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Equivalent Electrical Network for the Transversely Vibrating Uniform Bar

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The equivalent network for the transversely vibrating uniform bars under various end conditions are given, the value of each element of the networks is determined, and graphs showing the frequency characteristics of each element in the matrices for the bars are given. Some considerations such as the effect of the neglect of the distributed mass are given. The analysis by the method of normalized functions is briefly explained by citing a few examples.

INTRODUCTION

A TREATMENT by network theory of the linear mechanical-vibrating system is very useful for not only the analysis but also the synthesis of electromechanical and electroacoustical instruments.

Recently, many papers on analyses and treatments of vibration bars by the analogy of impedance or mobility have been reported.[1–8] Especially, Bishop has published the results of his excellent work in collected form.[1]

This paper deals with the equivalent networks for the transversely vibrating bars under various end conditions. The value of each element of the networks is determined so as to be convenient for practical use. Then, the equivalent circuits for the bars of which the distributed mass is neglected are shown; and, by employing the equivalent circuits, the possibility of easily analyzing some problems in the strength of materials is pointed out.

The paper, moreover, explains how resonating bars can be easily analyzed by simple circuits, using the normalized function for vibration modes.

These results are very useful for analyzing not only a unit mechanical system but also a composite one combined with unit systems. Furthermore, graphs showing the frequency characteristics of each element in the impedance or admittance matrix on the bar are given.

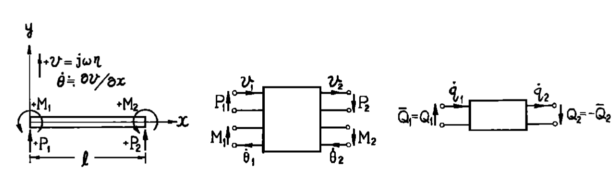
In this paper, the method of impedance analogy is adopted for analysis.

I. EXPRESSION BY MATRICES $\mathfrak{Z}$, $\mathfrak{Y}$, AND Γ

A. Open-Circuit Impedance Matrix $\mathfrak{Z}$

In this section, the relation between each element of the matrix $\mathfrak{Z}$ and the short-circuit admittance matrix $\mathfrak{Y}$ is explained by means of the new function F.

The equivalent network of the transversely vibrating bar shown in Fig. 1 (a) is that in Fig. 1 (b), and can be



(a) (b) (c)

Fig. 1. (a) Uniform bar vibrating transversely. (b) Equivalent four-terminal-pair network. (c) Equivalent two-terminal-pair network.

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[7] T. Yasuda, "Multi-Terminal-Network Treatment of Linear, Mechanically-Vibrating Systems," J. Inst. Elec. Commun. Engrs. Japan **37**, 37–42, 109–113 (1954).

[8] M. Yonekawa, "The Analysis of Frame Vibrating Systems," Elec. Commun. Lab. Tech. J. **9**, No. 6, 653–680 (1960).

more simply represented as in Fig. 1 (c) and by Eqs. 1.

$$|Q_1|=\begin{vmatrix}P_1\\M_1\end{vmatrix}=|\bar{Q}_1|,\quad |Q_2|=\begin{vmatrix}P_2\\M_2\end{vmatrix}=-|\bar{Q}_2|,$$
$$|\dot{q}_1|=\begin{vmatrix}v_1\\\dot{\theta}_1\end{vmatrix},\qquad |\dot{q}_2|=\begin{vmatrix}v_2\\\dot{\theta}_2\end{vmatrix},\tag{1}$$

where P, M, $v(=j\omega\eta)$, and $\dot{\theta}(\doteqdot\partial v/\partial x)$ are the impressed force, the impressed bending moment, the velocity of displacement, and the velocity of angular displacement, respectively; η is the displacement and θ is the angular displacement; Q is the general impressed force, $\bar{Q}$ is the general terminal force, and $\dot{q}$ is the general terminal velocity.

The fundamental equation is written with the aid of network theory as Eq. 2, and each element of the matrix $\mathfrak{Z}$ is concisely and systematically expressed in Eq. 3 by employing the functions F defined in Eqs. 4 and 5 following cf., Appendix A:

$$\begin{vmatrix}Q_1\\Q_2\end{vmatrix}=\begin{vmatrix}\mathfrak{Z}_{11}&\mathfrak{Z}_{12}\\\mathfrak{Z}_{21}&\mathfrak{Z}_{22}\end{vmatrix}\begin{vmatrix}\dot{q}_1\\\dot{q}_2\end{vmatrix},\tag{2}$$

$$\begin{vmatrix}P_1\\M_1\\P_2\\M_2\end{vmatrix}=\begin{vmatrix}z_{11}&\mathbf{Z}_{11}&z_{12}&\mathbf{Z}_{12}\\\overline{\mathbf{Z}}_{11}&Z_{11}&\overline{\mathbf{Z}}_{12}&Z_{12}\\z_{21}&\mathbf{Z}_{21}&z_{22}&\mathbf{Z}_{22}\\\overline{\mathbf{Z}}_{21}&Z_{21}&\overline{\mathbf{Z}}_{22}&Z_{22}\end{vmatrix}\begin{vmatrix}v_1\\\dot{\theta}_1\\v_2\\\dot{\theta}_2\end{vmatrix}=j(\rho SK)^{\frac{1}{2}}\begin{vmatrix}F_a/l&F_b&F_c/l&-F_d\\F_b&-F_e\cdot l&F_d&F_f\cdot l\\F_c/l&F_d&F_a/l&-F_b\\-F_d&F_f\cdot l&-F_b&-F_e\cdot l\end{vmatrix}\begin{vmatrix}v_1\\\dot{\theta}_1\\v_2\\\dot{\theta}_2\end{vmatrix},\tag{3}$$

where

$$\left\{\begin{aligned}
F_a&=\frac{\alpha(\mathrm{S\cdot c+C\cdot s})}{(\mathrm{C\cdot c}-1)}=\frac{\alpha H_6}{H_3},\\
F_b&=\frac{\mathrm{S\cdot s}}{(\mathrm{C\cdot c}-1)}=\frac{H_1}{H_3},\\
F_c&=\frac{-\alpha(\mathrm{s+S})}{(\mathrm{C\cdot c}-1)}=\frac{-\alpha H_7}{H_3},\\
F_d&=\frac{(\mathrm{c-C})}{(\mathrm{C\cdot c}-1)}=\frac{H_{10}}{H_3},\\
F_e&=\frac{(\mathrm{S\cdot c-C\cdot s})}{\alpha(\mathrm{C\cdot c}-1)}=\frac{H_5}{\alpha H_3},\\
F_f&=\frac{-(\mathrm{s-S})}{\alpha(\mathrm{C\cdot c}-1)}=\frac{-H_8}{\alpha H_3},\\
&\mathrm{S:\sinh\alpha},\quad \mathrm{C:\cosh\alpha},\quad \mathrm{s:\sin\alpha},\quad \mathrm{c:\cos\alpha}.
\end{aligned}\right.\tag{4}$$

$$\left\{\begin{aligned}
H_1&=\sinh\alpha\cdot\sin\alpha\equiv\mathrm{S\cdot s}, & H_6&=\sinh\alpha\cdot\cos\alpha+\cosh\alpha\cdot\sin\alpha\\
H_2&=\cosh\alpha\cdot\cos\alpha\equiv\mathrm{C\cdot c}, & &\equiv\mathrm{S\cdot c+C\cdot s},\\
H_3&=\cosh\alpha\cdot\cos\alpha-1\equiv\mathrm{C\cdot c}-1, & H_7&=\sin\alpha+\sinh\alpha\equiv\mathrm{s+S},\\
H_4&=\cosh\alpha\cdot\cos\alpha+1\equiv\mathrm{C\cdot c}+1, & H_8&=\sin\alpha-\sinh\alpha\equiv\mathrm{s-S},\\
H_5&=\sinh\alpha\cdot\cos\alpha-\cosh\alpha\cdot\sin\alpha & H_9&=\cos\alpha+\cosh\alpha\equiv\mathrm{c+C},\\
&\equiv\mathrm{S\cdot c-C\cdot s}, & H_{10}&=\cos\alpha-\cosh\alpha\equiv\mathrm{c-C}.
\end{aligned}\right.\tag{5}$$

$$\alpha^4=(\rho S/K)\omega^2l^4,\quad (K\alpha^2)/(j\omega l^2)=(\rho SK)^{\frac{1}{2}}/j.\tag{6}$$

In Eqs. 6, $K=EI$ expresses the bending strength, E is Young's modulus, and I is the second moment of area of the cross section about the neutral axis, and ρ, S, and l indicate the density, the sectional area, and the length of the bar, respectively.

B. Short-Circuit Admittance Matrix $\mathfrak{Y}$

Fundamental equations by the matrix $\mathfrak{Y}$ are shown in Eqs. 7 and 8 due to the network theory; that is,

$$\begin{vmatrix} \dot{q}_1 \\ \dot{q}_2 \end{vmatrix} = \begin{vmatrix} \mathfrak{Y}_{11} & \mathfrak{Y}_{12} \\ \mathfrak{Y}_{21} & \mathfrak{Y}_{22} \end{vmatrix} \begin{vmatrix} Q_1 \\ Q_2 \end{vmatrix}, \tag{7}$$

$$\begin{vmatrix} v_1 \\ \dot{\theta}_1 \\ v_2 \\ \dot{\theta}_2 \end{vmatrix} = \begin{vmatrix} y_{11} & \bar{\mathbf{Y}}_{11} & y_{12} & \bar{\mathbf{Y}}_{12} \\ \mathbf{Y}_{11} & Y_{11} & \mathbf{Y}_{12} & Y_{12} \\ y_{21} & \bar{\mathbf{Y}}_{21} & y_{22} & \bar{\mathbf{Y}}_{22} \\ \mathbf{Y}_{21} & Y_{21} & \mathbf{Y}_{22} & Y_{22} \end{vmatrix} \begin{vmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{vmatrix} = [j(\rho SK)^{\frac{1}{2}}]^{-1} \begin{vmatrix} F_e \cdot l & F_b & F_f \cdot l & -F_d \\ F_b & -F_a/l & F_d & F_c/l \\ F_f \cdot l & F_d & F_e \cdot l & -F_b \\ -F_d & F_c/l & -F_b & -F_a/l \end{vmatrix} \begin{vmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{vmatrix}. \tag{8}$$

C. Transfer Matrix $\mathbf{\Gamma}$

Fundamental equations by the matrix $\mathbf{\Gamma}$ are shown in Eqs. 9 and 10 as mentioned in Appendix A; that is,

$$\begin{vmatrix} \bar{Q}_1 \\ \dot{q}_1 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} \bar{Q}_2 \\ \dot{q}_2 \end{vmatrix}, \tag{9}$$

$$\begin{vmatrix} \bar{P}_1 \\ \bar{M}_1 \\ v_1 \\ \dot{\theta}_1 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_3 & D_4 \end{vmatrix} \begin{vmatrix} \bar{P}_2 \\ \bar{M}_2 \\ v_2 \\ \dot{\theta}_2 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} H_9 & -H_8\dfrac{\alpha}{l} & -H_7\dfrac{K\alpha^3}{j\omega l^3} & -H_{10}\dfrac{K\alpha^2}{j\omega l^2} \\ H_7\dfrac{l}{\alpha} & H_9 & H_{10}\dfrac{K\alpha^2}{j\omega l^2} & -H_8\dfrac{K\alpha}{j\omega l} \\ H_8\dfrac{j\omega l^3}{K\alpha^3} & H_{10}\dfrac{j\omega l^2}{K\alpha^2} & H_9 & -H_7\dfrac{l}{\alpha} \\ -H_{10}\dfrac{j\omega l^2}{K\alpha^2} & H_7\dfrac{j\omega l}{K\alpha} & H_8\dfrac{\alpha}{l} & H_9 \end{vmatrix} \begin{vmatrix} \bar{P}_2 \\ \bar{M}_2 \\ v_2 \\ \dot{\theta}_2 \end{vmatrix}. \tag{10}$$

II. BAR WITH END CONDITIONS

As to the one end of the bar, four ideal end conditions are to be considered—that is, the free end ($P=M=0$), the simple support ($v=0$), the sliding support ($\dot{\theta}=0$), and the fixed end ($v=\dot{\theta}=0$).

In Table I are shown the equivalent networks of the bars under various end conditions and the value of each element is given.

III. PURE STIFFNESS BAR

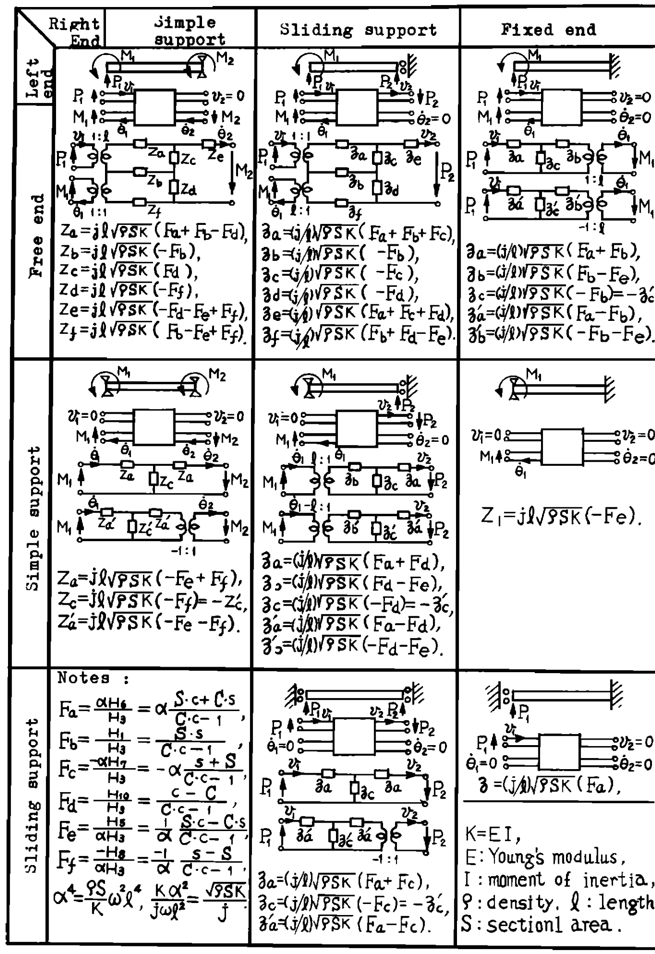
The fundamental equations for a stiffness bar, such as its distributed mass is neglected, are simply shown from Eqs. 3 and 10 as follows:

$$\begin{vmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{vmatrix} = \frac{K}{j\omega} \begin{vmatrix} 12/l^3 & 6/l^2 & -12/l^3 & 6/l^2 \\ 6/l^2 & 4/l & -6/l^2 & 2/l \\ -12/l^3 & -6/l^2 & 12/l^3 & -6/l^2 \\ 6/l^2 & 2/l & -6/l^2 & 4/l \end{vmatrix} \begin{vmatrix} v_1 \\ \dot{\theta}_1 \\ v_2 \\ \dot{\theta}_2 \end{vmatrix}, \tag{11}$$

or

$$\begin{vmatrix} P_1 \\ M_1/l \\ P_2 \\ M_2/l \end{vmatrix} = \frac{K}{j\omega l^3} \begin{vmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{vmatrix} \begin{vmatrix} v_1 \\ \dot{\theta}_1 \cdot l \\ v_2 \\ \dot{\theta}_2 \cdot l \end{vmatrix}, \tag{12}$$

TABLE I. Equivalent networks of the bars with end conditions.



$$\begin{vmatrix} P_1 \\ \bar{M}_1 \\ v_1 \\ \dot{\theta}_1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ l & 1 & 0 & 0 \\ \dfrac{-j\omega l^3}{6K} & \dfrac{-j\omega l^2}{2K} & 1 & -l \\ \dfrac{j\omega l^2}{2K} & \dfrac{j\omega l}{K} & 0 & 1 \end{vmatrix} \begin{vmatrix} \bar{P}_2 \\ \bar{M}_2 \\ v_2 \\ \dot{\theta}_2 \end{vmatrix}. \quad (13)$$

In Table II are shown the equivalent circuits for the stiffness bars with end conditions. It is found that some problems on the beam treated in the strength of materials can be considered as a problem of the network theory. Like Table I, this Table will be also useful for the design of the composite mechanical filter.

IV. EQUIVALENT NETWORKS BY THE NORMALIZED FUNCTION

A. Bar Impressed with Forces and Bending Moments

If the forces (P_{X_1}, P_{X_2}) and the bending moments (M_{X_1}, M_{X_2}) are impressed at any points (X_1, X_2) of the transversely vibrating uniform bar as shown in Fig. 2, then the following equations are given:

$$\begin{vmatrix} \begin{vmatrix} v_{(X_i)} \\ \dot{\theta}_{(X_i)} \end{vmatrix} \\ \begin{vmatrix} v_{(P_j)} \\ \dot{\theta}_{(X_j)} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} y_{X_iX_i} & \bar{\mathbf{Y}}_{X_iX_i} \\ \mathbf{Y}_{X_iX_i} & Y_{X_iX_i} \end{vmatrix} & \begin{vmatrix} y_{X_iX_j} & \bar{\mathbf{Y}}_{X_iX_j} \\ \mathbf{Y}_{X_iX_j} & Y_{X_iX_j} \end{vmatrix} \\ \begin{vmatrix} y_{X_jX_i} & \bar{\mathbf{Y}}_{X_jX_i} \\ \mathbf{Y}_{X_jX_i} & Y_{X_jX_i} \end{vmatrix} & \begin{vmatrix} y_{X_jX_j} & \bar{\mathbf{Y}}_{X_jX_j} \\ \mathbf{Y}_{X_jX_j} & Y_{X_jX_j} \end{vmatrix} \end{vmatrix} \begin{vmatrix} \begin{vmatrix} P_{X_i} \\ M_{X_i} \end{vmatrix} \\ \begin{vmatrix} P_{X_j} \\ M_{X_j} \end{vmatrix} \end{vmatrix}, \quad (14)$$

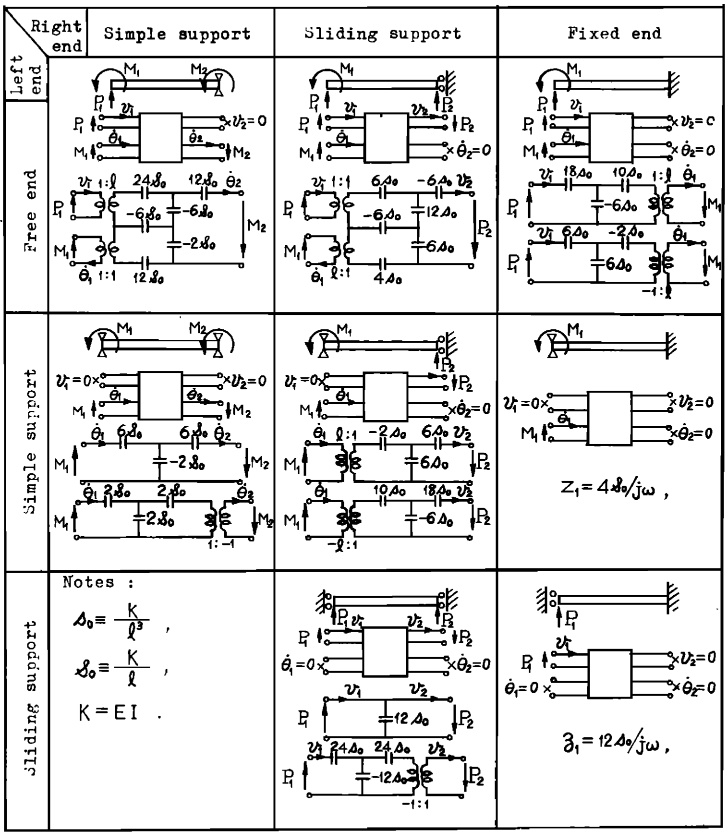


TABLE II. Equivalent networks of the stiffness bars with end conditions.

$$\begin{vmatrix} y_{X_iX_j} & \bar{\mathbf{Y}}_{X_iX_j} \\ \mathbf{Y}_{X_iX_j} & Y_{X_iX_j} \end{vmatrix} = Z_{0m}^{-1} \begin{vmatrix} \sum_m \Xi_{m(X_i)}\cdot\Xi_{m(X_j)}, & \frac{1}{l}\sum_m \Xi_{m(X_i)}\cdot\Xi_{m(X_j)}' \\ \frac{1}{l}\sum_m \Xi_{m(X_i)}'\cdot\Xi_{m(X_j)}, & \frac{1}{l^2}\sum_m \Xi_{m(X_i)}'\cdot\Xi_{m(X_j)}' \end{vmatrix}$$

$$= \sum_m \mathfrak{Y}_m, \tag{15}$$

where

$$\begin{cases} X = x/l, & Z_{0m}^{-1} = j\omega/[(\omega_m^2 - \omega^2)(\rho S l)], \\ \omega_m = (\alpha_m^2/l^2)(K/\rho S)^{\frac{1}{2}}, & \Xi_{m(X)}' = [\partial \Xi_{m(X)}]/\partial X. \end{cases} \tag{16}$$

And $\Xi_{m(X)}$ is the normalized function of the mth mode of the vibrating bar, and $m = 0, 1, 2, 3, \cdots, \infty$.

The sum of the short admittance matrices $\sum_m \mathfrak{Y}_m$ means that m networks are to be connected in parallel to one another, and therefore the resultant equivalent network is shown as Fig. 3.

In Fig. 4 is shown the equivalent network only for the mth vibration mode of the bar. And this simple equivalent circuit is very convenient for practical use, because the vibrating bars are frequently used in such a certain vibration mode as the mth. Another example is shown in Fig. 5.

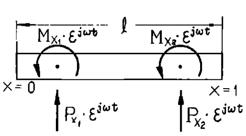


FIG. 2. Uniform bar impressed with forces and bending moments at any two points.

B. Bar with Loaded z_{X_j} and Z_{X_j}

In the case of the bar with load impedances, the shearing-force load z_{X_j} and the bending-moment load Z_{X_j} are to be put as follows: $P_{X_j} = -z_{X_j}\cdot v_{(X_j)}$, and $M_{X_j} = -Z_{X_j}\cdot\dot{\theta}_{(X_j)}$.

A few examples are shown in Figs. 6 and 7.

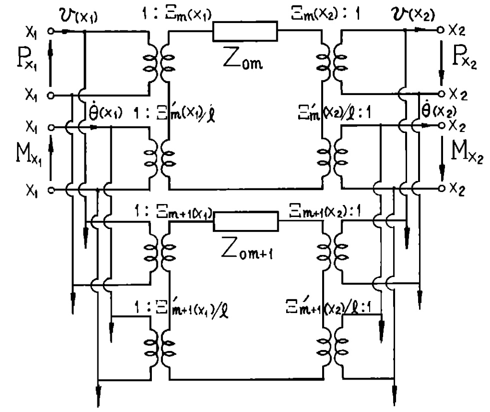


FIG. 3. General expression by the equivalent four-terminal-pair network on the bar shown in Fig. 2.

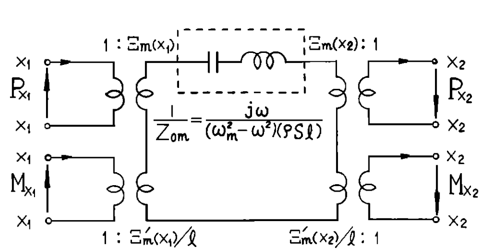


FIG. 4. Equivalent network of only the mth mode of vibration on the bar shown in Fig. 2.

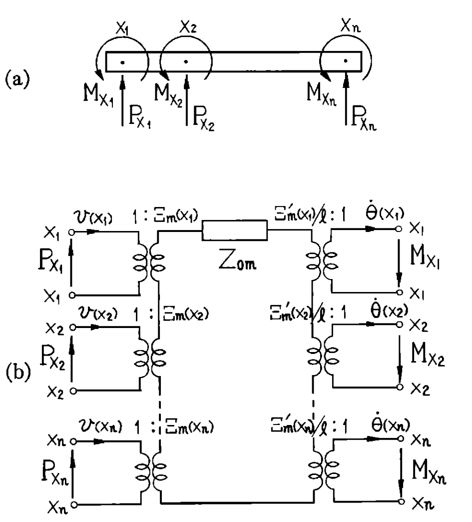


FIG. 5. (a) Bar impressed with forces and bending moments at any points $X_j (j=1, 2\ldots, n)$. (b) Equivalent $2\times n$ terminal-pair network of only the mth mode of vibration.

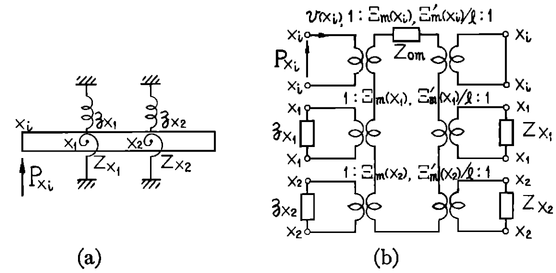


FIG. 6. (a) Bar impressed with a force P_{X_i} and loaded with z_{X_j} and Z_{X_j}. (b) Equivalent one-terminal-pair network.

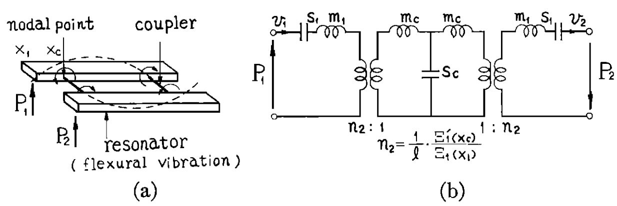


FIG. 7. (a) Example of the composite mechanical filter consisting of resonators and couplers. (b) Equivalent network.

V. CONCLUSION

In this paper, the fundamental equations on the transversely vibrating uniform bar are systematically presented and the equivalent networks of the bar under various end conditions are given, and then the value of each element of the networks is decided. Some considerations on the bar of which the distributed mass is neglected are also shown.

Moreover, an outline on the analysis by the method of normalized function is described by citing a few examples.

General considerations in the first half of this paper are available for an analysis of a composite mechanical system, while the method of normalized function in the second is very convenient for design of a resonator and the like.

Although considerations only on a unit mechanical system are explained in this paper, these results are developed to analysis and synthesis of a composite system combined with unit ones.

ACKNOWLEDGMENT

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Appendix A. Analysis by Matrices

This section describes a proof of the expression by matrix $\mathbf{\Gamma}$. And matrices $\mathfrak{Z}$ and $\mathfrak{Y}$ are deduced from the matrix $\mathbf{\Gamma}$.

The general terminal force $\bar{Q}$ and the general terminal velocity $\dot{q}$ shown in Fig. 1(c) are written as Eqs. A1

$$|\dot{q}|=\begin{vmatrix}\dot{\eta}\\ \dot{\theta}\end{vmatrix},\quad |\bar{Q}|=\begin{vmatrix}\bar{P}\\ \bar{M}\end{vmatrix}=\begin{vmatrix}\dfrac{K}{j\omega}\cdot\dfrac{\partial^3\dot{\eta}_{(x)}}{\partial x^3}\\ -\dfrac{K}{j\omega}\cdot\dfrac{\partial^2\dot{\eta}_{(x)}}{\partial x^2}\end{vmatrix}=\begin{vmatrix}\dfrac{K}{j\omega l^3}\cdot\dfrac{\partial^3\dot{\eta}_{(X)}}{\partial X}\\ -\dfrac{K}{j\omega l^2}\cdot\dfrac{\partial^2\dot{\eta}_{(X)}}{\partial X}\end{vmatrix},\tag{A1}$$

where $X=x/l$, $\theta\doteqdot\partial\eta/\partial x$, $\dot{q}=\partial q/\partial t$; and the notations η, θ, K, and so on, are indicated in Sec. I.

When the displacement η is considered as $\eta\cdot\epsilon^{j\omega t}$, the differential equation of the transverse vibration of the bar is written as follows:

$$K[\partial^4\eta_{(x)}/\partial x^4]+\rho S[\partial^2\eta_{(x)}/\partial t^2]=0,\tag{A2a}$$

or

$$[\partial^4\eta_{(X)}/\partial X^4]-\alpha^4\eta_{(X)}=0,\tag{A2b}$$

where

$$\alpha^4=(\rho S/K)\omega^2 l^4.\tag{A3}$$

Then, the solution of Eq. A2b is given as

$$\begin{aligned}\eta_{(X)}&=a_1\cos\alpha X+a_2\sin\alpha X+a_3\cosh\alpha X+a_4\sinh\alpha X\\ &\equiv a_1\mathrm{c}_X\quad+a_2\mathrm{s}_X\quad+a_3\mathrm{C}_X\quad+a_4\mathrm{S}_X,\end{aligned}\tag{A4}$$

where a_1, a_2, a_3, and a_4 are arbitrary constants.

From Eqs. A1 and A4, the following equations are obtained:

$$\begin{vmatrix}\bar{P}_X\\ \bar{M}_X\\ v_X\\ \dot{\theta}_X\end{vmatrix}=|H_X|\begin{vmatrix}a_1\\ a_2\\ a_3\\ a_4\end{vmatrix},\quad\text{or}\quad\begin{vmatrix}\bar{Q}_X\\ \dot{q}_X\end{vmatrix}=|H_X|\begin{vmatrix}a_n\\ a_m\end{vmatrix},\tag{A5}$$

where

$$|H_X|=\begin{vmatrix}\underline{\alpha}^3\mathrm{s}_X & -\underline{\alpha}^3\mathrm{c}_X & \underline{\alpha}^3\mathrm{S}_X & \underline{\alpha}^3\mathrm{C}_X\\ \underline{\alpha}^2\mathrm{c}_X & \underline{\alpha}^2\mathrm{s}_X & -\underline{\alpha}^2\mathrm{C}_X & -\underline{\alpha}^2\mathrm{S}_X\\ \mathrm{c}_X & \mathrm{s}_X & \mathrm{C}_X & \mathrm{S}_X\\ -\underline{\alpha}'\mathrm{s}_X & \underline{\alpha}'\mathrm{c}_X & \underline{\alpha}'\mathrm{S}_X & \underline{\alpha}'\mathrm{C}_X\end{vmatrix},\tag{A6}$$

$$\underline{\alpha}^n\equiv(K/j\omega)(\alpha/l)^n,\quad \underline{\alpha}'\equiv\alpha/l.\tag{A7}$$

If the input terminal ($X=0$) and the output terminal ($X=1$) are expressed with the suffices 1 and 2, respectively, then the following equations are deduced from Eqs. A5:

$$\begin{vmatrix}\bar{Q}_X\\ \dot{q}_X\end{vmatrix}=|H_x|\,|H_1|^{-1}\begin{vmatrix}\bar{Q}_1\\ \dot{q}_1\end{vmatrix}\tag{A8}$$

and, accordingly,

$$\begin{vmatrix}\bar{Q}_2\\ \dot{q}_2\end{vmatrix}=|\mathbf{\Gamma}_{2.1}|\begin{vmatrix}\bar{Q}_1\\ \dot{q}_1\end{vmatrix},\tag{A9}$$

$$\begin{vmatrix}\bar{Q}_1\\ \dot{q}_1\end{vmatrix}=|\mathbf{\Gamma}_{1.2}|\begin{vmatrix}\bar{Q}_2\\ \dot{q}_2\end{vmatrix},\tag{A10}$$

where

$$|\mathbf{\Gamma}_{1.2}|=|H_1|\;|H_2|^{-1}=|\;|H_2|\;|H_1|^{-1}\;|^{-1}=|\mathbf{\Gamma}_{2.1}|^{-1},\tag{A11}$$

$$|\mathbf{\Gamma}_{1.2}|=\tfrac{1}{2}\begin{vmatrix}(\mathrm{c}+\mathrm{C}) & -(\mathrm{s}-\mathrm{S})\underline{\alpha}' & -(\mathrm{s}+\mathrm{S})\underline{\alpha}^3 & -(\mathrm{c}-\mathrm{C})\underline{\alpha}^2\\ (\mathrm{s}+\mathrm{S})/\underline{\alpha}' & (\mathrm{c}+\mathrm{C}) & (\mathrm{c}-\mathrm{C})\underline{\alpha}^2 & -(\mathrm{s}-\mathrm{S})\underline{\alpha}\\ (\mathrm{s}-\mathrm{S})/\underline{\alpha}^3 & (\mathrm{c}-\mathrm{C})/\underline{\alpha}^2 & (\mathrm{c}+\mathrm{C}) & -(\mathrm{s}+\mathrm{S})/\underline{\alpha}'\\ -(\mathrm{c}-\mathrm{C})/\underline{\alpha}^2 & (\mathrm{s}+\mathrm{S})/\underline{\alpha} & (\mathrm{s}-\mathrm{S})\underline{\alpha}' & (\mathrm{c}+\mathrm{C})\end{vmatrix}.\tag{A12}$$

Consequently, the matrices $\mathfrak{Z}$ and $\mathfrak{Y}$ are deduced from Eq. A12 and written as Eqs. 3 and 8 in Sec. I.

Appendix B. Analysis by the Normalized Function $\Xi_{m(X)}$

An analysis by the normalized function on vibration mode of the transversely vibrating bar was already studied by T. Hayasaka[6] and amplified by T. Yasuda.[7]

Thereafter, the authors developed this analyzing method to a treatment for multiterminal networks.

When the forces P_{X_j} are impressed at any points $X_j(j=1,2,\cdots,n)$ of the uniform bar as shown in Fig. 2, the velocity of displacement $v_{(X_i)}$ at the point X_i is given as

$$v_{(X_i)}=\sum_m\frac{j\omega}{(\omega_m{}^2-\omega^2)(\rho Sl)}\Xi_{m(X_i)}\cdot\sum_{j=1}^{n}\Xi_{m(X_j)}\cdot P_{X_j}\tag{B1}$$

$$\equiv\sum_m\frac{1}{Z_{0m}}\Xi_{m(X_i)}\cdot\sum_{j=1}^{n}\Xi_{m(X_j)}\cdot P_{X_j}$$

$$\equiv(y_{X_iX_j})\cdot P_{X_j}.\tag{B2}$$

And then, the short admittances $(y_{X_iX_i})$ and $(y_{X_iX_j})$ are defined as the coefficient of P_{X_j} in Eq. B1.

For example, if only two forces P_{X_1} and P_{X_2} are impressed, the short admittances are shown as

$$\begin{cases} (y_{X_iX_i})=\sum_m \dfrac{1}{Z_{0m}}\Xi_{m(X_i)}\cdot\Xi_{m(X_i)}\equiv\sum_m(y_{mX_iX_i}), \\ (y_{X_iX_j})=\sum_m \dfrac{1}{Z_{0m}}\Xi_{m(X_i)}\cdot\Xi_{m(X_j)}\equiv\sum_m(y_{mX_iX_j}). \end{cases} \quad \text{(B3)}$$

By differentiating Eq. B1 with x, the velocity of angular displacement $\dot{\theta}_{(X_i)}(=\partial v_{(X)}/(\partial X\cdot l))$ is given as

$$\dot{\theta}_{(X_i)}=\sum_m \frac{1}{Z_{0m}\cdot l}\Xi_{m(X_i)}'\cdot\sum_{j=1}^{n}\Xi_{m(X_j)}\cdot P_{X_j}\equiv(\mathbf{Y}_{X_iX_j})\cdot P_{X_j}, \quad \text{(B4)}$$

where

$$\Xi_{m(X)}'=\partial\Xi_{m(X)}/\partial X. \quad \text{(B5)}$$

And similarly, the short admittances $(\mathbf{Y}_{X_iX_i})$ and $(\mathbf{Y}_{X_iX_j})$ are defined as the coefficient of P_{X_j} in Eq. B4.

Next, the bending moment M_{X_j} at the point X_j of the bar is substituted for

$$M_{X_j}=\lim_{\delta X\to 0} P_{X_j}\cdot l\cdot\delta X. \quad \text{(B6)}$$

Equation B6 means that the bending moment M_{X_j} is equal to the impression of two forces having such opposite signs as $+P_{X_j}$ and $-P_{X_j}$ at two points $(X_j+\frac{1}{2}\delta X)$ and $(X_j-\frac{1}{2}\delta X)$, respectively.

Accordingly, Eq. B7 is obtained.

$$\sum_{j=1}^{n}\left[\Xi_{m(X_j+\frac{1}{2}\delta X)}\cdot P_{X_j}-\Xi_{m(X_j-\frac{1}{2}\delta X)}\cdot P_{X_j}\right] \doteqdot \sum_{j=1}^{n}(P_{X_j}\cdot\delta X)\cdot\Xi_{m(X_j)}'=\sum_{j=1}^{n}\frac{M_{X_j}}{l}\Xi_{m(X_j)}'. \quad \text{(B7)}$$

Consequently, when the bending moments M_{X_j} are impressed, the velocities $v_{(X_i)}$ and $\dot{\theta}_{(X_i)}$ are written as follows:

$$v_{(X_i)}=\sum_m \frac{1}{Z_{0m}\cdot l}\Xi_{m(X_i)}\cdot\sum_{j=1}^{n}\Xi_{m(X_j)}'\cdot M_{X_j}\equiv(\bar{\mathbf{Y}}_{X_iX_j})\cdot M_{X_j}, \quad \text{(B8)}$$

$$_{(X_i)}=\sum_m \frac{1}{Z_{0m}\cdot l}\Xi_{m(X_i)}'\cdot\sum_{j=1}^{n}\Xi_{m(X_j)}'\cdot M_{X_j} \equiv(Y_{X_iX_j})\cdot M_{X_j}. \quad \text{(B9)}$$

The values of the normalized functions for various vibration bars are given by T. Hayasaka.[6]

Appendix C. Frequency Characteristics of Functions $F_a \sim F_f$

Diagrams of the calculated values of F functions defined in Eqs. 4 for the frequency constant α are shown in the Figs. C-1/C-6. In these Figures, the solid lines show the positive values and the dotted lines show the negative values of the functions.

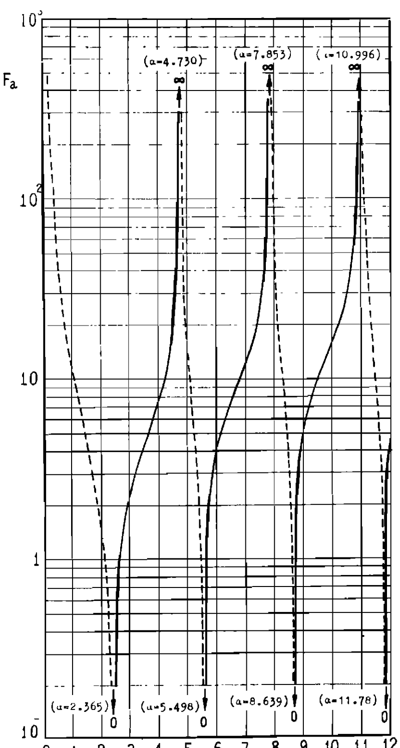

FIG. C-1. F_a.

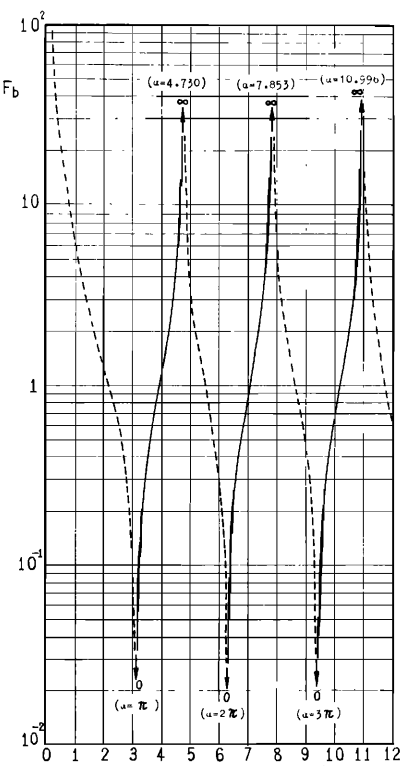

FIG. C-2. F_b.

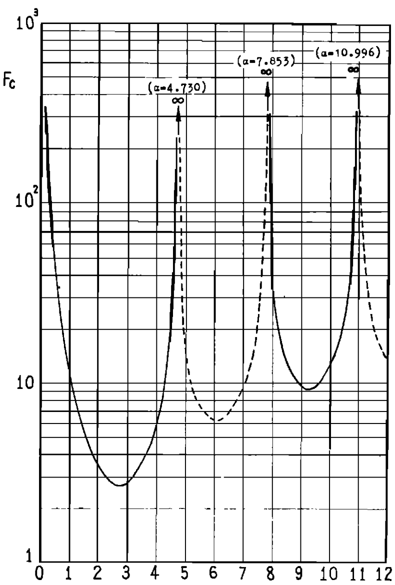

FIG. C-3. F_c.

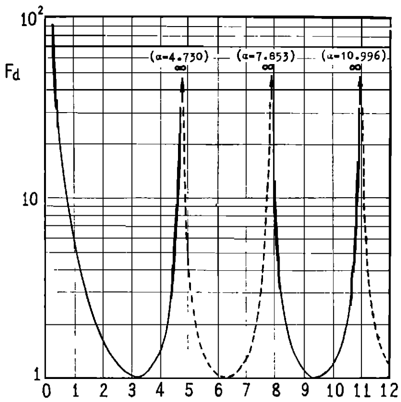

FIG. C-4. F_d.

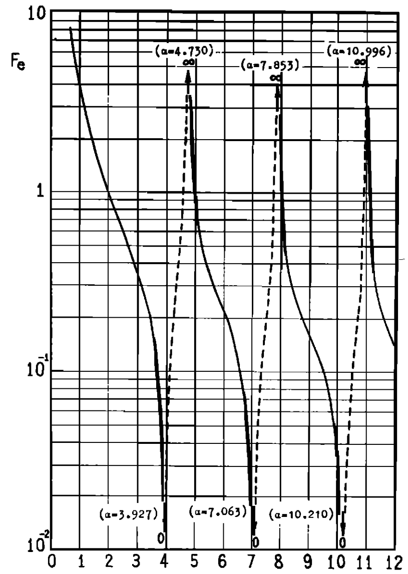

FIG. C-5. F_e.

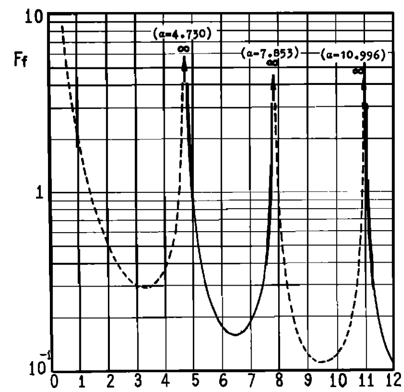

FIG. C-6. F_f.

Moreover, the physical meaning of F functions is given in Eqs. C1 or C2.

$$\begin{vmatrix} P_1 \\ M_1/l \\ P_2 \\ M_2/l \end{vmatrix} = \frac{j(\rho SK)^{\frac{1}{2}}}{l} \begin{vmatrix} F_a & F_b & F_c & -F_d \\ F_b & -F_e & F_d & F_f \\ F_c & F_d & F_a & -F_b \\ -F_d & F_f & -F_b & -F_e \end{vmatrix} \begin{vmatrix} v_1 \\ \dot{\theta}_1 \cdot l \\ v_2 \\ \dot{\theta}_2 \cdot l \end{vmatrix}, \qquad \text{(C1, cf. Eq. 3)}$$

$$\begin{vmatrix} v_1 \\ \dot{\theta}_1 \cdot l \\ v_2 \\ \dot{\theta}_2 \cdot l \end{vmatrix} = \frac{l}{j(\rho SK)^{\frac{1}{2}}} \begin{vmatrix} F_e & F_b & F_f & -F_d \\ F_b & -F_a & F_d & F_c \\ F_f & F_d & F_e & -F_b \\ -F_d & F_c & -F_b & -F_a \end{vmatrix} \begin{vmatrix} P_1 \\ M_1/l \\ P_2 \\ M_2/l \end{vmatrix}, \qquad \text{(C2, cf. Eq. 8)}$$