}\, \|f\|\, \epsilon_{N-1}^* \tag{56}$$

the $\epsilon_N{}^*$ satisfying the same properties as the ϵ_N and

$$\nu^*(f) = \frac{\int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(x)\, f(x)}{\int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(x)\, \Psi_0(-x)}\,. \tag{57}$$

The proof of (56)–(57) is exactly the same as the one of (33) which is given in the appendix.

Proceeding as in Section 1, we get:

$$\lim_{\substack{N\to\infty \\ N\alpha-[N\alpha]=m}} \rho_N(y_1, y_2) = \rho^2 n(\rho y_1 + s(\alpha - \tfrac{1}{2}) + m + \tfrac{1}{2}, \rho y_2 + s(\alpha - \tfrac{1}{2}) + m + \tfrac{1}{2}) \tag{58}$$

when $y_1 < y_2$ where

$$n(x_1, x_2) = \left[z_0 \int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(x)\, \Psi_0(-x)\right]^{-1} \sum_{l=-\infty}^{+\infty} e^{-\pi\lambda^2(x_1+l)^2} \Psi_0(-x_1 - l)$$
$$\times \sum_{n=0}^{\infty} z_0^{-n}(\mathbb{K}^n g_{x_2+l-n})(x_1 + l)\, e^{-\pi\lambda^2(x_2+l-n-1)^2} \Psi_0(x_2 + l - n - 1). \tag{59}$$

Since

$$\nu^*(f_{(1-s)/2}) = \left[\int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(x)\, \Psi_0(-x)\right]^{-1} z_0 \Psi_0\left(\frac{1-s}{2}\right). \tag{60}$$

It is easy to see from (59) that

$$n(x_1 + 1, x_2 + 1) = n(x_1, x_2). \tag{61}$$

Hence the two point correlation functions given by (58) are invariant under the group of discrete translations of length ρ^{-1}. This broken symmetry confirms that these states describe a crystal with a fixed center of gravity (whose position is related to $s(\alpha - \frac{1}{2}) + m + \frac{1}{2}$). This property will hold again for almost all λ, since $n(x_1, x_2)$ is a *real analytic function of* λ *on* $(0, +\infty)$. We can see this by noting first that

$$|(\mathbb{K}^n g_{x_2+l-n})(x_1 + l)| < (\mathrm{Re}\, \lambda^2)^{-n/2}$$

and then by applying the same reasoning as in Section 1., for the case of $n(x)$.

4. *Cluster Properties*

We want now to investigate the behavior of $n(x_1, x_2)$ when the distance between the two points x_1 and x_2 becomes large.

We note first of all that because of (61), we can choose $0 \leqslant x_1 \leqslant 1$. Let us then take

$$x_2 = m + x_2' \tag{62}$$

where $0 \leqslant x_2' \leqslant 1$. If we write

$$n(x_1, x_2) - n(x_1)\, n(x_2) = n(x_1, x_2) - c(x_1, x_2) + c(x_1, x_2) - n(x_1)\, n(x_2)$$

where

$$\begin{aligned} c(x_1, x_2) &= \left[z_0 \int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(x)\, \Psi_0(-x)\right]^{-1} \sum_{l=-\infty}^{+\infty} e^{-\pi\lambda^2(x_1+l\,^2} \Psi_0(-x_1 - l) \\ &\quad \times \sum_{n=1}^{\infty} \nu(g_{x_2+l-n})\, \Psi_0(x_1 + l)\, e^{-\pi\lambda^2(x_2+l-n-1)^2} \Psi_0(x_2 + l - n - 1) \\ &= \left[\int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(x)\, \Psi_0(-x)\right]^{-1} \sum_{l=-\infty}^{+\infty} e^{-\pi\lambda^2(x_1+l)^2} \Psi_0(x_1 + l)\, \Psi_0(-x_1 - l) \\ &\quad \times \sum_{r=-\infty}^{m+l-2} e^{-\pi\lambda^2(x_2'+r)^2} \Psi_0(x_2' + r)\, \Psi_0(-x_2' - r). \end{aligned} \tag{63}$$

Then we see that

$$\begin{aligned} &| c(x_1, x_2) - n(x_1)\, n(x_2)| \\ &\quad < Dn(x_2) \left[\sum_{l=m-1}^{+\infty} e^{-\pi\lambda^2(x_1-l)^2} + e^{-\pi\lambda^2(x_1+l)^2}\right] \\ &\quad\quad + Dn(x_2) \sum_{l=[(m-1)/2]}^{m-1} e^{-\pi\lambda^2(x_1-l)^2} + 2Dn(x_1) \sum_{r=[(m-1)/2]}^{+\infty} e^{-\pi\lambda^2(x_2'+r)^2} \end{aligned} \tag{64}$$

where

$$D = \left[\int_{-\infty}^{+\infty} d\sigma(x)\, \Psi_0(x)\, \Psi_0(-x)\right]^{-1}. \tag{65}$$

This estimate was obtained by using the following inequality:

$$\begin{aligned} \left| \sum_{l=-\infty}^{+\infty} f(l)\, g(m - l)\right| &\leqslant \sup_{n \in \mathbb{Z}} | g(n)| \left[\sum_{l=m}^{+\infty} | f(l)| + \sum_{l=m}^{+\infty} | f(-l)| + \sum_{l=[m/2]}^{m} | f(l)|\right] \\ &\quad + \sup_{m \leqslant n \leqslant 2m} | g(n)| \sum_{l=0}^{+\infty} | f(-l)| + \sup_{[m/2] \leqslant n \leqslant m} | g(n)| \sum_{l=0}^{+\infty} | f(l)| \end{aligned} \tag{66}$$

valid if $\sum_{l=-\infty}^{+\infty} f(l)$ and $\sum_{l=-\infty}^{+\infty} g(l)$ converge absolutely. It remains to estimate

$$| n(x_1, x_2) - c(x_1, x_2)| \leqslant z_0^{-1} D \sum_{l=-\infty}^{+\infty} e^{-\pi\lambda^2(x_1+l)^2 - \pi\lambda^2(x_2-l)^2} + z_0^{-1} D \sum_{l=-\infty}^{+\infty} e^{-\pi\lambda^2(x_1+l)^2} \times \sum_{n=1}^{\infty} | z_0^{-n}(\mathbb{K}^n g_{x_2+l-n})(x_1+l) - \nu(g_{x_2+l-n})\,\Psi_0(x_1+l)|\, e^{-\pi\lambda^2(x_2+l-n-1)^2}.$$

Using (66) and (33) we get

$$\begin{aligned} | n(x_1, x_2) - c(x_1, x_2)| &\leqslant z_0^{-1} D \left[\sum_{l=m-1}^{+\infty} e^{-\pi\lambda^2(x_1+l)^2} + \sum_{l=m-1}^{+\infty} e^{-\pi\lambda^2(x_1-l)^2} \right] \\ &+ z_0^{-1} D \left[\sum_{l=[(m-1)/2]}^{m-1} e^{-\pi\lambda^2(x_1+l)^2} + e^{-\pi\lambda^2(x_2'+[(m-1)/2])^2} \sum_{l=-\infty}^{+\infty} e^{-\pi\lambda^2(x_1+l)^2} \right] \\ &+ z_0^{-1} D \lambda^{-1/2} \overline{m} \left[\sum_{l=-\infty}^{+\infty} e^{-\pi\lambda^2(x_1+l)^2} \sum_{n=1}^{\infty} \epsilon_{n-1} e^{-\pi\lambda^2(x_2'+m-1+l-n)^2} \right]. \end{aligned} \tag{67}$$

Using again (66), the last term of this inequality can be bounded by:

$$\begin{aligned} & z_0^{-1} D \lambda^{-1/2} \overline{m} \left(\sum_{n=0}^{+\infty} \epsilon_n \right) \left[\sum_{l=m-1}^{+\infty} e^{-\pi\lambda^2(x_1-l)^2} + \sum_{l=m-1}^{+\infty} e^{-\pi\lambda^2(x_1+l)^2} + \sum_{l=[(m-1)/2]}^{m-1} e^{-\pi\lambda^2(x_1-l)^2} \right] \\ &+ z_0^{-1} D \lambda^{-1/2} \overline{m} \left(\sum_{l=-\infty}^{+\infty} e^{-\pi\lambda^2(x_1+l)^2} \right) [\sup_{m-1 \leqslant k \leqslant 2(m-1)} g(k) + \sup_{[(m-1)/2] \leqslant k \leqslant m-1} g(k)], \end{aligned} \tag{68}$$

where

$$g(k) = \sum_{n=1}^{+\infty} \epsilon_{n-1} e^{-\pi\lambda^2(x_2'+k-n)^2}. \tag{69}$$

And if we use (66) once more, we see that:

$$\begin{aligned} g(k) &\leqslant \left(\sum_{n=0}^{+\infty} \epsilon_n \right) \left[\sum_{s=k}^{+\infty} e^{-\pi\lambda^2(x_2'-1+s)^2} + \sum_{s=k}^{+\infty} e^{-\pi\lambda^2(x_2'-1-s)^2} \right] + \left(\sum_{n=0}^{+\infty} \epsilon_n \right) \\ &\times \sum_{s=[k/2]}^{k} e^{-\pi\lambda^2(x_2'-1+s)^2} + \left[\sum_{s=-\infty}^{+\infty} e^{-\pi\lambda^2(x_2'+s)^2} \right] [\sup_{[k/2] \leqslant n \leqslant k} \epsilon_n + \sup_{k \leqslant n \leqslant 2k} \epsilon_n]. \end{aligned} \tag{70}$$

Collecting all these inequalities together (from (64) to (70)), we can conclude that

$$\lim_{|x_2-x_1|\to\infty} n(x_1, x_2) = n(x_1)\, n(x_2). \tag{71}$$

However, our careful estimates allows us to determine also the rate of approach to this limit. It is governed indeed by the rate of approach to zero of

$$\sum_{l=[(m-1)/2]}^{+\infty} e^{-\pi\lambda^2(x\pm l)^2} \quad \text{and} \quad \sup_{\frac{1}{2}[(m-1)/2]\leqslant n\leqslant 2(m-1)} \epsilon_n \leqslant \left(\sum_{n=0}^{+\infty} \epsilon_n\right) \epsilon_{\frac{1}{2}[(m-1)/2]}$$

and the first expression decays like a gaussian, the second one like an exponential. (This is shown in the appendix, Lemma 1(b), Eq. (A.10)). Hence we see that

$$| n(x_1, x_2) - n(x_1)\, n(x_2)| < Ae^{-\xi|x_2-x_1|}, \qquad \xi > 0 \tag{72}$$

when $| x_2 - x_1 |$ is sufficiently large. A and ξ are some constants which could be determined from our inequalities.

Let us first note that this cluster property, as well as the other ones we have proved here for the one and two-point correlation functions could have been obtained with more effort, but by using the same techniques, for the higher order ones.

A consequence of general interest of this cluster property is that the equilibirum states we have built by taking the thermodynamic limit along appropriate subsequences of N and Λ (see (40)) are *extremal* [9].

A more specific property, like the *exponential rate of clustering*, appears especially interesting in connection with the fact that we deal here with a system of particles interacting via a long range potential. We can interpret this effect, by saying that some *screening* of the discrete charges occur, even in this crystalline state.

B. Periodic Boundary Conditions

Because of their computational convenience, these boundary conditions, are those which are usually considered in many body theory. In our case they lead to the well-known expression for the potential energy [10]

$$H = \frac{4\pi\sigma^2}{2} L^{-1} \sum_{k\neq 0} \frac{1}{k^2} \rho_k \rho_{-k} \tag{73}$$

where

$$k = \frac{2\pi n}{L}, \qquad n \in \mathbb{Z}, \tag{74}$$

and ρ_k is the Fourier transform of the density

$$\rho_k = \sum_{i=1}^{N} e^{ikx_i}. \tag{75}$$

It is also useful to express this Hamiltonian in terms of an effective two body potential, periodic of period L:

$$v_L(x) = \frac{4\pi}{L} \sum_{k\neq 0} \frac{1}{k^2} e^{ikx} \tag{76}$$

or

$$v_L(x) = 2\pi L \left[\left(\frac{x}{L}\right)^2 - \left|\frac{x}{L}\right| + \frac{1}{6}\right] \tag{77}$$

when

$$x \in [-L, +L],$$

so that

$$H = \frac{1}{2} N\sigma^2 v_L(0) + \frac{\sigma^2}{2} \sum_{i\neq j} v_L(|x_i - x_j|). \tag{78}$$

This expression enables us to compute the amount of energy due to the interaction of the particles with their image. We find [11]

$$H = H_0 - \frac{2\pi\sigma^2}{L}\left(\sum_{i=1}^{N} x_i - \frac{NL}{2}\right)^2 \quad \text{when} \quad 0 < x_i < L \tag{79}$$

where

$$H_0 = -2\pi\sigma^2 \sum_{i<j} |x_i - x_j| + 2\pi\sigma^2 \left(\frac{N}{L}\right) \sum_{i=1}^{N} \int_0^L dx\, |x - x_i| - \frac{2\pi\sigma^2}{2}\left(\frac{N}{L}\right)^2 \int_0^L \int_0^L dx\, dy\, |x - y|. \tag{80}$$

A direct attack of such a Hamiltonian by methods similar to those used in the case of free boundary conditions is made difficult by the "long range" character of the correction

$$-\frac{2\pi\sigma^2}{L}\left(\sum_{i=1}^{N} x_i - \frac{NL}{2}\right)^2.$$

It appears possible however to circumvent this difficulty, by using the following identity.

$$e^{\pi a^2} = \left(\frac{L\beta\sigma^2}{2}\right)^{1/2} \int_{-\infty}^{+\infty} dz\, e^{-\pi(L\beta\sigma^2/2)s^2 + 2\pi(L\beta\sigma^2/2)^{1/2} s a}. \tag{81}$$

Let us choose

$$a = \left(\frac{2\sigma^2\beta}{L}\right)^{1/2} \left(\sum_{i=1}^{N} x_i - \frac{NL}{2}\right). \tag{82}$$

Then the Boltzmann factor corresponding to periodic boundary conditions reads

$$e^{-\beta H} = \left(\frac{L\beta\sigma^2}{2}\right)^{1/2} \int_{-\infty}^{+\infty} ds\, e^{-\beta H_0} e^{-\pi(L\beta\sigma^2/2)s^2 + 2\pi\beta\sigma^2 s(\sum_{i=1}^{N} x_i - NL/2)}. \tag{83}$$

This identity simplifies considerably when we order the particles:

$$0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant L$$

$$e^{-\beta H} = \left(\frac{L\beta\sigma^2}{2}\right)^{1/2} e^{-\pi\lambda^2(N/12)} \int_{-\infty}^{+\infty} ds\, e^{-\pi\lambda^2 \sum_{i=1}^{N}(\rho x_i - i - (s-1)/2)^2} \tag{84}$$

where

$$\lambda^2 = 2\beta\sigma^2\rho^{-1}, \tag{85}$$

$$\rho = \frac{N}{L}. \tag{86}$$

In this way we see that we have reduced this problem to one discussed before: the plasma with free boundary conditions, but with a charge defect s. Note however that the density of the background is no more $\rho = (N + s)/L$, but $\rho = N/L$. However if $s/L \to 0$, they are asymptotically the same.

5. *Thermodynamics*

We can easily compute the partition function

$$Q_L^{\,N} = \left(\frac{L\beta\sigma^2}{2}\right)^{1/2} e^{-(\pi\lambda^2/12)N} \rho^{-N} \int_{-\infty}^{+\infty} ds\, (\mathbb{K}^N g_{(1-s)/2}) \left(\frac{1-s}{2}\right), \tag{87}$$

the only difference in the integrand with the case of free boundary conditions coming from the factor $g_N(\rho x_N)$ instead of $g_{N+s}(\rho x_N)$, since $L = N\rho^{-1}$ in our case.

In order to compute

$$\lim_{N\to\infty} z_0^{-N} \int_{-\infty}^{+\infty} ds\, (\mathbb{K}^N g_{(1-s)/2}) \left(\frac{1-s}{2}\right) = \int_{-\infty}^{+\infty} ds\, \nu(g_{(1-s)/2})\, \Psi_0 \left(\frac{1-s}{2}\right) \tag{88}$$

we use the inequality

$$|\, z_0^{-N}(\mathbb{K}^N f)(x) - \nu(f)\, \Psi_0(x)| < m(x)\, \|f\|\, \epsilon_{N-1}\,, \tag{89}$$

proved in the appendix (Lemma 1(b)). $m(x)$ is a uniformly bounded continuous function of x such that

$$\int_0^{+\infty} m(x)\, dx < \infty. \tag{90}$$

Indeed, using (89) we get

$$\left| z_0^{-N} \int_{-\infty}^{+\infty} ds\, (\mathbb{K}^N g_{(1-s)/2}) \left(\frac{1-s}{2}\right) - \int_{-\infty}^{+\infty} ds\, \nu(g_{(1-s)/2})\, \Psi_0 \left(\frac{1-s}{2}\right) \right|$$
$$< \epsilon_{N-1} \left[\sup_{x\in R} m(x) \int_{-\infty}^{0} ds\, \| g_{(1-s)/2} \| + \lambda^{-1/2} \int_0^{\infty} ds\, m\left(-\frac{s+1}{2}\right)\right].$$

We thus see, that with these boundary conditions the free energy density is given by

$$-\beta f(\rho, \beta) = -\beta f_0(\rho, \beta) - (\pi\lambda^2/12) + \ln(z_0(\lambda)/e), \tag{91}$$

and is equal to the free energy in the case of free boundary conditions, with strict charge neutrality ($s = 0$). This shows that the symmetry restoring term that we subtracted from the hamiltonian is a "surface" one.

6. *Correlation Functions*

Let us begin again with the one-particle correlation function.
Using (84), we get

$$\rho_L{}^N(x) = \rho \left[\int_{-\infty}^{+\infty} ds\, (\mathbb{K}^N g_{(1-s)/2}) \left(\frac{1-s}{2}\right)\right]^{-1} \sum_{j=1}^{N} \int_{-\infty}^{+\infty} ds\, e^{-\pi\lambda^2(\rho x - j - (s-1)/2)^2}$$
$$\times (\mathbb{K}^{j-1} g_{\rho x - j + (1-s)/2 + 1}) \left(\frac{1-s}{2}\right) (\mathbb{K}^{N-j} g_{(1-s)/2}) \left(\rho x - j + \frac{1-s}{2}\right). \tag{92}$$

In order to avoid surface effects as before, we choose

$$\rho x = \rho y + N\alpha. \tag{93}$$

Making the change of variables

$$(1-s)/2 = t,$$

we get

$$\left[\int_{-\infty}^{+\infty} dt\, (\mathbb{K}^N g_t)(t)\right] \rho_N(y) = \rho \int_{-\infty}^{+\infty} dt \sum_{j=1}^{N} e^{-\pi\lambda^2(\rho y + t + N\alpha - j)^2}$$
$$\times (\mathbb{K}^{j-1} g_{\rho y + t + N\alpha - j + 1})(t) (\mathbb{K}^{N-j} g_t)(\rho y + t + N\alpha - j). \tag{94}$$

As in Section 2, the next step consists in showing that we can replace asymptotically at the right-hand side of this equality:

$$(\mathbb{K}^{j-1} g_{\rho y+t+N\alpha-j+1})(t) \qquad \text{by} \quad z_0^{j-1} \nu(g_{\rho y+t+N\alpha-j+1})\, \Psi_0(t)$$

and

$$(\mathbb{K}^{N-j} g_t)(\rho y + t + N\alpha - j) \qquad \text{by} \quad z_0^{N-j} \nu(g_t)\, \Psi_0(\rho y + t + N\alpha - j).$$

This is done by proving the analog of inequality (38). In order to see how it works, let us show that the corresponding second sum at the right-hand side of this inequality tends to zero. We have in our case, using (89):

$$\begin{aligned} &z_0^{-1}\rho \int_{-\infty}^{+\infty} dt \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+t+N\alpha-j)^2} m(t)\, \| g_{\rho y+t+N\alpha-j} \|\, \epsilon_{j-2} \nu(g_t)\, \psi_0(\rho y + t + N\alpha - j) \\ &\quad < z_0^{-1}\rho\lambda^{-1/2} \int_{-\infty}^{+\infty} dt\, m(t)\, \nu(g_t) \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+t+N\alpha-j)^2} \epsilon_{j-2} \\ &\quad < z_0^{-1}\rho\lambda^{-1/2} \int_{-\infty}^{+\infty} dt\, m(t)\, \nu(g_t) \left(\sum_{j=1}^{\infty} \epsilon_j\right) \\ &\quad \times \left[\sum_{k=[N\alpha]-[N\alpha^2]}^{[N\alpha]-2} e^{-\pi\lambda^2(\rho y+t+N\alpha-[N\alpha]+k)^2} + \epsilon_{[N\alpha^2]-2} \sum_{k=-\infty}^{+\infty} e^{-\pi\lambda^2 k^2}\right]. \end{aligned} \tag{95}$$

But it can easily be seen that

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} dt\, f(t) \sum_{k=n}^{+\infty} e^{-\pi\lambda^2(x+t+k)^2} = 0, \tag{96}$$

uniformly with respect to x on all compact subsets of R, if $\int_{-\infty}^{+\infty} dt |f(t)| < \infty$. Since $\int_{-\infty}^{+\infty} dt \mid m(t)\, \nu(g_t)| < \infty$, we see that (95) tends uniformly to zero, when y belongs to a compact subset of R.

We can then conclude that

$$\begin{aligned} \lim_{N\to\infty} \rho_N(y) = z_0^{-1}\rho \left[\int_{-\infty}^{+\infty} dt\, \nu(g_t)\, \Psi_0(t)\right]^{-1} \lim_{N\to\infty} \int_{-\infty}^{+\infty} dt\, \nu(g_t)\, \Psi_0(t) \\ \times \sum_{j=2}^{N-1} e^{-\pi\lambda^2(\rho y+t+N\alpha-j)^2} \nu(g_{\rho y+t+N\alpha-j+1})\, \Psi_0(\rho y + t + N\alpha - j). \end{aligned} \tag{97}$$

In order to get a limit we once more need to choose some subsequences; keeping $N\alpha - [N\alpha] = m$ fixed, we get (making the change of variables $k = [N\alpha] - j$)

$$\lim_{\substack{N\to\infty \\ N\alpha-[N\alpha]=m}} \rho_N(y) = \rho \int_{-\infty}^{+\infty} d\mu(t)\, n(\rho y + t + m) \tag{98}$$

where

$$d\mu(t) = \frac{\nu(g_t)\,\Psi_0(t)}{\int_{-\infty}^{+\infty} dt\,\nu(g_t)\,\Psi_0(t)}\,dt = \frac{\Psi_0(1-t)\,\Psi_0(t)}{\int_{-\infty}^{+\infty} dt\,\Psi_0(1-t)\,\Psi_0(t)}\,dt \tag{99}$$

and $n(x)$ is given by (41). In order to see this, we apply (96) to (97) after we have replaced $\nu(g_{\rho y+t+m+k+1})$ by g and $\Psi_0(\rho y + t + m + k)$ by 1. The analysis, we ran through, for the one-particle correlation function remains valid, in the case of the many points one. The final answer will be

$$\lim_{\substack{N\to\infty \\ N\alpha-[N\alpha]=m}} \rho_N(y_1, y_2,\dots, y_k) = \rho^k \int_{-\infty}^{+\infty} d\mu(t)\, n(\rho y_1 + t + m,\dots, \rho y_k + t + m) \tag{100}$$

where $n(x_1, x_2,\dots, x_k)$ is the basic correlation fuction (see (59), e.g.), obtained from sequences of finite systems with free boundary conditions. We have already seen from this result that the equilibrium state obtained by taking periodic boundary conditions is *not extremal,* since it can be decomposed into the extremal states obtained by taking free boundary conditions. (Note that the measure describing this decomposition $d\mu(t)$ is temperature and density dependent). This property has the important consequence that in such a state the correlation functions *do not cluster.* Let us investigate the symmetry properties of such a state and decompose it in a more natural way into extremal states.

We need the following property of the measure μ.

$$\int_{-\infty}^{+\infty} d\mu(x)\, e^{-2\pi inx} = \delta_{n,0}\,. \tag{101}$$

When $n = 0$, this is a mere consequence of the definition (99). If $n \neq 0$, then

$$\begin{aligned}\int_{-\infty}^{+\infty} d\mu(x)\, e^{-2\pi inx} &= \frac{1}{2\pi in}\int_{-\infty}^{+\infty} dx\, e^{-2\pi inx}\,\frac{d}{dx}\,[\Psi_0(1-x)\,\Psi_0(x)]\\ &= z_0^{-1}\,\frac{1}{2\pi in}\int_{-\infty}^{+\infty} dx\, e^{-2\pi inx}[e^{-\pi\lambda^2x^2}\Psi_0(x)\,\Psi_0(-x)\\ &\quad - e^{-\pi\lambda^2(x-1)^2}\Psi_0(x-1)\,\Psi_0(1-x)] = 0\end{aligned}$$

as can be seen by changing $x - 1$ into x in the last integral.

Now consider the function

$$\bar{n}(x_1,\dots, x_k) = \int_{-\infty}^{+\infty} d\mu(a)\, n(x_1 + a,\dots, x_k + a) \tag{102}$$

where $n(x_1, \ldots, x_k)$ (given in the text for $k = 1, 2, \ldots$) is a function invariant under $\mathbb{Z}$, that is

$$n(x_1 + 1, x_2 + 1, \ldots, x_k + 1) = n(x_1, x_2, \ldots, x_k). \tag{103}$$

In other words considered as a function of $x \in R^k$, $n(x)$ is periodic of period 1 in the direction $e = (1, 1, \ldots, 1)$

$$n(x + e) = n(x). \tag{104}$$

Let us decompose x as follows

$$x = x_{\parallel} + x_{\perp} \tag{105}$$

with

$$x_{\parallel} = \frac{(x, e)}{(e, e)}\, e \tag{106}$$

and

$$(x_{\perp}, x_{\parallel}) = 0. \tag{107}$$

Since $n(x)$ is C^1 on the unit circle in the variable $(x, e)/(e, e)$, we can Fourier decompose it:

$$n(x) = \sum_{p=-\infty}^{+\infty} n_p(x_{\perp})\, e^{2\pi i p (x,e)/(e,e)} \tag{108}$$

where

$$n_p(x_{\perp}) = \frac{1}{(e, e)} \int_0^{(e,e)} d(x, e)\, n(x)\, e^{-2\pi i p (x,e)/(e,e)} \tag{109}$$

with (108) converging uniformly, we see that

$$\int_{-\infty}^{+\infty} d\mu(a)\, n(x + ae) = n_0(x_{\perp}),$$

by using (101).

This implies that $\bar{n}(x + ae) = \bar{n}(x)$, $\forall a \in R$, or in other words that $\bar{n}(x_1, \ldots, x_k)$ is translation invariant. Moreover,

$$\bar{n}(x_1, \ldots, x_k) = \frac{1}{k} \int_0^k dt\, n\left(x_1 - \frac{\sum_{i=1}^k x_i}{k} + t, \ldots, x_k - \frac{\sum_{i=1}^k x_i}{k} + t\right)$$

and since it is tranlation invariant, we can shift each x_i, by $1/k \sum_{i=1}^k x_i$ and we get

$$\bar{n}(x_1, \ldots, x_k) = \int_0^1 dc\, n(x_1 + c, \ldots, x_k + c).$$

In conclusion, we have shown that the limiting correlation functions for periodic boundary conditions are independent of the subsequences chosen and can be expressed as

$$\rho(y_1, \ldots, y_k) = \rho^k \int_0^1 dc\, n(\rho y_1 + c, \ldots, y_k + c). \tag{102}$$

Moreover, they are *translation invariant*.

Thus we see that if we use such boundary conditions it is not possible to see that the system is in a crystalline state by looking at the one-point correlation function. Instead, we need to first investigate the clustering properties of the two-point correlation function and then to try to decompose it as in (102).

We also notice that the decomposition (102) is the most natural one for a crystal in one dimension, since it represents an average over the possible positions of the center of gravity of our system [12].

Appendix

In this section, we want to prove a number of properties of the operator $\mathbb{K}$ and its eigenfuctions, which were assumed in the text.

Let us write again the operator $\mathbb{K}$ in the following form.

$$(\mathbb{K}f)(x) = \int_{-\infty}^{+\infty} \theta(y - x + 1) f(y)\, d\sigma(y) \tag{A.1}$$

where we have introduced the measure σ, given by

$$\sigma([y, -\infty)) = \int_{-\infty}^{y} e^{-\pi\lambda^2 t^2}\, dt. \tag{A.2}$$

It appears useful to define this operator as one acting on the Hilbert space $\mathscr{L}^2(R, \sigma)$, defined by the scalar product

$$(f, g) = \int_{-\infty}^{+\infty} d\sigma(x)\, \bar{f}(x)\, g(x). \tag{A.3}$$

On $\mathscr{L}^2(R, \sigma)$, $\mathbb{K}$ has an adjoint $\mathbb{K}^*$ defined in the following way.

$$(\mathbb{K}^* f)(x) = \int_{-\infty}^{+\infty} \theta(x - y + 1) f(y)\, d\sigma(y). \tag{A.4}$$

We will first prove the following properties of the operators $\mathbb{K}$ and $\mathbb{K}^*$.

LEMMA 1 (a). *$\mathbb{K}$ and $\mathbb{K}^*$ have one common positive eigenvalue z_0. To this eigenvalue corresponds the eigenvectors $\Psi_0(x)$ and $\varphi_0(x)$ respectively, which are positive almost everywhere. The eigenvalue z_0 is simple and larger in modulus than all the other ones. z_0, $\Psi_0(x)$, and $\varphi_0(x)$ considered as functions of λ are real analytic in* $(0, +\infty)$.

(b) *If $\mathbb{P}$ denotes the projector, defined by*:

$$(\mathbb{P}f)(x) = \nu(f)\, \Psi_0(x) \tag{A.5}$$

where

$$\nu(f) = \frac{\int_{-\infty}^{+\infty} d\sigma(y)\, \varphi_0(y) f(y)}{\int_{-\infty}^{+\infty} d\sigma(y)\, \varphi_0(y)\, \Psi_0(y)} \tag{A.6}$$

then

$$\lim_{n\to\infty} z_0^{-n}\mathbb{K}^n = \mathbb{P} \qquad \textit{uniformly in } \mathscr{L}^2(R, \sigma) \tag{A.7}$$

More precisely the following estimate holds.

$$| z_0^{-n}(\mathbb{K}^n f)(x) - (\mathbb{P}f)(x)| < m(x)\, \|f\|\, \epsilon_{n-1} \tag{A.8}$$

where

$$\sup_{x\in R} m(x) = \bar{m} < \infty, \qquad \int_0^{+\infty} m(x)\, dx < \infty, \tag{A.9}$$

and the ϵ_n satisfy the following properties.

$$0 < \epsilon_{n+m} \leqslant \epsilon_n \epsilon_m \leqslant \sum_{n=1}^{\infty} \epsilon_n , \tag{A.10}$$

$\exists n_0(\gamma)$ such that $\forall n \geqslant n_0(\gamma)$ $\epsilon_n \leqslant e^{-\gamma n}$ with $\gamma > 0$.

Proof. (a) Note first the following properties of $\mathbb{K}$ and $\mathbb{K}^*$. They are compact, because they are of the Hilbert-Schmidt type:

$$\int_{-\infty}^{+\infty} d\sigma(x)\, d\sigma(y)\, | K(x, y)|^2 = \int_{-\infty}^{+\infty} d\sigma(x)\, d\sigma(y)\, | K^*(x, y)|^2 < \infty.$$

Here $K(x, y)$ (respectively) $K^*(x, y)$) denotes the kernel of the integral equation corresponding to $\mathbb{K}$ (respectively $\mathbb{K}^*$):

$$K(x, y) = \theta(y - x + 1),$$
$$K^*(x, y) = K(y, x).$$

$\mathbb{K}$ and $\mathbb{K}^*$ map the cone of the nonnegative functions into itself, because the kernels $K(x, y) \geqslant 0$. Despite the fact that the kernels are not strictly positive, the following important property holds.

$$\mathbb{K}^n(x, y) = 0 \quad \text{if} \quad y - x + n \leqslant 0,$$
$$\mathbb{K}^n(x, y) > 0, \quad \text{otherwise.}$$

This means that the σ-measure of the set of points for which $\mathbb{K}^n(x, y) = 0$ can be made arbitrary small if we taken n sufficiently large.

The property itself is evident for $n = 1$, and can easily be proved by induction.

$$\begin{aligned}\mathbb{K}^n(x, y) &= \int_{-\infty}^{+\infty} d\sigma(z)\, K(x, z)\, \mathbb{K}^{n-1}(z, y) \\ &= \int_{-\infty}^{+\infty} d\sigma(z)\, \mathbb{K}^{n-1}(z, y)\, \theta(z - x + 1)\, \theta(y - z + n - 1) \\ &= \theta(y - x + n) \int_{x+1 \leqslant z \leqslant y-x+n-1} d\sigma(z)\, \mathbb{K}^{n-1}(z, y).\end{aligned}$$

These properties allow us to apply theorem (β') [13, p. 274]. Indeed all the conditions of the theorem are satisfied, once we have made the change of variables:

$$\sigma(y) = \int_{-\infty}^{y} e^{-\pi\lambda^2 t^2}\, dt \qquad \text{or} \quad y = E(\sigma),$$

and in these new variables, the integral equation reads

$$zf(\mu) = \int_{0}^{\lambda^{-1}} d\sigma\, \theta(E(\sigma) - E(\mu) + 1) f(\sigma)$$

where

$$f(\mu) = \Psi(E(\mu)).$$

This proves the first part of the lemma, except the statement about the analyticity. But this one is merely a consequence of the fact that z_0, is a simple root of the Fredholm determinant $D(z)$, which is analytic in λ, and that Ψ_0 and φ_0 are given by the ratio of two determinants again analytic in λ.

In order to prove (b) let us introduce the following operator.

$$\mathbb{K}_1 = \mathbb{K} - z_0\mathbb{P}.$$

It is easily seen that $\mathbb{K}_1$ has the same eigenvalues as the operator $\mathbb{K}$ with the exception of z_0. In fact if

$$\mathbb{K}f = zf \qquad (f \neq 0, z \neq z_0)$$

then $\nu(f) = 0$ since $z\nu(f) = \nu(\mathbb{K}f) = z_0\nu(f)$. Conversely, if

$$\mathbb{K}_1 f = zf \qquad (f \neq 0, z \neq 0)$$

then $z\nu(f) = \nu(\mathbb{K}_1 f) = \nu(\mathbb{K}f) - z_0\nu(f)\,\nu(\Psi_0) = 0$ and consequently

$$\mathbb{K}f = \mathbb{K}_1 f + z_0\nu(f)\,\Psi_0 = \mathbb{K}_1 f = zf$$

in which $z \neq z_0$ since under the contrary assumption from (a) we would deduce that $f = c\Psi_0$, which would imply that

$$\mathbb{K}_1 f = \mathbb{K}f - z_0\nu(f)\,\Psi_0 = c[z_0\Psi_0 - z_0\nu(\Psi_0)\,\Psi_0] = 0.$$

Hence the eigenvalues of $\mathbb{K}_1$ all lie inside the circle $|z| = z_0$; that is,

$$\lim_{n\to\infty} \| \mathbb{K}_1{}^n \|^{1/n} = \inf_n \| \mathbb{K}_1{}^n \|^{1/n} = z_1 < z_0$$

according to (a).

It is easily seen that $\mathbb{P}\mathbb{K}_1 = \mathbb{K}_1\mathbb{P} = 0$; hence

$$\mathbb{K}^n = z_0{}^n\mathbb{P}^n + \mathbb{K}_1{}^n,$$

so that

$$\| z_0^{-n}\mathbb{K}^n - \mathbb{P} \| = z_0^{-n} \| \mathbb{K}_1{}^n \| = \epsilon_n$$

which tends to zero, when $n \to \infty$.

However, we need a more precise estimate that we can get as follows.

$$z_0^{-n}(\mathbb{K}^n f)(x) - (\mathbb{P}f)(x) = z_0^{-1} \int_{x-1}^{+\infty} d\sigma(y)[z_0^{-(n-1)}(\mathbb{K}^{n-1}f)(y) - (\mathbb{P}f)(y)];$$

hence, by Schwartz inequality

$$| z_0^{-n}(\mathbb{K}^n f)(x) - (\mathbb{P}f)(x)| < z_0^{-1} \left(\int_{x-1}^{+\infty} d\sigma(y)\right)^{1/2} z_0^{-(n-1)} \| \mathbb{K}_1^{n-1} \| \, \|f\|$$

that is,

$$| z_0^{-n}(\mathbb{K}^n f)(x) - (\mathbb{P}f)(x)| \leqslant m(x)\, \epsilon_{n-1} \|f\|$$

where

$$m^2(x) = \int_{x-1}^{+\infty} d\sigma(y)$$

satisfies the properties (A.9).

Equation (A.10) follows from the definition of ϵ_n

$$\epsilon_n = z_0^{-n} \| \mathbb{K}_1{}^n \|$$

and the fact that

$$\lim_{n\to\infty} \epsilon_n^{1/n} = \inf_n \epsilon_n^{1/n} < 1.$$

LEMMA 2. (1)

$$\varphi_0(x) = c\Psi_0(-x), \qquad c > 0. \tag{A.11}$$

(2) $\Psi_0(x)$ *is a decreasing function of* x, *such that*

$$0 \leqslant \Psi_0(x) \leqslant \Psi_0(-\infty) = 1. \tag{A.12}$$

The functions $\Psi_0^{(N)}(x)$ *defined by*

$$\Psi_0^{(N)}(x) = 1 + \sum_{k=1}^{N} (-1)^k z_0^{-k} \int_{-\infty}^{x-1} d\sigma(x_1) \int_{-\infty}^{x_1-1} d\sigma(x_2) \cdots \int_{-\infty}^{x_{k-1}-1} d\sigma(x_k) \tag{A.13}$$

constitute a converging sequence of upper and lower bounds to $\Psi_0(x)$. *More precisely*

$$\Psi_0^{(2p-1)}(x) \leqslant \Psi_0(x) \leqslant \Psi_0^{(2p)}(x) \tag{A.14}$$

(3) *Considered as a function of* x, $\Psi_0(x)$ *can be extended to an entire function of* x. *This function is uniformly bounded in* x *on any strip*:

$$a \leqslant \operatorname{Im} x \leqslant b$$

of the complex x*-plane.*

(4) *Keeping* x *real,* $\Psi_0(x)$ *can be extended to an analytic function of* λ *in a neighborhood D of the positive real axis. Moreover this function is uniformly bounded in* λ *and* x, *on all the compact subsets of D.*

Proof. (1)

$$z_0\varphi_0(x) = \int_{-\infty}^{x+1} dy\, e^{-\pi\lambda^2 y^2} \varphi_0(y) = \int_{-x-1}^{+\infty} dy\, e^{-\pi\lambda^2 y^2} \varphi_0(-y);$$

hence

$$z_0\varphi_0{}'(+x) = \int_{x-1}^{+\infty} dy\, e^{-\pi\lambda^2 y^2} \varphi_0{}'(+y) \qquad \varphi_0{}'(x) = \varphi_0(-x).$$

But this is nothing else than the integral equation $z_0\varphi_0{}' = \mathbb{K}\varphi_0{}'$, which we know to possess only one solution corresponding to the eigenvalue z_0, namely,

$$\varphi_0{}'(x) = \varphi_0(-x) = c\Psi_0(x).$$

(2) We already know that $\Psi_0(x) \geqslant 0$. It is decreasing, since

$$z_0[\Psi_0(x+h) - \Psi_0(x)] = \int_{x+h-1}^{x-1} d\sigma(y)\, \Psi_0(y).$$

It is uniformly bounded, because it belongs to $\mathscr{L}^2(R, \sigma)$

$$z_0 \mid \Psi_0(x)|^2 = \left| \int_{x-1}^{+\infty} d\sigma(y)\, \Psi_0(y) \right|^2 < \left(\int_{-\infty}^{+\infty} d\sigma(y) \right) \| \Psi_0 \|^2.$$

For convenience we have chosen the normalization $\Psi_0(-\infty) = 1$. The functions $\Psi_0^{(N)}$ converge uniformly to $\Psi_0(x)$ on all the compact subsets of R. Indeed,

$$\Psi_0(x) = 1 - z_0^{-1} \int_{-\infty}^{x-1} d\sigma(y)\, \Psi_0(y),$$

so that

$$\begin{aligned} &\Psi_0(x) - \Psi_0^{(N)}(x) \\ &\quad = (-1)^{N+1} z_0^{-(N+1)} \int_{-\infty}^{x-1} d\sigma(x_1) \int_{-\infty}^{x_1-1} d\sigma(x_2) \cdots \int_{-\infty}^{x_N-1} d\sigma(x_{N+1})\, \Psi_0(x_{N+1}), \end{aligned}$$

and since $0 \leqslant \Psi_0(x) \leqslant 1$,

$$| \Psi_0(x) - \Psi_0^{(N)}(x)| < \frac{z_0^{-(N+1)}}{(N+1)!} \left(\int_{-\infty}^{+\infty} d\sigma(x) \right)^{N+1}$$

and

$$\Psi_0^{(2p)}(x) \geqslant \Psi_0(x) \geqslant \Psi_0^{(2p-1)}(x).$$

(3) We have just seen that $\Psi_0(x)$ is given by the uniformly convergent series

$$\Psi_0(x) = 1 + \sum_{k=1}^{+\infty} (-1)^k z_0^{-k} \int_{-\infty}^{x-1} d\sigma(x_1) \int_{-\infty}^{x_1-1} d\sigma(x_2) \cdots \int_{-\infty}^{x_{k-1}-1} d\sigma(x_k)$$

when x belongs to a compact subset of R.

Now allowing x to be complex, let us rewrite the term of order k of this series as follows.

$$a_k = \int_{-\infty}^{-1} dy_1 \int_{-\infty}^{-1} dy_2 \cdots \int_{-\infty}^{-1} dy_k\, e^{-\pi\lambda^2(y_1+x)^2 - \cdots - \pi\lambda^2(y_1+\cdots+y_k+x)^2}.$$

After we have made the change of variables

$$x_1 - x = y_1 , \qquad x_{i+1} - x_i = y_{i+1} , \qquad i = 1 \cdots N - 1,$$

we get

$$\begin{aligned} | a_k | &\leqslant \int_{-\infty}^{-1} dy_1 \cdots \int_{-\infty}^{-1} dy_k \, e^{+\pi\lambda^2 k (\mathrm{Im} x)^2 - \pi\lambda^2 (y_1 + \mathrm{Re} x)^2 - \cdots - \pi\lambda^2 (y_1 + \cdots + y_k + \mathrm{Re} x)^2} \\ &= [e^{\pi\lambda^2 (\mathrm{Im} x)^2}]^k \int_{-\infty}^{\mathrm{Re} x - 1} d\sigma(x_1) \int_{-\infty}^{x_1 - 1} d\sigma(x_2) \cdots \int_{-\infty}^{x_{k-1} - 1} d\sigma(x_k) \\ &\leqslant \frac{[e^{\pi\lambda^2 (\mathrm{Im} x)^2}]^k}{k!} \left(\int_{-\infty}^{+\infty} dy \, e^{-\pi\lambda^2 y^2} \right)^k . \end{aligned}$$

This shows that $a_k(x)$ can be extended to an analytic function on any strip of the form $a \leqslant \mathrm{Im}\, x \leqslant b$, and that

$$| \Psi_0(x) | \leqslant \exp \left\{ z_0^{-1} \exp \left[\pi\lambda^2 (\mathrm{Im}\, x)^2 \int_{-\infty}^{+\infty} d\sigma(y) \right] \right\}.$$

Hence the series converge uniformly on any strip to an analytic function and $\Psi_0(x)$ is uniformly bounded on such strips. In this way we have extended $\Psi_0(x)$ to an entire function of x.

(4) We already know that $z_0(\lambda)$ is real analytic on $(0, +\infty)$ and strictly positive on this axis. Hence it can be extended to an analytic function in a neighborhood D of this axis and such that $z_0(\lambda) \neq 0$ in D.

Proceeding as before, we get

$$| \Psi_0(x) | < \exp \left\{ | z_0 |^{-1} \int_{-\infty}^{+\infty} dx \, e^{-\pi (\mathrm{Re} \lambda^2) x^2} \right\}. \tag{A.15}$$

Hence the series converges uniformly to an analytic function of λ on all the compact subsets of D. Equation (A.15) also shows that $\Psi_0(x)$ is uniformly bounded in λ and x, on such subsets of D.

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