ELECTROSTATIC STABILITY AND INSTABILITY OF N EQUAL CHARGES IN A CIRCLE

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Received 10 June 1985; in final form 26 August 1985

The Coulomb energy of N equal charges placed in a circle is calculated. Contrary to the common belief, the configuration with the minimum energy does not always correspond to the symmetrical placement of all N charges on the circumference. This is true only for $N \leqslant 11$; for $N > 11$ the configuration with $N-1$ charges on the circumference and one (Nth) charge in the center of the circle is energetically more preferable. Therefore, for $N > 11$ there will be a spontaneous "electrostatic ejection" of the Nth electron from the circumference into the center of the circle. The related problem of the equilibrium configuration of N point charges on a sphere is also briefly discussed.

What is the configuration with the minimum energy for the system of N equal point charges confined within a circle?

A natural answer seems to be that, due to the Coulomb repulsion, the charges will arrange themselves in equally spaced positions on the circumference, i.e. N charges will be in the vertices of the regular N-side polygon inscribed into the circle. This statement appears to be so obvious that, to my best knowledge, nobody ever bothered to verify it by a direct calculation of the total electrostatic energy (W). Such a calculation, performed in the present paper, shows that the answer to this question is not so obvious as it might seem at first glance [1].

Consider a system of N equal charges Q which are free to move inside the circle of radius R. Let W be the total Coulomb energy of the configuration when all N charges are symmetrically spaced on the circumference of the circle:

$$W = \sum_{1 \leqslant i < j \leqslant N} Q^2 / |r_i - r_j| \,. \qquad (1)$$

Similarly, one can consider the electrostatic energy (E) of another configuration, when only $N-1$ charges are at the vertices of the inscribed regular polygon ("$N-1$-gon") and one "extra" charge is "ejected" to the center of the circle. A common belief is that always (i.e. for *all* N) $W < E$, i.e. that the configuration which has all N charges on the circumference is energetically the most preferable one, regardless of the value of N.

Table 1 gives the values of W and E for various N. One can see that $W < E$ only for $N \leqslant 11$ (fig. 1). Starting from $N = 12$ the *opposite* statement is true, i.e. $E < W$ (fig. 2). Using a simple BASIC program on the TRS-80 pocket calculator, I checked the non-equaliy $E < W$ to $N = 400$ [1], but as was later proved by Webb [2] it holds for any integer $N > 11$.

This result can be interpreted in the following way. For $N > 11$ it will be energetically beneficial for the system of N charges placed originally on the circumference of the circle to "expell" one charge to the center of the circle. By analogy one might conclude that a similar "effect" should also take place for the system of N charges placed on the inner surface of a sphere. This is, however, not so. As was pointed out in several follow up letters [3–5], the above result for the circle follows from the application of three-dimensional Coulomb potential to the pseudo-two-dimensional system. Our circle should properly be regarded as a thin disk embedded in three-dimensional space and then the charge at the center of the circle is actually at the center of one of the flat surfaces of the disk.

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Table 1
Energy of the configuration with all N charges on the circumference (W), with one charge "ejected" to the center of the circle (E), and their difference $W - E$ (in units Q^2/R)

$N = 3$	1.732050808(W)	$N = 4$	3.828427125	$N = 5$	6.881909605
	2.5(E)		4.732050808		7.828427125
	−0.767949192($W - E$)		−0.903623683		−0.946517520
$N = 6$	10.96410162	$N = 7$	16.13335410	$N = 8$	22.43892677
	11.88190961		16.96410162		23.13335410
	−0.91780799		−0.83074752		−0.69442733
$N = 9$	29.92344920	$N = 10$	38.62449898	$N = 11$	48.57567512
	30.43892677		38.92344920		48.62449898
	−0.51547757		−0.29895022		*−0.04882386*
$N = 12$	59.80736155	$N = 13$	72.34728955	$N = 14$	86.22096475
	59.57567512		71.80736155		85.34728955
	+0.23168643		+0.53992800		+0.87367520
$N = 15$	101.4519980	$N = 20$	198.6904722	$N = 100$	7529.28523
	100.2209648		195.3689723		7462.77474
	+1.2310332		+3.3214999		+66.51049

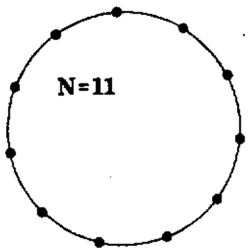

Fig. 1. The equilibrium configuration for 11 equal point charges in a circle (all 11 charges are on the circumference).

A further elaboration of the problem of N *discrete* point charges confined to a disk will be an interesting exercise. It will likely establish the gradual appearence of concentric circles of charges and in the limit of $N \to \infty$ one should expect a continuum charge distribution of the type $\propto (R^2 - r^2)^{-1/2}$ [6,7]. By all means some kind of "periodical law" should be expected: at some values of N the addition of a next charge will result in the appearance of the next circle of point charges.

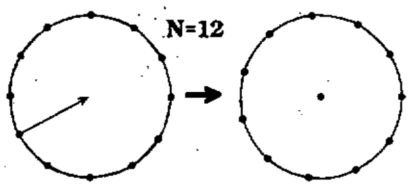

Fig. 2. For 12 equal charges the same configuration (left) is unstable (non-equilibrium). The stable configuration (right) has one charge at the center of the circle.

Recently I discussed a rather peculiar situation which may originate from the Coulomb repulsion between two electrons in a system with four trapping sites [8,9]. For some specific geometries of a four-site impurity complex the only allowed spontaneous radiative transition will be a *simultaneous* jump of both electrons. The system considered in the present paper provides another, purely classical, illustration of unexpected peculiarities of the Coulomb interaction.

In conclusion I would like to mention another interesting and related problem, namely: "What will be the stable (least energy) configuration of N equal point charges on a sphere?" This problem dates back to the classical atomic model of J.J. Thomson (e.g. ref. [10]). Thomson's model has, of course, been abandoned, but the mathematical problem still remains unsolved, except for some special values of N [11–14]. Despite its purely classical nature and unambiguous formulation, this problem is very intricate and its general solution (i.e. a common algorithm valid for any integer N) is yet to be found. This problem is of importance in stereochemistry, botany, virology, information theory, nuclear theory, and elsewhere [14–16].

Even for the "simple" cases: $N = 4$, 6, 8, 12, and

20 (numbers of vertices of five regular Platonic polyhedrons) there is a very interesting curiosity which seems to contradict common sense. While the tetrahedron ($N = 4$), octahedron ($N = 6$) and icosahedron ($N = 12$) do indeed provide the required minimum energy configurations, the cube ($N = 8$) and dodecahedron ($N = 20$) *do not*! Some degree of the "self-twisting" is needed for the inscribed cube and dodecahedron to reach the global minimum of the potential energy [10,11,14]. With some imagination this may be seen as a classical analog of the Jahn–Teller effect. It, perhaps, may be attributed to the fact that tetrahedron, octahedron and icosahedron all have triangle faces (triangle is a rigid figure), while the cube and dodecahedron have "soft" deformable faces (square and pentagon faces, respectively).

It is also worth noting that the quantum treatment of both problems (charges on a thin disk and on the spherical surface) may appear viable for the variety of chemical physics and other problems mentioned above.

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