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A novel iterative method for the reconstruction of signals from nonuniformly spaced samples†

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Abstract

In this paper we address the problem of reconstruction of signals from their nonequally spaced samples. Exploiting the close-to-band structure of the composing matrix, a two-stage procedure for the recovery of uniform samples from nonuniform samples has been suggested by Plotkin and Swamy (1987). In order to reduce the computational complexity, a special procedure of partitioning the composing matrix into a set of overlapping submatrices was used. Then the error in the estimate was reduced by applying an iterative procedure. The present paper is an extension of results presented by Plotkin and Swamy (1987). In this we propose a modification to their procedure, so as to recover an equal number of uniformly spaced samples as are those in the nonuniform set. We show that the iterative algorithm converges conditionally and the conditions are weak and may be implemented easily. Computer simulation results have been presented which show that the proposed technique performs well even for deviations of nonuniform sample positions well beyond the corresponding uniform positions. The proposed method is attractive from computational point of view also.

Zusammenfassung

In dieser Arbeit sprechen wir das Problem der Signalrekonstruktion aus nicht äquidistanten Abtastwerten an. Unter Ausnutzung der 'Closed-to-Band'-Struktur der Composing-Matrix wurde von Plotkin und Swamy [1987] eine zweistufige Prozedur zur Berechnung von äquidistanten Abtastwerten aus nicht-äquidistanten vorgeschlagen. Um den Rechenaufwand zu reduzieren, wird eine spezielle Prozedur benutzt, bei der die Composing-Matrix in einen Satz von überlappenden Submatrizen zerlegt wird. Dann wird der Schätzfehler durch die Anwendung einer iterativen Prozedur reduziert. Die vorliegende Arbeit ist eine Erweiterung der von Plotkin und Swamy (1987) angegebenen Resultate. Dabei schlagen wir eine Modifikation ihrer Prozedur vor, indem die gleiche Anzahl von äquidistanten Abtastwerten wie die des nicht äquidistanten Satzes ermittelt wird. Wir zeigen, daß der iterative Algorithmus bedingt konvergent ist, und daß die Bedingungen schwach sind und leicht implementiert werden können. Es werden Simulationsresultate wiedergegeben, die zeigen, daß die vorgeschlagene Technik gut arbeitet selbst für große Abweichungen der nicht-äquidistanten

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Abtast-Positionen gegenüber den zugehörigen äquidistanten Positionen. Die vorgeschlagene Methode ist auch unter dem rechnerischen Gesichtspunkt attraktiv.

Résumé

Nous nous intéressons dans cet article au problème de la reconstruction de signaux à partir d'échantillons espacés irrégulièrement. Une procédure en deux étapes exploitant la structure proche d'une structure à bande de la matrice de composition pour le recouvrement d'échantillons uniformes à partir d'échantillons non uniformes a été suggérée par Plotkin et Swamy (1987). Dans le but de réduire la complexité des calculs, une procédure spéciale de partitionnement de la matrice de composition en un ensemble de sousmatrices se recouvrant était utilisée. L'erreur d'estimation était alors réduite par application d'une procédure itérative. Le présent article constitue une extension des résultats présentés par Plotkin et Swamy (1987). Nous proposons dans celui-ci une modification de leur procédure permettant d'obtenir un nombre d'échantillons uniformément espacés égal à celui des échantillons non uniformes. Nous montrons que l'algorithme itératif converge conditionnellement, que les conditions sont peu restrictives, et que l'implantation est aisée. Des simulations sur ordinateur sont présentées afin de montrer que la technique proposée marche bien même lorsque les déviations des positions des échantillons non uniformes sont bien supérieures aux positions uniformes correspondantes. La méthode proposée est également attrayante du point de vue calcul.

Key words: Nonuniform sampling; Signal reconstruction; Iterative algorithms

1. Introduction

The significance of signal reconstruction from their nonequally spaced samples has been well recognized in the context of filter design, speech processing, power spectral estimation, holography, astronomy, and data compression. In most of these problems, either equally spaced samples are not available or variations in the instantaneous bandwidth of a signal necessitates nonuniform sampling rates corresponding to local characteristics of the signal. The classical Shannon sampling theorem and the numerous modifications show that a finite energy signal $x(t)$, band-limited to $|f| \leqslant f_0$ and sampled at or above its Nyquist rate $(2f_0)$, is uniquely determined by its uniformly spaced samples $\{x(nT)\}$ where $T = 1/2f_0$ and its samples at any instant t_m can be reconstructed as

$$\boldsymbol{x}(t_m) = \boldsymbol{A}\boldsymbol{x}(nT), \tag{1}$$

where $\boldsymbol{x}(t_m)$ is the vector of reconstructed samples, $\boldsymbol{A}$ is a matrix with (m, n)th element of the composing functions,

$$A_{m,n} = \Phi(t_m, nT) = \frac{\sin \omega_0(t_m - nT)}{\omega_0(t_m - nT)}, \quad m, n = 1, \ldots, N, \tag{2}$$

and

$$\boldsymbol{x}(nT) = [x(T), x(2T), \ldots, x(NT)]^{\mathrm{T}} \tag{3}$$

is a given set of equally spaced samples with $(\cdot)^{\mathrm{T}}$ denoting the transpose operation. In Eq. (2) $\omega_0 = 2\pi f_0$ and t_m is the mth time instant. One may note that the reconstruction of $x(t)$ is almost exact in the interval $\{T - NT\}$ {i.e. for the finite order of matrix $\boldsymbol{A}$) and the number of reconstructed samples need not necessarily be equal to N, to achieve a certain value of the r.m.s. error.

The problem of reconstructing $x(t)$ does, however, become a difficult one when the available signal samples $\{x(\tau_m)\}$ are not uniformly spaced, i.e. $\tau_m \neq mT$, and one cannot use Eq. (1) for recovering $\{x(t_m)\}$. A number of algorithms have been proposed in the past for the recovery of the signal when $\{x(\tau_m)\}$ are available.

A typical reconstruction procedure may be found in [12] whose main disadvantage is that in practice a set of composing functions has to be generated for every data block. Wiley [11] applied a theorem due to Sandberg [9] concerning the convergence of nonlinear mappings in Hilbert spaces to the design of an iterative algorithm for the reconstruction of band-limited functions from nonuniformly spaced samples. This method recovers the function

without error provided the nonuniform sample positions do not deviate too much from the corresponding equally sample positions. Marvasti [3] suggested a simple iterative procedure which involves low-pass filtering of the nonuniform samples followed by sampling and low-pass filtering cycles. It has been shown that the technique works well when the average sampling rate is higher than the Nyquist rate. In their recent papers, Marvasti et al. [4, 5] have applied an algorithm based on the generalized iterative deconvolution to obtain equally spaced samples from the given nonequally spaced samples.

Plotkin et al. [7] have proposed a method in which the set of nonuniformly spaced samples were conformed to a special correcting function and this function was then used as a multiplicity factor to correct the signal, reconstructed by the direct application of Eq. (1). It was also shown that under certain restricted conditions, such as the deviation of sampling positions $|\tau_m|$ around the corresponding synchronous positions $\{nT\}$ does not exceed one-half of the average sampling step T, the reconstruction technique allows one to use the standard composing functions. Clark et al. [2] have shown that a nonuniform sample sequence can be obtained by applying a coordinate transformation on a uniform sequence and then by using an invertible transformation one can recover the original signal. All these methods try to recover the original signal directly from the nonuniform sequences.

Following the classical Shannon's theorem, Plotkin and Swamy [6] argued that since $x(t_m)$ in Eq. (1) could be any arbitrary set, the given set of nonuniform samples could be one of the possibilities. Then the relation between the nonuniformly spaced samples $y = x(\tau_m)$ and the corresponding uniformly spaced samples $x = x(nT)$ is given by Eq. (1) where A is the composing matrix with elements

$$A_{m,n} = \frac{\sin \omega_0(\tau_m - nT)}{\omega_0(\tau_m - nT)}, \quad m,n = 1, \ldots, N, \tag{4}$$

with

$$\tau_m = mT + \delta t, \quad m = 1, \ldots, N, \tag{5}$$

as the nonuniform sample instants. In Eq. (5), δt is a random variable distributed in the range $(\pm JT/2)$ where J is called as the jitter parameter. Once the nonequally spaced samples y are available, one can find the corresponding uniform samples x by solving the system of linear equations $(y = Ax)$. Then Eq. (1) can be invoked to reconstruct the original continuous signal. Thus, the reconstruction of a signal becomes a two-step procedure in which the first important step is to solve a set of linear equations.

In [6], authors suggested a special iterative procedure for the reconstruction of uniform samples $x(nT)$. This procedure has been also extended to the two-dimensional (2-D) case [1]. By exploiting the close-to-band structure of the matrix A, they reduced the number of computations in the recovery of uniform samples.

The present paper is an extension of results presented in [6]. This paper is organized as follows. Section 2 introduces a modified version of the iterative procedure, which unlike in [6] recovers an equal number of uniformly spaced samples as are those in the nonuniform set. We show that the iterative algorithm converges conditionally and the conditions can be met easily. The performance of the proposed method is evaluated in Section 3 and is compared with that of the recently proposed iterative algorithm of Marvasti [5], using the mean squared error (MSE) in recovering the uniform sample set as the criterion. Finally, in Section 4 we present some concluding remarks.

2. Iterative technique for the recovery of uniformly spaced samples from their nonuniformly distributed counterparts

In this section we first examine the nature of the matrix A and show that it has a structure that is close to a band matrix. We then propose an algorithm to solve the set of simultaneous equations in (1). The proposed algorithm exploits the inherent structure in A to reduce the number of computations. We also show that an iterative procedure, which reduces the error in the recovery of uniform samples, converges conditionally.

2.1. Nature of matrix A

Recall the matrix A with elements as in Eq. (4). Using Eq. (5) in (4), we obtain

$$A_{m,n} = \frac{\sin \pi[(m-n)+q(m)]}{\pi[(m-n)+q(m)]}, \quad m,n = 1, \ldots, N, \quad (6)$$

where we have used the relation $\omega_0 = \pi/T$ and $q(m) = \delta t/T$ is the normalized random sampling position, distributed uniformly in the range $(\pm J/2)$, where J is the jitter parameter. One may note here that a jitter value of unity implies that the nonuniform sampling positions $\{\delta t\}$ deviate from their corresponding uniform sampling positions by a maximum of $\pm T/2$. From Eq. (6) we note that the absolute values of the off-diagonal elements of matrix A decay as the column index moves away from the main diagonal. The rate of decay, however, depends on the value of $q(m)$. To see the nature of this decay, we have plotted in Fig. 1 the normalized element values as a function of the off-diagonal element number for various values of $q(m)$. We observe from these plots that even for a value of $q(m)$ as large as 0.995 (which implies that the nonuniform sample position deviates from the uniform sample position by an approximate time equal to $T/2$), the element values drop below 10 dB (from the maximum value) for off-diagonal positions beyond 9. Thus, the matrix A has an inherent structure that is close to a band matrix where the bandwidth that contains the significant elements is dependent on the value of $q(m)$, which in turn is dependent on the jitter parameter J.

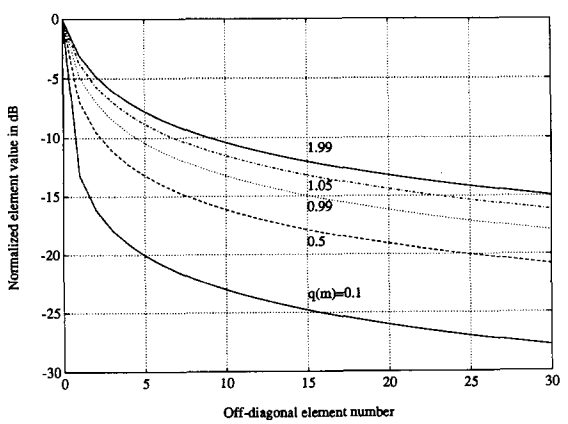

Fig. 1. Magnitude of elements in matrix A.

The other important aspect of the matrix A is with respect to its ill-conditioning. Recall the matrix A with elements as in Eq. (4). The matrix will be singular if $A_{m,n} = A_{m+1,n}$, for $n = 1, 2, \ldots, N$ and this can happen only if $\tau_m = \tau_{m+1}$ which in turn implies that adjacent nonuniform sample locations are identical. Although this occurrence has a finite probability while generating the nonuniform sample locations, it can be and should be avoided as one does not convey any extra information by transmitting the same sample twice! Thus, the potential problem of ill-conditioning while solving the matrix equations in the recovery of uniform samples from the nonuniform ones can be avoided.

2.2. The basic procedure

We note from the discussion in Section 1 that equally spaced samples of a signal can be recovered from the corresponding nonuniformly spaced ones by solving a system of linear equations $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. Higher dimensionality of A (in the case of large data lengths) increases the computational complexity and one way to reduce this is to use a system of equations of reduced size. In what follows we propose a method in which the matrix A is partitioned into overlapped submatrices each with a reduced dimension (see Fig. 2). Since A is close to a band matrix, each of the elements of vector $\boldsymbol{y}$ is a result of a few elements of vector $\boldsymbol{x}$, and the effect of rest of the elements is rather weak. Thus, we can reduce the number of simultaneous equations required to solve for each element of $\boldsymbol{x}$ by considering only those that are strongly coupled. However, this increases the number of matrix equations to be solved but each with a reduced dimension. As it will be shown below, the complexity of the whole procedure is reduced significantly as a result of the proposed method.

There is one important aspect of this partitioning that needs attention. This is regarding the ill-conditioning of each of the submatrices. Recall the discussion on the ill-conditioning of the matrix A where we have suggested a nonuniform sampling

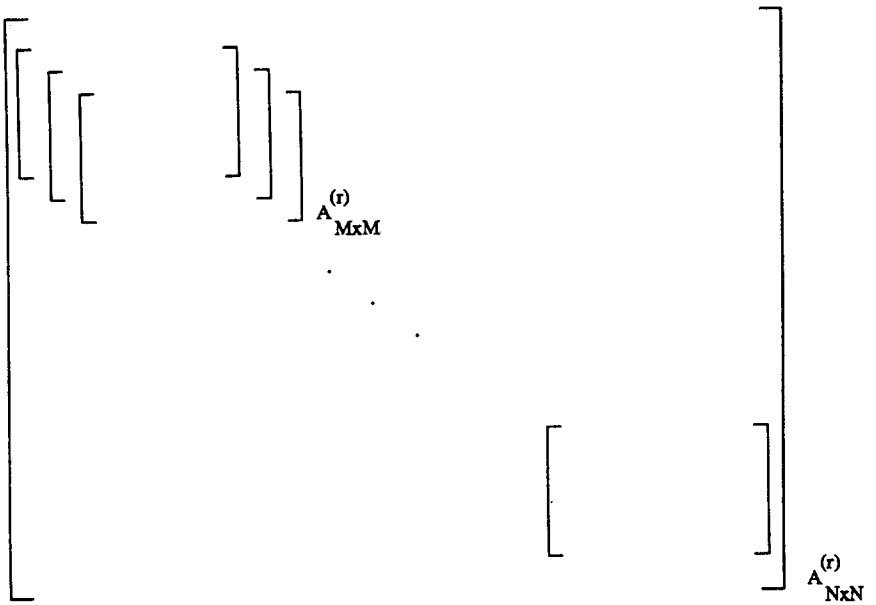

Fig. 2. Partitioning of matrix A into overlapping submatrices.

methodology which avoids the coincidence of two adjacent nonuniform sample locations. The other possible nonuniform sample location which causes singularity of the submatrices is the coincidence of mth nonuniform sample location with $(m+1)$th uniform sample location. Although, the probability of occurrence of such an event is very small, it can lead to very large errors as explained below. It is easy to see from Eq. (4) that such an occurrence makes the mth row of A all zeros except at $(m+1)$th location where it is unity. Since the $(m, m+1)$th element is not a diagonal element, there are at least $(M-1)$ submatrices (of size $M \times M$) which have a row with $(M-1)$ elements including the diagonal one as zeros. Further, one cannot rule out the possibility of one submatrix whose last row is all zeros! It is obvious that such submatrices can lead to very large reconstruction errors. To avoid this problem, one needs to see that the nonuniform sample locations do not coincide with their adjacent uniform ones. However, this can be done either at the transmitting or receiving location. At the receiver location, the nonuniform sample location that is problematic, can be altered by a small amount. Our experiments show that even a shift as small as $10^{-3}\,T$ can reduce the reconstruction error substantially. The proposed technique is described below:

Step 1. Make the necessary alterations to the nonuniform locations to avoid ill-conditioning of the submatrices.

Step 2. From the nonuniform sample positions form an $N \times N$ matrix A with elements as in Eq. (4). Then form $(N-M+1)$ overlapping submatrices of size $(M \times M)$, where M takes on an odd value (starting with 3) such that

$$A^{(r)} = (A_{m,n}), \quad m, n = 1, 2, \ldots, M \text{ and}$$
$$r = 1, 2, \ldots, (N+M-1). \tag{7}$$

Step 3. From the given nonuniform sample vector $y = x(\tau_m)$ of length N, form $(N-M+1)$ overlapping subvectors, each of length M, such that

$$y^{(r)} = [y(r), y(r+1), \ldots, y(M+r-1)]^{\mathrm{T}},$$
$$r = 1, 2, \ldots, (N+M-1). \tag{8}$$

Step 4. Solve for vector $\hat{x}^{(r)} = \hat{x}^{(r)}(nT)$ from the matrix equation:

$$y^{(r)} = A^{(r)} x^{(r)} \tag{9}$$

either by using a direct inverse or Gaussian elimination method. In the above $(\hat{\cdot})$ denotes the estimate of the quantity.

Step 5. Form the solution vector $\hat{x}$ from $\hat{x}^{(r)}$, $r = 1,2,\ldots,(N + M - 1)$, as

$$\hat{x} = [\hat{x}(1),\ldots,\hat{x}^{(1)}(\beta),\hat{x}^{(2)}(\beta),\ldots,\hat{x}^{(N-M)}(\beta), \hat{x}^{(N-M+1)}(\beta),\ldots,\hat{x}^{(N-M+1)}(M)]^{T}, \quad (10)$$

where $\beta = (M + 1)/2$. The rationale for forming the vector $\hat{x}$ as in Eq. (10) is as follows: we observe from Eqs. (7)–(9) that there are multiple solutions for each of the elements of vector x (except for the first and last $(M - 1)/2$) and these are available as elements of vectors $\hat{x}^{(r)}$. Out of these multiple solutions for each of the elements of x, *we propose to choose the middle element of each of* $\hat{x}^{(r)}$ *as the solution element*. The number of such solution elements is $(N - M + 1)$. The first and last $[(M - 1)/2]$ elements of $\hat{x}$ are, however, taken from $\hat{x}^{(1)}$ and $\hat{x}^{(N+M-1)}$, respectively. Note that this procedure of using the overlapped submatrices $A^{(r)}$ instead of A and choosing the middle element of each of $\hat{x}^{(r)}$ to form $\hat{x}$ is in essence the one suggested in [7], whereas the recovery of the first and last $(M - 1)/2$ elements of $\hat{x}$ is the modification proposed in the present paper.

To see how well the above procedure recovers the uniform samples from a set of nonuniform samples, let us consider the following example.

EXAMPLE 1. A single sinusoid with frequency $f = 4300$ Hz was considered. The sampling interval was chosen as $T = 10^{-4}$ s and the number of samples considered was $N = 50$. A jitter value of $J = 1$ was considered for generating the nonuniform sample positions. A uniform random sequence of instants in the range $\pm J/2$ was generated and the random sampling points deviating from the uniform sample positions were identified. Then the matrix A was formed with elements as in Eq. (4). The set of nonuniform samples y were obtained from the uniform sample vector x using the equation $y = Ax$. Then the proposed method was applied with a submatrix size $M = 7$ to estimate $\hat{x}$.

Fig. 3 shows that the recovered signal from which we observe that $\hat{x}$ is almost same as x. To

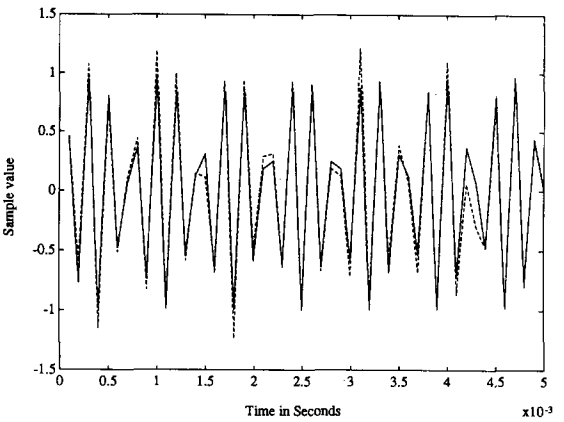

Fig. 3. Uniform and recovered samples. Signal: single sinusoid of frequency 4300 Hz. Sampling time: 10^{-4} s. Jitter value: 1. (——): equally spaced samples of the signal. (– – –) equally spaced samples, recovered from the nonequally spaced samples, using the proposed method.

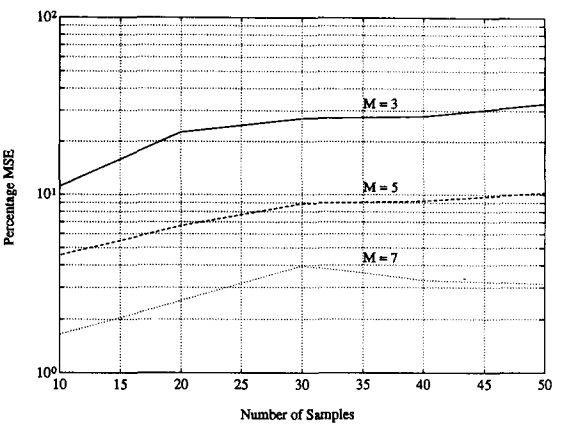

Fig. 4. Performance of the Proposed Method. Signal: single sinusoid of frequency 4300 Hz. Sampling time: 10^{-4} s. Jitter value: 1.

quantify the closeness of the recovered signal with the original one, we use the MSE, η, as the criterion:

$$\eta = \|x - \hat{x}\|^2 / \|x\|^2, \quad (11)$$

where $\|\cdot\|$ denotes the L_2 norm of the vector. In Fig. 4 we have plotted η as a function of the number of samples for various submatrix sizes, from which we note that the MSE is about 4% for a data length of 30 for an $M = 7$ and the MSE increases as

M takes values as 5 and 3. This is expected because, with a smaller dimension of the submatrix, one will not be able to capture the elements of A that have significant magnitude.

2.2.1. Computational complexity

In spite of the substantial error in the recovery of the uniform samples, the proposed method is attractive from the computational complexity point of view. Assuming that we use the Gaussian elimination to solve the matrix equation, we would need μ_D multiplications, where

$$\mu_D = (2N^3 + 3N^2 - 5N + 6)/6, \tag{12}$$

for a data length of N [8], whereas the number of multiplications in the proposed technique will be

$$\mu_P = (N - M + 1)(2M^3 + 3M^2 - 5M + 6)/6. \tag{13}$$

To compare the number of multiplications in the proposed method with that of the direct solution we computed μ_D and μ_P for $N = 50$ and $M = 7$. The number of multiplications in each case are $\mu_P = 5896$, whereas $\mu_D = 42876$.

Although the proposed method has an advantage in terms of the computations, the MSE may become unacceptably large as the jitter value J increases especially for the case $J > 1$. In order to reduce the MSE, in what follows we apply an iterative algorithm.

2.3. Iterative improvement

The proposed method of Section 2.2 {Ref. Steps 1–4} estimates the vector $\hat{x}$ from A and y. Once $\hat{x}$ is available one can obtain an estimate of $\hat{y}$ based on given values of $\hat{x}$:

$$\hat{y} = A\hat{x} \tag{14}$$

and the error in the estimation of y as

$$e_y = y - \hat{y}. \tag{15}$$

Note that the error in y is a direct result of the error in x. Thus, expressing y and $\hat{y}$ in terms of x and $\hat{x}$, respectively, we obtain an estimate of the error in $\hat{x}$ as

$$e_y = Ae_x. \tag{16}$$

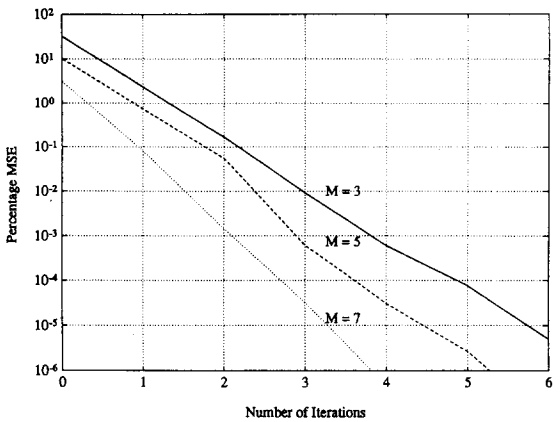

Fig. 5. Mean squared error in the proposed method. Signal: single sinusoid of frequency 4300 Hz. Sampling time: 10^{-4} s. Jitter value: 1. Number of samples: 50.

Thus, the error in the estimation of x is related to the error e_y (which can be computed) and the relation is same as the matrix equation (1). This enables us to obtain an estimate of e_x, using steps 2–5 of Section 2.2 and correct the error in $\hat{x}$ as

$$\hat{x}_2 = \hat{x}_1 + \hat{e}_x, \tag{17}$$

where the subscript indicates the iteration number. The steps in Eqs. (14)–(17) together with those in Section 2.2 could be repeated till the normalized error $\|\hat{e}_x\|/\|\hat{x}\|$ is reduced to a small acceptable value ε.

To see how the iterations, suggested above, help in reducing the MSE we again consider the signal of Example 1. Fig. 5 shows the MSE(η) as a function of iteration index k for submatrix sizes $M = 3$, 5 and 7. An iteration index 0 is for the basic recovery steps (7)–(10). From Fig. 5 we note that $\eta \leqslant 10^{-2}$ even for an $M = 3$ and $k = 3$. Thus, the proposed method together with the iterations recovers the uniform samples with a very low MSE.

2.3.1. Additional computations

Recall Eq. (13) where the number of multiplications μ_P required for Steps 2–5 of Section 2.2 was given. The iterative technique of Section 2.3 requires μ_I additional multiplications for each iteration, where μ_I is given by

$$\mu_I = N^2 + \mu_P. \tag{18}$$

In (18), N^2 is due to the multiplications required in obtaining $\hat{y}$ and μ_P is for estimating $\hat{e}_x$. Thus, the total number of multiplications for the whole procedure (basic recovery plus k iterations) is given by

$$\mu = kN^2 + (k+1)\mu_P . \tag{19}$$

In what follows we establish that the proposed method together with the iterative procedure does converge to the true solution of x, subject to certain conditions.

2.4. Convergence analysis

Recall the matrix equation (9). Let the solution to these linear equations be written as

$$x^{(r)} = A^{(r)} y^{(r)} , \tag{20}$$

where

$$A^{(r)} = [A^{(r)}]^{-1} . \tag{21}$$

It is then easy to see that the solution vector $\hat{x}$ of (10) can be expressed as

$$\hat{x} = Ty , \tag{22}$$

where

$$T = HP. \tag{23}$$

In Eq. (23) H is an $N \times (MN)$ matrix given by

$$H = \begin{bmatrix} A^{(1)}(1) & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{(1)}(\beta-1) & 0 & \cdots & \cdots & 0 & 0 \\ A^{(1)}(\beta) & 0 & \cdots & \cdots & 0 & 0 \\ 0 & A^{(2)}(\beta) & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & A^{(N-M)}(\beta) & 0 \\ 0 & 0 & \cdots & \cdots & 0 & A^{(N-M+1)}(\beta) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & A^{(N-M+1)}(M) \end{bmatrix} , \tag{24}$$

where $A^{(r)}(i)$ denotes ith row of the matrix $A^{(r)}$. In (23), the matrix P is an $(MN) \times N$ matrix given by

$$P = \begin{bmatrix} I_M & 0 & 0 & \cdots & 0 \\ z_M & I_M & 0 & \cdots & 0 \\ z_M & z_M & I_M & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_M & z_M & z_M & \cdots & I_M \end{bmatrix} , \tag{25}$$

with I_M as an $M \times M$ identity matrix and z_M as a column vector of M zeros. Recall Eq. (17) where the estimate $\hat{x}$ is corrected by an estimate of the error $\hat{e}_x$. Using Eq. (22) in (17) the new estimate of x can be expressed as

$$\hat{x}_1 = Ty + T(y - \hat{y}) . \tag{26}$$

Using the relations between y, $\hat{y}$, x and $\hat{x}$, it is easy to show that

$$\hat{x}_1 = T \sum_{i=0}^{1} (I - AT)^i Ax , \tag{27}$$

where I is an $N \times N$ identity matrix. It can be shown after some work that the estimate $\hat{x}_k$ at the kth iteration is given by

$$\hat{x}_k = T \sum_{i=0}^{k} (I - AT)^i Ax . \tag{28}$$

The terms in the summation of above equation form a geometric series and Eq. (28) can be further simplified to give

$$\hat{x}_k = T[I - (I - AT)^k] T^{-1} x. \tag{29}$$

One may note here that T is a full rank matrix and hence invertible. We can further simplify Eq. (29) to give

$$\hat{x}_k = x - (I - TA)^k x , \tag{30}$$

which shows that $\hat{x}_k$ approaches x as k increases, provided all the eigenvalues of $(I - TA)$ are less than unity in the magnitude sense [10]. Thus, the proposed method has a conditional convergence.

The complexity of the matrices T and A does not permit us to draw any conclusions on the behavior of the eigenvalues of $(I - TA)$. However, we took the help of computer simulations to look at the nature of these eigenvalues for a given A and T which in turn is a function of the submatrix size

Table 1
Number of trials in which the maximum eigenvalue of $(I - TA)$ is less than unity

	$N = 50$			$N = 100$			$N = 150$		
J	$M = 3$	$M = 5$	$M = 7$	$M = 3$	$M = 5$	$M = 7$	$M = 3$	$M = 5$	$M = 7$
0.5	100	100	100	100	100	100	100	100	100
1.0	100	100	100	100	100	100	100	100	100
1.5	100	100	100	100	100	100	99	100	100
2.0	69	99	100	37	97	100	32	98	100
2.5	7	83	100	1	60	95	0	48	92

M and jitter value J. One hundred trials, each with a different random sequence for the nonuniform sample instants, were conducted for various values of N, M and J and the results are presented in Table 1. The entries show the number of trials in which the maximum eigenvalue of $(I - TA)$ is less than unity. We note from these results that the iterations converge in most of the cases and one may have to use a larger submatrix (M) as J is increased.

3. Performance evaluation

In this section we present some computer simulation results to evaluate the performance of the proposed technique. The criterion used for the evaluation is the MSE in recovering the uniform samples from the given nonuniform set. We also compare these results with those obtained with the recently proposed Marvasti's technique [5].

3.1. Results on MSE

Recall Eqs. (11) and (30). We can express the MSE as

$$\eta = \|(I - TA)^k x\|^2 / \|x\|^2. \tag{31}$$

From (31) we note that η is a function of the matrices T and A and the number of iterations k. The important parameters that govern the properties of T and A are (i) the matrix dimension N, (ii) the jitter parameter J and (iii) the submatrix size M. To bring out clearly the inter relations among these parameters and the dependence of η on k, we assume a tolerance limit ε on η, and find the combinations of M, J and k (for a given N) that satisfy the condition $\eta \leqslant \varepsilon$.

For these simulations we consider three types of signals:

- Single sinusoid $x_{ss}(t) = \cos(2\pi ft)$, $f = 4300$ Hz.
- A chirp signal $x_{ch}(t) = \cos(2\pi(f_1 t + ((f_2 - f_1)/2nT)t^2))$, where $f_1 = 100$ Hz and $f_2 = 4500$ Hz.
- Low-pass random signal $x_{lp}(t)$, obtained by filtering a white Gaussian sequence in a low-pass filter with cut-off frequency 4300 Hz.

The rationale behind selecting the above set of signals is to demonstrate the capability of the technique for various cases: A single sinusoid is for the monotone close to the maximum frequency. A chirp signal is to simulate a case where the signal covers the entire low-pass band up to maximum frequency. Chirp signal, although covers the entire frequency band, it has some regularity in its structure. To represent a real-world signal, a low-pass random signal was also considered. In all these cases the sampling interval was chosen as $T = 10^{-4}$ s. A set of 50 uniform samples were drawn from the above signals and nonuniform samples were obtained for various jitter values. The technique proposed in Sections 2.2 and 2.3 was used to recover the uniform samples and the MSE was computed for each iteration. The iterations were stopped when the condition, $\eta \leqslant \varepsilon$ ($\varepsilon = 0.1\%$), was met. In Fig. 6 we have plotted k_{min} (minimum number of iterations required for $\eta \leqslant \varepsilon$) as a function of J for $M = 3, 5$ and 7, respectively. From these plots we note that the results are almost identical for the three types of signals considered, and for a moderate jitter value $J = 1$ the number of iterations could at the most be 3 for $M = 3$.

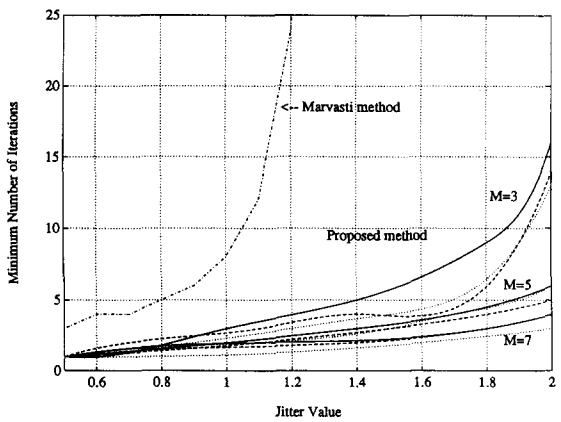

Fig. 6. Performance of the proposed method. Sampling time: 10^{-4} s. Number of samples: 50. (——): single sinusoid; (– – –): chirp signal; ··· : low-pass random signal.

3.2. Comparison with Marvasti's technique

Marvasti et al. [5] have applied an iterative method for the recovery of uniform samples from their nonuniform counterparts, using the following procedure:

$$\boldsymbol{x}_{k+1} = \lambda PS\boldsymbol{x} + (P - \lambda PS)\boldsymbol{x}_k\,, \tag{32}$$

where λ, $\boldsymbol{x}$ and $\boldsymbol{x}_k$ are the convergence constant, the original finite energy signal and the recovered signal at the kth iteration, respectively. P and S are the band limiting and ideal nonuniform sampling operators. $PS\boldsymbol{x}$ is the given nonuniform sample vector. It was also shown in [5] that the iterations converge for $0.5 \leqslant \lambda \leqslant 1$. To be in tune with the notation of the present paper P is taken as an identity matrix (as the signals considered are already band-limited) and $S = A$. Thus, (32) can be rewritten as

$$\boldsymbol{x}_{k+1} = \lambda \boldsymbol{y} + (I - \lambda A)\boldsymbol{x}_k\,. \tag{33}$$

The iterations of (33) with $\lambda = 0.95$ were applied for the three types of signals considered earlier and it was found that the results are identical for all the three signals. The plots of $k_{\min}$ are superimposed in Fig. 6 from which we observe that $k_{\min}$ is large even for a jitter value $J = 1.2$.

It is easy to see that the number of multiplications in Marvasti's technique are kN^2 which is comparatively less than the proposed method for a given k (see Eq. (19)). However, the $k_{\min}$ for the same MSE is less for the proposed method, and hence it appears to be advantageous computationally also.

4. Conclusions

In this paper we have addressed the problem of reconstruction of signals from their nonequally spaced samples. Exploiting the close-to-band structure of the composing matrix, a two-stage procedure for the recovery of uniform samples from nonuniform samples has been investigated. The first step consists of the special procedure to solve a set of linear equations to estimate the uniform sample vector. This procedure utilizes a set of overlapping submatrices to reduce the dimensionality of the system of equations thus reducing the computational complexity. Then the error in the estimate is reduced by applying an iterative procedure. It has been shown that the proposed technique converges conditionally to the true solution. The simulation results, however, show that these conditions are weak and may be implemented easily. Computer simulation results have been presented which show that the proposed technique performs well even for deviations of nonuniform sample positions well beyond the corresponding uniform positions. The proposed method is attractive from a computational point of view also.

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