ANNALS OF PHYSICS: **24**, 118–173 (1963)

The Theory of a General Quantum System Interacting with a Linear Dissipative System

R. P. FEYNMAN

California Institute of Technology, Pasadena, California

AND

F. L. VERNON, JR.*

Aerospace Corporation, El Segundo, California

A formalism has been developed, using Feynman's space-time formulation of nonrelativistic quantum mechanics whereby the behavior of a system of interest, which is coupled to other external quantum systems, may be calculated in terms of its own variables only. It is shown that the effect of the external systems in such a formalism can always be included in a general class of functionals (influence functionals) of the coordinates of the system only. The properties of influence functionals for general systems are examined. Then, specific forms of influence functionals representing the effect of definite and random classical forces, linear dissipative systems at finite temperatures, and combinations of these are analyzed in detail. The linear system analysis is first done for perfectly linear systems composed of combinations of harmonic oscillators, loss being introduced by continuous distributions of oscillators. Then approximately linear systems and restrictions necessary for the linear behavior are considered. Influence functionals for all linear systems are shown to have the same form in terms of their classical response functions. In addition, a fluctuation-dissipation theorem is derived relating temperature and dissipation of the linear system to a fluctuating classical potential acting on the system of interest which reduces to the Nyquist-Johnson relation for noise in the case of electric circuits. Sample calculations of transition probabilities for the spontaneous emission of an atom in free space and in a cavity are made. Finally, a theorem is proved showing that within the requirements of linearity all sources of noise or quantum fluctuation introduced by maser-type amplification devices are accounted for by a classical calculation of the characteristics of the maser.

I. INTRODUCTION

Many situations occur in quantum mechanics in which several systems are coupled together but one or more of them are not of primary interest. Problems

---

* This report is based on a portion of a thesis submitted by F. L. Vernon, Jr. in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the California Institute of Technology, 1959.

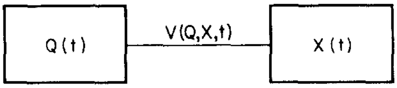

FIG. 1. General quantum systems Q and X coupled by a potential $V(Q, X, t)$

in the theory of measurement and in statistical mechanics present good examples of such situations. Suppose, for instance, that the quantum behavior of a system is to be investigated when it is coupled to one or more measuring instruments. The instruments in themselves are not of primary interest. However, their effects are those of perturbing the characteristics of the system being observed. A more concrete example is the case of an atom in an excited state which interacts with the electromagnetic field in a lossy cavity resonator. Because of the coupling there will be energy exchange between the field and the atom until equilibrium is reached. If, however, the atom were not coupled to any external disturbances, it would simply remain unperturbed in its original excited state. The cavity field, although not of central interest to us, influences the behavior of the atom.

To make the discussion more definite, let us suppose there are two nonrelativistic quantum systems whose coordinates are represented in a general way by Q and X, as in Fig. 1, coupled together through some interaction potential which is a function of the parameters of the two systems. It is desired to compute the expectation value of an observable which is a function of the Q variables only. As is well known, the complete problem can be analyzed by taking the Hamiltonian of the complete system, forming the wave equation as follows:

$$\{H(Q) + H(X) + V(Q, X)\}\psi(Q, X) = -\left(\frac{\hbar}{i}\right)\frac{\partial}{\partial t}\psi(Q, X),$$

and then finding its solution. In general, this is an extremely difficult problem. In addition, when this approach is used, it is not easy to see how to eliminate the coordinates of X and include its effect in an equivalent way when making computations on Q. A satisfactory method of formulating such problems as this in a general way was made available by the introduction of the Lagrangian formulation of quantum mechanics by Feynman. He applied the techniques afforded by this method extensively to studies in quantum electrodynamics. Thus, in a problem where several charged particles interact through the electromagnetic field, he found that it was possible to eliminate the coordinates of the field and recast the problem in terms of the coordinates of the particles alone. The effect of the field was included as a delayed interaction between the particles (*1*, *2*).

The central problem of this study is to develop a general formalism for finding all of the quantum effects of an environmental system (the interaction system) upon a system of interest (the test system), to investigate the properties of this formalism, and to draw conclusions about the quantum effects of specific interaction systems on the test system. Cases where the interaction system is composed of various combinations of linear systems and classical forces will be considered in detail. For the case in which the interaction system is linear, it will be found that parameters such as impedance, which characterize its classical behavior, are also important in determining its quantum effect on the observed system. Since this linear system may include dissipation, the results have application in a study of irreversible statistical mechanics.

In Section II, after a brief discussion of the Lagrangian formulation of quantum mechanics, a general formulation of the problem is made and certain functionals, called influence functionals, will be defined, which contain the effect of the interaction system (such as system X in Fig. 1) on the test system in terms of the coordinates of the test system only. General properties of these functionals will be derived and their relationship to statistical mechanics will be discussed. To obtain more specific information about the properties of the formalism, we then specialize the discussion to cases where well-defined systems are involved. In Section III, the special cases are considered in which the interaction system is a definite classical force and a random classical force. In Section IV, the influence functionals for exactly linear systems at zero temperatures are derived and then extended to the case that the linear systems are driven by classical forces. In addition, the effect of finite temperatures of linear systems is considered. Then, in Section V, the unobserved systems are again assumed to be general but weakly coupled to the observed system. Within the approximation of weak coupling these general systems also behave as if they were linear. Then finally in Section VI, the results of the analysis are used to prove a general result concerning maser noise.

It is to be emphasized that although we shall talk of general test and interaction systems, the Lagrangian formulation is restricted to cases involving momentum or coordinate operators. Therefore, strictly speaking, systems in which the spin is of importance are not covered by this analysis. However, this has no bearing on the results since their nature is such that their extension to the case where spins are important can be inferred.

An equivalent approach can be made to the problem using the Hamiltonian formulation of quantum mechanics by making use of the ordered operator calculus developed by Feynman (*3*). This approach has been used to some extent by Fano (*4*) and has been developed further by Hellwarth (*5*).[1] Some advantages of this method are that many results may be obtained more simply than by the Lagrangian method and nonclassical concepts such as spin enter the formalism naturally. However, the physical significance of the functions being dealt with are often clearer in the Lagrangian method.

[1] Many of the results obtained in this work have also been obtained by him using ordered operator techniques.

II. GENERAL FORMULATION—INFLUENCE FUNCTIONAL

A. Lagrangian Formulation of Quantum Mechanics

We shall begin the discussion with a brief introduction to the Lagrangian or space-time approach to quantum mechanics and the formal way in which one may set up problems of many variables.[2] Let us suppose that we are considering a single system which has coordinates that are denoted by Q, and that for the time being it is not acted on by any other quantum system. It can be acted on by outside forces, however. The system may be very complicated, in which case Q represents all the coordinates in a very general way. If at a time t the variable Q is denoted by Q_t, then the amplitude for the system to go from position Q_τ at $t = \tau$ to Q_T at $t = T$ is given by

$$K(Q_T, T; Q_\tau, \tau) = \int \exp[(i/\hbar)S(Q)]\mathfrak{D}Q(t) \tag{2.1}$$

in integral which represents the sum over all possible paths $Q(t)$ in coordinate space from Q_τ to Q_T of the functional $\exp[(i/\hbar)S(Q)]$.[3] $S(Q) = \int_\tau^T L(\dot{Q}, Q, t)\,dt$ is the action calculated classically from the Lagrangian for the trajectory $Q(t)$. For the case that Q is a single linear coordinate of position, this is represented in the diagram in Fig. 2. The magnitude of the amplitude for all paths is equal but the phase for each path is given by the classical action along that path in units of $\hbar$. Thus, amplitudes for neighboring paths which have large phases tend to cancel. The paths which contribute the greatest amount are those whose amplitudes have stationary phases for small deviations around a certain path. This is the path for which the classical action is at an extremum and is, therefore, the classical path. Remarkably enough, for free particles and harmonic oscillators, the result of the path integration is

$$K(Q_T, Q_\tau) = (\text{Smooth Function}) \exp[(i/\hbar)S_{\text{cl}}]$$

where S_{cl} is the action evaluated along the classical path between the two end points Q_τ, Q_T. However, for more complicated systems this simple relation does not hold. A discussion of the methods of doing integrals of this type is not included here since methods appropriate for the purposes here are already contained in the literature (*1*, *2*).

Since $K(Q_T, Q_\tau)$ is the amplitude to go from coordinate Q_τ to Q_T, it follows that at $t = T$ the amplitude that the system is in a state designated by $\phi_m(Q_T)$ when initially in a state $\phi_n(Q_\tau)$ is given by

$$\begin{aligned} A_{mn} &= \int \phi_m^*(Q_T)K(Q_T, Q_\tau)\phi_n(Q_\tau)\,dQ_T\,dQ_\tau \\ &= \int \phi_m^*(Q_T)\exp[(i/\hbar)S(Q)]\phi_n(Q_\tau)\mathfrak{D}Q(t)\,dQ_T\,dQ_\tau \end{aligned} \tag{2.2}$$

---

[2] For a more complete treatment, see ref. *1*.

[3] In subsequent equations $K(Q_T, T; Q_t, t)$ will be written $K(Q_T, Q_t)$.

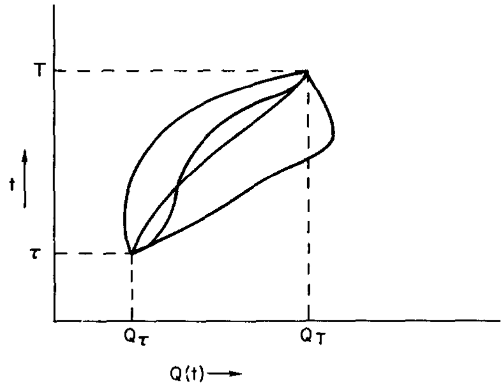

FIG. 2. Space-time diagram showing possible paths for particle to proceed from Q_τ to Q_T

The probability of the transition from $n \rightarrow m$ is given by $|A_{mn}|^2$ and from Eq. (2.2) this can be written in the form of multiple integrals as follows:

$$P_{mn} = \int \phi_m^*(Q_T)\phi_m(Q'_T) \exp\{(i/\hbar)[S(Q) - S(Q')]\} \times \phi_n(Q_\tau)\phi_n^*(Q'_\tau)\mathfrak{D}Q(t)\mathfrak{D}Q'(t)\, dQ_\tau\, dQ'_\tau\, dQ_T\, dQ'_T \tag{2.3}$$

As a example of a more complicated case let us consider two systems whose coordinates are Q and X.[4] The systems are coupled by a potential which can be designated as $V(Q, X)$ and incorporated in the total Lagrangian. We assume that when $V = 0$ the states of Q and X can be described by sets of wave functions $\phi_k(Q)$ and $\chi_p(X)$ respectively. If, initially, Q is in a state $\phi_n(Q_\tau)$ and X is in a state $\chi_i(X_\tau)$, then the amplitude that Q goes from state n to m while X goes from state i to f can be formed in a similar way to that of Eq. (2.2),

$$A_{mf,ni} = \int \phi_m^*(Q_T)\chi_f^*(X_T) \exp[(i/\hbar)S(Q, X)]\phi_n(Q_\tau)\chi_i(X_\tau) \times \mathfrak{D}Q(t)\mathfrak{D}X(t)\, dQ_\tau\, dX_\tau\, dQ_T\, dX_T \tag{2.4}$$

where $S(Q, X)$ represents the classical action of the entire system including both Q and X. The important property of separability afforded by writing the amplitude in this way is now apparent.[5] For instance, if one wishes to know the effect that X has on Q when X undergoes a transition from state i to f, then all

---

[4] Each system will be denoted by the coordinates that characterize it. Where Q or X means specifically a coordinate, it will be so designated by a statement if it is not obvious.

[5] If system Q represents a harmonic oscillator and the interaction of Q with X were linear and of the form $-\gamma(t, X)Q(t)$, then that part of Eq. (2.4) which involves the Q variables corresponds to the function G_{mn} defined and used by Feynman to eliminate the electromagnetic field oscillators. See ref. *2*.

of the integrals on the X variables may be done first. What is left is an expression for A_{mn} for Q and in terms of Q variables only but with the effect of X included. The extension of writing transition amplitudes for large numbers of systems is obvious. In principle the order in which the variables are eliminated is always arbitrary.

B. Definition of Influence Functional

A functional can now be defined which can be used to describe mathematically the effect of external quantum systems upon the behavior of a quantum system of interest.[6]

The fundamental theorem for this work may be stated as follows: For any system, Q, acted on by external classical forces and quantum mechanical systems as discussed above, the probability that it makes a transition from state $\psi_n(Q_\tau)$ at $t = \tau$ to $\psi_m(Q_T)$ at $t = T$ can be written

$$P_{mn} = \int \psi_m^*(Q_T)\psi_m(Q'_T) \exp\{(i/\hbar)[S_0(Q) - S_0(Q')]\}\mathfrak{F}(Q, Q') \\ \times \psi_n^*(Q'_\tau)\psi_n(Q_\tau)\mathfrak{D}Q(t)\mathfrak{D}Q'(t)\, dQ_\tau\, dQ'_\tau\, dQ_T\, dQ'_T \tag{2.5}$$

where $\mathfrak{F}(Q, Q')$ contains all the effects of the external influences on Q, and $S_0(Q) = \int_\tau^T L(\dot{Q}, Q, t)\, dt$, the action of Q without external disturbance. The proof of this is straightforward. Let us examine two coupled systems characterized by coordinates Q and X as represented diagrammatically in Fig. 1. Q will represent the test system and X the quite general interaction system, (excepting only the effects of spin) coupled by a general potential $V(Q, X, t)$ to Q. Assume Q to be initially $(t = \tau)$ in state $\psi_n(Q_\tau)$ and X to be in state $\chi_i(X_\tau)$, a product state. The probability that Q is found in state $\psi_m(Q_T)$ while X is in state $\chi_f(X_T)$ at $t = T$ can be written in the manner discussed above and is

$$\begin{aligned} P_{mf,ni} &= |A_{mf,ni}|^2 \\ &= \int \psi_m^*(Q_T)\psi_m(Q'_T)\chi_f^*(X_T)\chi_f(X'_T) \\ &\quad \times \exp\{(i/\hbar)[S_0(Q) - S_0(Q') + S(X) - S(X') + S_I(Q, X) \\ &\qquad - S_I(Q', X')]\} \\ &\quad \times \psi_n^*(Q'_\tau)\psi_n(Q_\tau)\chi_i^*(X'_\tau)\chi_i(X_\tau)\, dX_\tau \\ &\qquad \cdots dQ'_T\mathfrak{D}X'(t) \cdots \mathfrak{D}Q(t) \end{aligned} \tag{2.6}$$

The primed variables were introduced when the integrals for each $A_{mf,ni}$ were combined. Now if all of Eq. (2.6) which involves coordinates other than Q or

---

[6] Hereafter, the system of interest will be referred to as the test system. Conversely, the system not of primary interest will be called the interaction or environmental system.

Q' is separated out and designated as $\mathfrak{F}(Q, Q')$, then the following expression is obtained

$$\mathfrak{F}(Q, Q') = \int \chi_f^*(X_T)\chi_f(X'_T) \cdot \exp\{(i/\hbar)[S(X) - S(X') + S_I(Q, X) - S_I(Q', X')]\} \times \chi_i^*(X'_\tau)\chi_i(X_\tau)\, dX_\tau\, dX'_\tau \mathfrak{D}X(t)\mathfrak{D}X'(t) \tag{2.7}$$

Incorporation of this expression into Eq. (2.6) yields the desired form of Eq. (2.5). If the path integrals are written in terms of kernels, Eq. (2.7) becomes

$$\mathfrak{F}(Q, Q') = \int \chi_f^*(X_T)\chi_f(X'_T)K_Q(X_T, X_\tau)K^*_{Q'}(X'_T, X'_\tau) \times \chi_i^*(X'_\tau)\chi_i(X_\tau)\, dX_\tau \cdots dX'_T \tag{2.8}$$

where the subscript Q means that the kernel includes the effect of a potential $V(Q, X)$ acting on X during the interval $T > t > \tau$. As can be seen, $\mathfrak{F}$ is a functional whose form depends upon the physical system X, the initial and final states of X, and the coupling between Q and X.

It is to be emphasized that the formulation of $\mathfrak{F}$ is such that it includes all the effects of the interaction system in influencing the behavior of the test system. Thus, if there are two systems A and B which can act on Q, and if

$$\mathfrak{F}_{A \text{ on } Q} = \mathfrak{F}_{B \text{ on } Q},$$

then the effects of A on Q are the same as those of B on Q. It follows that if simplifying assumptions are necessary in finding $\mathfrak{F}_{A \text{ on } Q}$ and $\mathfrak{F}_{B \text{ on } Q}$ (due to the complicated nature of A and B) and if the resulting functions are equal, then within the approximations the effects of A and B on the test system are the same. In the situation where the interaction system is composed of a linear system or combinations of linear systems we shall see that the same form of $\mathfrak{F}$ is always appropriate. To adapt this general form of $\mathfrak{F}$ to a particular linear system it is only necessary to know such quantities as impedance and temperature which determine its classical behavior. In still other situations, very weak coupling between systems is involved. The approximate $\mathfrak{F}$ which can be used in this case to represent the effect of the interaction systems has a form which is independent of the nature of the interaction system. This form is the same as for linear systems. These cases will be considered in more detail in later sections.

C. General Properties of Influence Functionals

There are several general properties of influence functionals which are of interest and which will be useful in subsequent arguments. The first three of these (1, 2, 3) follow directly from the definition of $\mathfrak{F}(Q, Q')$. The last two (4, 5) will require more discussion.

1. If the physical situation is unsure (as for instance if the type of interaction system X, or the initial or final states are not known precisely) but if the probability of the pth situation is w_p and the corresponding influence functional is $\mathfrak{F}_p$, then the effective $\mathfrak{F}$ is given by

$$\mathfrak{F}_{\text{eff}} = \sum_p w_p \mathfrak{F}_p \equiv \langle \mathfrak{F} \rangle \tag{2.9}$$

Thus, in Eq. (2.6) if the initial state of X were not certain but the probability of each initial state were w_i, then P_{mn} for system Q would be given by $\sum_i w_i P_{mf,ni}$. Since the summation involves only the part of Eq. (2.6) involving the X variables, it is a sum over the influence functions for each possible initial state and results in an average influence functional of the type given above.

2. If a number of statistically and dynamically independent partial systems act on Q at the same time and if $\mathfrak{F}^{(k)}$ is the influence of the kth system alone, the total influence of all is given by the product of the individual $\mathfrak{F}^{(k)}$:

$$\mathfrak{F} = \prod_{k=1}^{N} \mathfrak{F}^{(k)} \tag{2.10}$$

Again referring to Eq. (2.6), if there were N subsystems interacting with Q, then the probability that Q makes a transition from state n to m while each of the subsystems makes a transition from its initial to its final state is given by an expression of the same form as Eq. (2.6). The difference in this case being that the term involving the X variables would be replaced by a product of N similar terms—one for each subsystem. Thus, when the term involving all the $X^{(k)}$ variables is separated out the complete influence functional is recognized as a product of the functionals $\mathfrak{F}^{(k)}(Q, Q')$ for each subsystem.

Definition: In many cases it will be convenient to write $\mathfrak{F}$ in the form $\exp[i\Phi(Q, Q')]$. Φ is then called the influence phase. For independent disturbances as considered in 2, the influence phases add. In the event that $i\Phi(Q, Q')$ is a real number we will continue to use the notation Φ; the phase simply becomes imaginary. It will frequently be more convenient to work with Φ rather than $\mathfrak{F}$.

3. The influence functional has the property that

$$\mathfrak{F}^*(Q, Q') = \mathfrak{F}(Q', Q) \tag{2.11}$$

Referring to Eq. (2.7), the definition of the influence functional, this fact follows immediately upon interchanging Q and Q'.

4. In the class of problems in which the final state of the interaction system is arbitrary, which means the final states are to be summed over, then $\mathfrak{F}(Q, Q')$ is independent of $Q(t)$ if $Q(t) = Q'(t)$ for all t. All of the problems we will be concerned with here are of this type.

The validity of this statement can be ascertained by observing Eq. (2.7), the general definition of the influence functional. In particular, for the case where the initial and final states of the interaction system X are i and f respectively, as in

Eq. (2.7), we denote the influence functional by ${}_f\mathfrak{F}_i(Q, Q')$. Let us assume we have no interest in the final state of X which means that ${}_f\mathfrak{F}_i(Q, Q')$ must be summed over all such states. The initial state i can be quite general. Thus, the influence functional for the case of an arbitrary final state is

$$\mathfrak{F}_i(Q, Q') = \textstyle\sum_f {}_f\mathfrak{F}_i(Q, Q')$$

For clarity in finding the result of letting $Q(t) = Q'(t)$ for all t in $\mathfrak{F}_i(Q, Q')$, we will write out the expression explicitly from Eq. (2.7). It is

$$\begin{aligned}\mathfrak{F}_i(Q, Q) = \int \textstyle\sum_f \chi_f^*(X_T)\chi_f(X'_T)& \\ \cdot\exp\{(i/\hbar)[S(X) - S(X') + S_I(Q, X) - S_I(Q, X')]\}& \\ \times \chi_i^*(X'_\tau)\chi_i(X_\tau)\, dX_\tau \cdots \mathfrak{D}X'(t)&\end{aligned}$$

Since Q appears in the interaction potentials acting on the X and X' variables respectively, it loses its identity as the coordinate of a quantum system and becomes just a number (which may be, of course, a function of time). Thus $S_I(Q, X)$ may be interpreted as the action of an external potential which drives the X system. The above expression then represents the probability that X, which is in state i initially, is finally in any one of its possible states after being acted on by an external potential (as, for instance, in Eq. (2.3) summed over the final states, m). This result is unity. We have then that $\mathfrak{F}(Q, Q) = 1$ and is independent of $Q(t)$.

5. A more restrictive statement of the property in the above paragraph (4) can be made. In this same class of problems in which the final states are summed over, if $Q(t) = Q'(t)$ for all $t > r$ then $\mathfrak{F}(Q, Q')$ is independent of $Q(t)$ for $t > r$. To see this we write down the influence functional from Eq. (2.8) breaking up the time interval into two parts, before and after r. Setting $Q = Q'$ for $t > r$ and utilizing the closure relation for the sum over final states we have,

$$\begin{aligned}\mathfrak{F}_r(Q, Q') = \int \delta(X_T - X'_T)K_Q(X_T, X_r)K_Q^*(X'_T, X'_r)& \\ \times K_Q(X_r, X_\tau)K^*_{Q'}(X'_r, X'_\tau)\chi_i^*(X'_\tau)\chi_i(X_\tau)\, dX_\tau \cdots dX'_T&\end{aligned}$$

Examining the parts of the above integral which contain the effects of $t > r$:

$$\begin{aligned}\int \delta(X_T - X'_T)K_Q(X_T, X_r)K_Q^*(X'_T, X'_r)\, dX_T\, dX'_T& \\ = \int K_Q(X'_r, X_T)K_Q(X_T, X_r)\, dX_T& \\ = K_Q(X'_r, X_r) = \delta(X_r - X'_r)&\end{aligned}$$

The expression for $\mathfrak{F}_r(Q, Q')$ becomes then

$$\begin{aligned}\mathfrak{F}_r(Q, Q') = \int \delta(X_r - X'_r)K_Q(X_r, X_\tau)K^*_{Q'}(X'_r, X'_\tau)& \\ \times \chi_i^*(X'_\tau)\chi_i(X_\tau)\, dX_\tau \cdots dX'_r,&\end{aligned}$$

which is independent of $Q(t)$ for $t > r$. As will be seen later in the specific case of linear systems, this leads to a statement of causality.

D. STATISTICAL MECHANICS

Finally it is appropriate to point out explicitly the significance of the influence functional in a study of quantum statistical mechanics. In the class of problems considered here we are only interested in making measurements on the test system and not on the interacting system. Thus, when the expectation value of an operator which acts only on the test system variables is taken, the final states of the interaction system must be summed over. It is equivalent to taking the expectation value of the desired operator in the test system and simultaneously the unit operator in the interaction system. Therefore, only the influence functional where the final states of the interaction system are summed over will be of interest to us.

Starting with the coordinate representation of the density matrix (*6*) for the test and interaction systems, $\rho(Q, X; Q', X')$, we will show the part played by the influence functional in obtaining an expression for $\rho(Q_T, Q'_T)$, that is, with the X coordinates eliminated, in terms of its value at an earlier time τ, $\rho(Q_\tau, Q'_\tau)$. First, we recall that the definition of ρ is as follows:

$$\rho(Q, X; Q', X') = \langle \psi(Q, X)\psi^*(Q', X')\rangle_{\text{av}} \tag{2.12}$$

where $\psi(Q, X)$ represents the wave function for one of the systems in an ensemble of systems each representing one of the possible states of the Q, X system (*7*). The average, represented by $\langle\ \rangle_{\text{av}}$, is taken over the ensemble. The trace of the density matrix is

$$\operatorname{Tr} \rho(Q, X; Q', X') = \textstyle\int\int \rho(Q, X; Q, X)\, dQ\, dX \tag{2.13}$$

and the expectation value of an operator A which operates on the Q variables only is

$$\langle A\rangle = \textstyle\int\int\int \rho(Q, X; Q', X) A(Q, Q')\, dQ\, dQ'\, dX \tag{2.14}$$

In the above

$$\begin{aligned} A(Q', Q) &= \textstyle\sum_{i,j} A_{ij}\phi_i^*(Q)\phi_j(Q'), \\ A_{ij} &= \textstyle\int \phi_i^*(Q) A \phi_j(Q)\, dQ, \end{aligned} \tag{2.15}$$

and $\phi_i(Q)$ is one of a set of complete orthonormal eigenfunctions. From Eq. (2.14) then we see that the formal expression which we wish to derive is

$$\textstyle\int \rho(Q_T, X_T; Q'_T, X_T)\, dX_T = \rho(Q_T, Q'_T) \tag{2.16}$$

in terms of $\rho(Q_\tau, Q'_\tau)$. From the rules given in Section II.A for propagation of a wave function with time we can easily find ρ_T in terms of ρ_τ. Thus

$$\begin{aligned}\rho(Q_T, X_T; Q'_T, X'_T) = \int \exp\{(i/\hbar)[S_0(Q) &- S_0(Q') \\ + S(X) - S(X') + S_I(Q, X) &- S_I(Q', X')]\} \\ \times\, \rho(Q_\tau, X_\tau; Q'_\tau, X'_\tau)&\mathcal{D}Q(t) \cdots dX'_\tau\end{aligned} \tag{2.17}$$

Now, for simplicity let us assume that initially the two systems are independent so that

$$\rho(Q_\tau, X_\tau; Q'_\tau, X'_\tau) = \rho(Q_\tau, Q'_\tau)\rho(X_\tau, X'_\tau)$$

Then eliminating the X_T coordinate as indicated in Eq. (2.16) we have

$$\begin{aligned}\rho(Q_T, Q'_T) = \int\{\int \delta(X_T - X'_T) \exp[(i/\hbar)[S(X) &- S(X') \\ + S_I(Q, X) - S_I(Q', X')]]&\rho(X_\tau, X'_\tau) \\ \times\, \mathcal{D}X(t) \cdots dX'_\tau\} \exp[(i/\hbar)[S_0(Q) - S_0(Q')]]&\rho(Q_\tau, Q'_\tau)\mathcal{D}Q(t) \cdots dQ'_\tau\end{aligned}$$

The expression inside the braces is identified as $\mathcal{F}(Q, Q')$ for the case in which the final state of X is summed over. Therefore, the following result is obtained:

$$\begin{aligned}&\rho(Q_T, Q'_T) \\ &= \int \mathcal{F}(Q, Q') \exp[(i/\hbar)[S_0(Q) - S_0(Q')]]\rho(Q_\tau, Q'_\tau)\mathcal{D}Q(t) \cdots dQ'_\tau\end{aligned} \tag{2.18}$$

Thus, if the density matrix of the test system Q is represented by $\rho(Q_\tau, Q'_\tau)$ at some initial instant τ, the density matrix $\rho(Q_T, Q'_T)$ at some later time T is given by Eq. (2.18). The entire influence of the interaction system is contained in $\mathcal{F}(Q, Q')$.

E. Use of Influence Functionals

At this point we need to consider how influence functionals can be used in the analysis of a problem. For clarity the discussion will be specialized to a particular problem but the principle is valid more generally. Suppose we wish to know the probability that a test system Q makes a transition from an initial state $\phi_n(Q_\tau) \exp[(-i/\hbar)E_n\tau]$ to a final state $\phi_m(Q_T) \exp[(-i/\hbar)E_mT]$ when coupled to an interaction system. The formal expression for this probability is, from Eq. (2.5),

$$\begin{aligned}P_{nm} = \int \phi_m^*(Q_T)\phi_m(Q'_T) \exp\{(i/\hbar)[S_0(Q) &- S_0(Q')]\}\mathcal{F}(Q, Q') \\ &\times \phi_n^*(Q'_\tau)\phi_n(Q_\tau)\, dQ_\tau \cdots dQ'(t)\end{aligned} \tag{2.19}$$

This is formally exact but except in special cases it cannot be evaluated exactly. Furthermore, to obtain any specific answers to the problem the characteristics of Q must be known as well as knowing the influence functional. However, by using perturbation theory we may find general expressions for transition probabilities to as many orders as desired. For example, if the interaction system is a linear system at zero temperature, we will find that $\mathfrak{F}_0(Q, Q')$ is of the form $\exp[i\Phi_0(Q, Q')]$. The perturbation expansion is obtained by writing $\exp[i\Phi_0(Q, Q')]$ in terms of a power series and evaluating the path integral corresponding to each term in the expansion. In many cases the coupling between Q and the interaction system is small enough that only a few terms in the expansion are necessary. In Appendix I the basic procedure for finding the perturbation expansion is demonstrated by finding the specific expression up to second order in the potentials involved for transition probability of a test system when acted on by a linear interaction system at zero temperature. Calculation of transition probabilities represent only one piece of information that one might desire to know about a test system. For instance, it is more usually desired to find the expectation value of an operator in the test system. To calculate this one needs to know the density matrix describing the test system when it is coupled to an interaction system. The exact expression for the required density matrix is given in Section II,D. Again in the general case, one runs into the difficulty of making an exact calculation and is forced to make calculations using perturbation theory. The same procedure of expanding the influence functional into a power series and performing the required path integrations yields useful perturbation expressions.

III. INFLUENCE FUNCTIONALS FOR CLASSICAL POTENTIALS

In this section we will derive specific forms and properties of influence functionals for the effects of classical potentials on the test system. These represent the simplest form of influence functionals and their properties follow directly from the general properties obtained in the previous section. These forms will then be extended to the case where the classical potential represents Brownian noise.

A. PROPERTIES OF INFLUENCE FUNCTIONALS FOR CLASSICAL POTENTIALS

The first step is to find the influence functional for a definite classical potential acting on the test system, Q. If the potential energy term in the Lagrangian is of the form $V(Q, t)$, then it can be ascertained readily by referring to the fundamental definition of $\mathfrak{F}(Q, Q')$ that

$$\mathfrak{F}(Q, Q') = \exp\{-(i/\hbar)\int_\tau^T[V(Q, t)-V(Q', t)]\,dt\} \tag{3.1}$$

or equivalently the influence phase is

$$\Phi(Q, Q') = -(1/\hbar)\int_\tau^T[V(Q, t)-V(Q', t)]\,dt \tag{3.2}$$

The next degree of complication is to have several potentials, $\sum_k V_k(Q, t)$ acting on Q simultaneously. However, since the sum of all these potentials represents an equivalent potential, say $V(Q, t) = \sum_k V_k(Q, t)$, then it is obvious that the total influence functional $\mathfrak{F}(Q, Q')$ is the product of the individual $\mathfrak{F}_k(Q, Q')$. More specifically,

$$\begin{aligned} \mathfrak{F}(Q, Q') &= \exp\{-(i/\hbar)\textstyle\int_\tau^T[V(Q, \quad t)-V(Q', \quad t)]\,dt\} \\ &= \exp \textstyle\sum_k\{-(i/\hbar)\int_\tau^T[V_k(Q, \quad t)-V_k(Q', \quad t)]\,dt\} \\ &= \textstyle\prod_k \mathfrak{F}_k(Q, Q'), \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} \Phi(Q, Q') &= -(1/\hbar)\textstyle\sum_k\int_\tau^T[V_k(Q, \quad t)-V_k(Q', \quad t)]\,dt \\ &= \textstyle\sum_k\Phi_k(Q, Q') \end{aligned} \tag{3.4}$$

The same result follows directly from Section II.C.3 which gives the total influence functional for several statistically and dynamically independent systems acting on Q. The total influence functional for all the systems (in this case potentials) is the product of the functionals for the individual systems.

Another property of the classical influence functionals is obtained by inspection of Eq. (3.1). We notice that for any classical $\mathfrak{F}(Q, Q')$ if conditions are such that $Q(t) = Q'(t)$, then $\mathfrak{F}(Q, Q') = 1$ and is independent of t for all times that the two variables are equal. It follows that the influence phase is zero for this condition.

Finally, from Section II.C.1 we find that if the potential is uncertain but the probability of each $V_r(Q, t)$ is w_r then the average functional is given by

$$\begin{aligned} \langle\mathfrak{F}(Q, Q')\rangle &= \textstyle\sum_r w_r \exp\{-(i/\hbar)\int_\tau^T[V_r(Q, \quad t)-V_r(Q', \quad t)]\,dt\} \\ &= \textstyle\sum_r w_r\mathfrak{F}_r(Q, Q') \end{aligned} \tag{3.5}$$

In the following Sections we will assume a probability distribution w_r appropriate to Brownian noise and will be able to derive a specific form for the average influence functional.

B. SPECIFIC FUNCTIONALS FOR RANDOM POTENTIALS

Let us now suppose that the potential has known form, $V(Q)$, but unknown strength $C(t)$ as a function of time so that the total potential is $V(Q, t) = C(t)V(Q)$. The average influence functional for two cases involving this type of potential will be particularly useful in the discussion contained in Sections IV and V. These cases are: (1) when $C(t)$ is characterized by any coupling strength (average magnitude of C) with a purely Gaussian distribution, and (2) when $C(t)$ is composed of large number of very weak potentials (acting on the test system simultaneously) whose distributions are stationary but not necessarily Gaussian.

1. *Gaussian Noise*

First, we consider the situation when $C(t)$ is Gaussian noise with a power spectrum $\Phi(\nu)$ and a correlation function $R(\tau) = (2/\pi)\int_0^\infty \Phi(\nu) \cos \nu\tau \, d\nu$ then $\langle \mathfrak{F} \rangle$ is given by

$$\begin{aligned} \langle \mathfrak{F} \rangle &= \langle \exp \{ (i/\hbar) \int_\tau^T C(t)[V(Q)-V(Q')] \, dt \} \rangle \\ &= \exp \{ -\hbar^{-2} \int_\tau^T \int_\tau^t R(t-s)[V(Q_t) - V(Q'_t)][V(Q_s) - V(Q'_s)] \, ds \, dt \} \end{aligned} \tag{3.6}$$

Expressed in Fourier transform notation this becomes

$$\langle \mathfrak{F} \rangle = \exp \{ -(\pi\hbar^2)^{-1} \int_0^\infty \phi(\nu) \mid V_\nu(Q) - V_\nu(Q') \mid^2 d\nu \} \tag{3.7}$$

where

$$V_\nu(Q) - V_\nu(Q') = \int_\tau^T [V(Q) - V(Q')] e^{-i\nu t} dt \tag{3.8}$$

Expressions of the type given in Eq. (3.6) are common for operations in which it is required to find the characteristic function, $F(i\zeta) = \langle e^{i\zeta f(T)} \rangle$ for $f(T)$ represented by integrals of the form $f(T) = \int_{a(T)}^{b(T)} A(T, t) x(t) \, dt$ where $x(t)$ is a Gaussian process. The result will not be worked out here as it may be found in standard references (*8*).[7] The equivalent expression for $\mathfrak{F}$ in terms of frequency components, Eq. (3.7), is obtained from Eq. (3.6) in a direct manner using the definitions for $R(t)$ and Eq. (3.8).

2. *Brownian Noise*

The Gaussian behavior of Brownian noise, characterized by the typical Gaussian probability distribution, may be the result of the cumulative effects of many small statistically independent sources, none of which is truly Gaussian. How that comes about can be seen as follows. The effect of these small sources on a test system may be represented by an influence functional of the same form as that of Eq. (3.6) where now $C(t) = \sum_{i=1}^N C_i(t)$, N is a very large number, and the $C_i(t)$ are independent random variables. Application of the central-limit theorem to this situation shows that the probability distribution appropriate to $C(t)$ is asymptotically normal subject to the following conditions (*9*):

(a) The average values,

$$\langle C_i(t) \rangle < \infty$$

and

$$\mu_{i,2} \equiv \langle | C_i(t) - \langle C_i(t) \rangle |^2 \rangle < \infty,$$

---

[7] See, for example, pp. 372–373 where it is shown that the characteristic function $F(i\xi)$ appropriate to the integral given above for a Gaussian process $x(t)$ is $\exp[-\frac{1}{2}\xi^2 \int\int_{a(T)}^{b(T)} \cdot A(T, t) A(T, s) K_x(t, s) \, ds \, dt]$ where the covariance $K_x(t, s) = \langle x_t x_s \rangle$ is the correlation function corresponding to $R(t - s)$ in Eq. (3.6).

(b) The absolute moments

$$\mu_{i,2+\delta} \equiv \langle | C_i(t) - \langle C_i(t) \rangle |^{2+\delta} \rangle$$

exist for some $\delta > 0$, and

(c) Making use of the definition

$$\mu_l = \textstyle\sum_{i=1}^{N} \mu_{i,l} ,$$

then

$$\lim_{N \to \infty} \frac{\mu_{2+\delta}}{(\mu_2)^{1+\delta/2}} \to 0$$

The condition of independence on the large number of variables and the finite average values required by (a) and (b) above assures that no one component dominates the total distribution. Condition (c) is sufficient to ensure that all higher order correction terms tending to deviate from a normal distribution vanish in the limit of large N. It should be recognized that if the C_i possesses finite third moments $\mu_{i,3}$ the correction terms arising from these moments decrease as $N^{-1/2}$. However, for the cases in which we are interested, the number of the component forces C_i is essentially infinite and higher order terms are negligible.

IV. INFLUENCE FUNCTIONALS FOR LINEAR SYSTEMS

Linear systems are of considerable interest both because of the large number of situations in which they are involved and because they are amenable to exact calculation. In this section the influence functional for arbitrary combinations of oscillators will be found by direct extension of the analysis of a single oscillator. All linear systems which are lossless and those which contain certain kinds of loss can be represented by distributions of oscillators. Situations in which dissipation arises from sources other than distributions of perfect oscillators will be covered in Section V. The same conclusions apply for all linear systems, however, as will be discussed subsequently. For clarity, we will restrict our attention initially to linear interaction systems at zero temperature and not acted on by classical forces. The effects of finite temperature and forces can then be included so that their significance is more apparent.

A. Zero Temperature Linear System

The result to be proven involves the assumption that the interaction system (X) is linearly coupled to the test system (Q). The total Lagrangian for the system is

$$L_{\text{total}} = L_0(\dot{Q}, Q, t) + L(\dot{X}, X, t) + L_I(Q, X) \tag{4.1}$$

where $L_I = \gamma QX$, and $L(\dot{X}, X, t)$ is the part of the Lagrangian involving the X system above. The situation is the same as that shown in Fig. 1 except the interaction potential is given by $V(Q, X, t) = -\gamma QX$ and the X system is linear. Our fundamental theorem for linear systems is as follows:

The influence phase for the effect of X on Q can be written as follows:

$$\Phi(Q\,Q') = \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{Q'_\nu(Q_{-\nu} - Q'_{-\nu})}{(i\nu Z_\nu)} + \frac{Q_{-\nu}(Q_\nu - Q'_\nu)}{(-i\nu Z_{-\nu})}\right] d\nu \tag{4.2}$$

$\Phi_0(Q, Q')$ is found by studying the properties of X alone.[8] *Q_ν is the Fourier transform of $\gamma(t)Q(t)$ and Z_ν is a classical impedance function which relates the reaction of X to an applied force. Z_ν is found by taking the classical system corresponding to X (that is, whose Lagrangian is $L(\dot{X}, X, t)$) and finding the response of the coordinate X to a driving force $f(t)$ which is derived from the potential $-f(t)X(t)$. $f(t)$ is considered to be applied at $T = 0$ subject to the initial conditions that $X(0) = \dot{X}(0) = 0$. Z_ν is defined by the expression*

$$Z_\nu = f_\nu/(i\nu X_\nu) \tag{4.3}$$

where

$$f_\nu = \int_0^\infty f(t)e^{-i\nu t}\,dt \quad \text{and} \quad X_\nu = \int_0^\infty X(t)e^{-i\nu t}\,dt$$

In the time domain, Eq. (4.2) can be expressed as

$$\begin{aligned} i\Phi(Q, Q') = -(1/2\hbar)\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{t} \gamma_t\gamma_s(Q_t - Q'_t)& \\ \cdot[Q_sF^*(t-s) - Q'_sF(t-s)]\,ds\,dt& \end{aligned} \tag{4.4}$$

where Im $F(t)$, which we will call $B(t)$ is, for $t > 0$, the classical response of X to a force $f(t) = \delta(t)$. Re $F(t)$, which for this zero temperature case we call $A_0(t)$, is the correlation function for the zero point fluctuation of the variable X, a point discussed at more length below. The relations connecting these quantities are then.

$$\begin{aligned} F(t) &= A_0(t) + iB(t) \\ \frac{1}{i\nu Z_\nu} &= \int_0^\infty B(t)e^{-i\nu t}\,dt \end{aligned} \tag{4.5a}$$

and the inverse relations

$$\begin{aligned} A_0(t) &= -\frac{2}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{1}{i\nu Z_\nu}\right)\cos \nu t\,d\nu \\ B(t) &= -\frac{2}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{1}{i\nu Z_\nu}\right)\sin \nu t\,d\nu \end{aligned} \tag{4.5b}$$

---

[8] More generally, the part of the interaction represented by Q could be represented by function of Q such as $V(Q)$. In this case Q in the influence phase would be replaced by $V_\nu(Q)$, the Fourier transform of $V(Q(t))$.

$A_0(t)$ and $B(t)$ are related as follows:

$$A_0(t) = \frac{2}{\pi}\int_0^\infty \frac{sB(s)}{s^2 - t^2}\,ds \tag{4.5c}$$

These relationships may be written in many forms. Two additional forms are

$$F(t) = \frac{2i}{\pi}\int_0^\infty \left(\frac{1}{i\nu Z_\nu}\right)\cos \nu t\, d\nu$$

and

$$\begin{aligned} \textstyle\int_{-\infty}^{\infty} F(|t|)e^{-i\nu t}dt &= 2i(i\nu Z_\nu)^{-1} && \text{for } \nu > 0 \\ &= 2i(-i\nu Z_{-\nu})^{-1} && \text{for } \nu < 0 \end{aligned} \tag{4.5d}$$

All the poles of $1/i\nu Z_\nu$ have positive imaginary parts and this impedance function has the additional property that

$$(1/i\nu Z_\nu) = (1/-i\nu Z_{-\nu})^*$$

In the case of finite temperatures, the influence phase can be written in the same form as Eq. (4.4) except that $\text{Re}F(t) = A(t)$, that is, without the subscript 0, and a more general relation exists connecting $A(t)$ and Im $(1/i\nu Z_\nu)$ (see Section IV,C).

$\mathfrak{F}(Q, Q')$ *for Single Lossless Harmonic Oscillator*

To prove the above theorem, we consider first a test system, Q, which is coupled to a simple harmonic oscillator whose mass is m, characteristic frequency ω, and displacement coordinate X. The complete Lagrangian for X and Q can be written

$$L_{\text{total}} = L_0(\dot{Q}, Q, t) + \tfrac{1}{2}m\dot{X}^2 - \tfrac{1}{2}m\omega^2X^2 + QX \tag{4.6}$$

and the total action is written similarly.[9]

$$S_{\text{total}} = S_0(Q) + \textstyle\int_\tau^T (\tfrac{1}{2}m\dot{X}^2 - \tfrac{1}{2}m\omega^2X^2 + QX)\,dt$$

If X is assumed to be initially in the ground state (corresponding to zero temperature) then to within a normalizing constant $\chi_i(X) = e^{-m\omega X^2/2\hbar}$. The final state of X is assumed to be arbitrary which means the final states are to be summed over. Therefore, in Eq. (2.7), the definite state $\chi_f^*(X_T)\chi_f(X'_T)$ will be replaced by the sum $\sum_n \Phi_n^*(X_T)\Phi_n(X'_T) = \delta(X_T - X'_T)$. The $\Phi_n(X)$ represent the energy eigenfunctions of the harmonic oscillator. With this in-

---

[9] The interaction Lagrangian QX could be written more transparently as γQX where Q and X are the coordinates of the system involved and γ is a coupling factor which may or may not be a function of time. For simplicity in writing the lengthy expressions to follow, γ has been incorporated into an effective coordinate Q since no loss in generality results.

formation available the influence functional is completely defined and from Section II.B can be written

$$\mathfrak{F}_0(Q, Q') = \int \delta(X_T - X'_T) K_Q(X_T, X_\tau) K^*_{Q'}(X'_T, X'_\tau) \times \exp[-(m\omega/2\hbar)(X_\tau^2 + X_\tau'^2)]\, dX_\tau \cdots dX'_T \tag{4.7a}$$

where the subscript Q, Q' refer to the interaction potentials $-QX$ and $-Q'X'$ acting on the X and X' systems respectively. The subscript 0 on $\mathfrak{F}_0(Q, Q')$ indicates zero temperature. For the harmonic oscillator

$$\begin{aligned} K_Q(X_T, X_\tau) &= N \exp\{(i/\hbar)[S(X) - S_I(Q, X)]_{\text{classical}} \\ &= N \exp\{[i\omega/2\hbar \sin\omega(T - \tau)][(X_T^2 + X_\tau^2)\cos\omega(T - \tau) \\ &\quad - 2X_T X_\tau + (2X_T/\omega)\textstyle\int_\tau^T Q_t \sin\omega(t - \tau)\,dt + (2X_\tau/\omega) \\ &\quad \cdot \textstyle\int_\tau^T Q_t \sin\omega(T - t)\,dt - (2/\omega^2)\int_\tau^T \int_\tau^t Q_t Q_s \sin\omega(T - t)\sin\omega(s - \tau)\,ds\,dt\} \end{aligned} \tag{4.7b}$$

where N is a normalizing factor depending only on ω and the time interval $T - \tau$.[10] Thus, Eq. (4.7a) represents a Gaussian integral over the four X variables since S is itself quadratic in the X variables. When the integrals are carried out the following result is obtained for the influence phase:

$$i\Phi_0(Q, Q') = -(2\hbar m\omega)^{-1} \int_\tau^T \int_\tau^t (Q_t - Q'_t) \times (Q_s e^{-i\omega(t-s)} - Q'_s e^{i\omega(t-s)})\, ds\, dt \tag{4.8}$$

Thus, $F(t - s)$ in Eq. (4.4) corresponds in this case to $e^{+i\omega(t-s)}/m\omega$ and from the definition given above $B(t - s) = (1/m\omega)\sin\omega(t - s)$.[11] Rewriting Eq. (4.8) in transform notation we have

$$\Phi_0(Q, Q') = (2\pi\hbar)^{-1} \int_0^\infty \left\{ \frac{Q'_\nu(Q_{-\nu} - Q'_{-\nu})}{-m[(\nu - i\epsilon)^2 - \omega^2]} + \frac{Q_{-\nu}(Q_\nu - Q'_\nu)}{-m[(\nu + i\epsilon)^2 - \omega^2]} \right\} d\nu \tag{4.9}$$

where

$$Q_\nu = \int_{-\infty}^{\infty} Q_t e^{-i\nu t}\, dt$$

The function $\{-m[(\nu - i\epsilon)^2 - \omega^2]\}^{-1} = (m\omega)^{-1}\int_0^\infty \sin\omega t\, e^{-i\nu t}\, dt$ corresponds to $1/i\nu Z_\nu$ of Eq. (4.2).[12]

---

[10] See ref. *2*, Section 3.

[11] The finite time interval indicated by the limits T and τ can be interpreted as turning the coupling (between Q and X) on at $t = \tau$ and off at $t = T$. However, since the interaction system is to be considered in most cases as part of the steady-state environment of Q, it is really more meaningful to extend these limits over an infinite range of time ($\tau \rightarrow -\infty$, $T \rightarrow +\infty$). The possibility of allowing X to interact with Q over a finite range of time can be taken care of by giving the coupling factor (already included in the variable Q) the proper time dependence.

[12] ϵ which occurs in $i\nu Z_\nu$ is a convergence factor which was inserted in taking the Fourier transform $(1(t)/m\omega)\sin\omega t$ where $1(t)$ is the unit step function and is kept to show the location of the poles with respect to the ν axis when doing integrations of the type $\int_0^\infty H(\nu)\,[i\nu Z_\nu]^{-1}\, d\nu$.

Having obtained the expression for the influence phase we now turn to the classical problem of finding the response of $X(t)$ to a driving force, $f(t)$, applied at $t = 0$ with the initial conditions $X(0) = \dot{X}(0) = 0$. Starting with the Lagrangian of the unperturbed oscillator from Eq. (4.6), we add to it a potential term $-f(t)X(t)$. This potential has the same form as the coupling potential $-QX$ used in the quantum calculation. However, it is to be emphasized that the response of X to a force has nothing to do with the system Q outside of the type of coupling involved; therefore, $f(t)$ will symbolize the force in the classical problem. The complete Lagrangian is

$$L(\dot{X}, X, t) = \tfrac{1}{2}m\dot{X}^2 - \tfrac{1}{2}m\omega^2X^2 + fX \tag{4.10}$$

and the equation of motion derived from it is,

$$m\ddot{X} + m\omega^2X = f \tag{4.11}$$

Its solution under the initial conditions stated above is

$$X(t) = (m\omega)^{-1}\int_0^t f(s) \sin \omega(t - s)\, ds \tag{4.12}$$

or alternatively, in terms of Fourier transforms, is

$$X_\nu = f_\nu\{-m[(\nu - i\epsilon)^2 - \omega^2]\}^{-1} \tag{4.13}$$

Therefore, $B(t - s)$ in this case is a Green's function which yields the response of $X(t)$ to an impulse force $f(s) = \delta(s)$ and its transform yields $1/i\nu Z_\nu$. Thus a classical calculation of the ratio X_ν/f_ν under quiescent initial conditions yields the proper function for $1/i\nu Z_\nu$.

Distribution of Oscillators—Representation of Loss

The results of the preceding section are easily extended to the situation where the interaction system is a distribution of oscillators. First, we consider the case of independent oscillators coupled to the test system. It is assumed that there is a distribution of oscillators such that $G(\Omega)\, d\Omega$ is the weight of oscillators whose natural frequency is in the range between Ω and $\Omega + d\Omega$. More specifically, $G(\Omega)\, d\Omega$ is the product of the number of oscillators and the square of their coupling constants divided by the mass in $d\Omega$. Thus, we have a situation represented by the diagram of Fig. 3. Each oscillator is assumed to be initially in the ground state and finally in an arbitrary state; the coupling is again assumed to be linear. The total action is then given by

$$S[Q, X(\Omega)] = S_0(Q) + \int_\tau^T \int_0^\infty G(\Omega)[\tfrac{1}{2}\dot{X}^2 - \tfrac{1}{2}X^2\Omega^2 + QX]\, d\Omega\, dt \tag{4.14}$$

For the general properties of influence functionals already described we know that when independent disturbances act on Q the influence functional is a product of the ones for each individual disturbance. Since $\mathfrak{F}_0(Q, Q') = \exp [i\Phi_0(Q, Q')]$

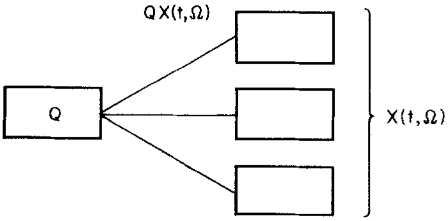

FIG. 3. Test system Q coupled to a distribution of oscillators

for the case of a single oscillator, the total influence phase for the distribution is the sum of the individual phases,

$$\Phi_0(Q, Q') = \int_0^\infty G(\Omega)\Phi_{0,\Omega}(Q, Q')\, d\Omega \tag{4.15}$$

More explicitly,

$$\Phi_0 = (2\pi\hbar)^{-1} \int_0^\infty G(\Omega)\, d\Omega \cdot \int_0^\infty \left\{ \frac{Q'_\nu(Q_{-\nu} - Q'_{-\nu})}{-[(\nu - i\epsilon)^2 - \Omega^2]} + \frac{Q'_{-\nu}(Q_\nu - Q'_\nu)}{-[(\nu + i\epsilon)^2 - \Omega^2]} \right\} d\nu \tag{4.16}$$

For this case then, the form of Eq. (4.2) is obtained if we put

$$(i\nu Z_\nu)^{-1} = \lim_{\epsilon \to 0} \int_0^\infty G(\Omega)[(\nu - i\epsilon)^2 - \Omega^2]^{-1}\, d\Omega \tag{4.17}$$

or[13]

$$(Z_\nu)^{-1} = (\pi/2)G(\nu) - i\nu\int_0^\infty G(\Omega)(\nu^2 - \Omega^2)^{-1}\, d\Omega \tag{4.18}$$

Thus the effects of all the oscillators are included in the influence phase through the expression for Z_ν, Eq. (4.17). Now, however, because of the continuous distribution of oscillators, Z_ν has a finite real part. We will now show that this real part represents dissipation by arriving at the same impedance function classically.

As before, we take the part of the Lagrangian from Eq. (4.14) having to do with the oscillators, except that the coupling potential $-Q(t)\int_0^\infty G(\Omega)X(\Omega, t)\, d\Omega$ is replaced by $-f(t)\int_0^\infty G(\Omega)X(\Omega, t)\, d\Omega$, a classical potential. $X(\Omega, t)$ is the coordinate of the oscillator in the distribution whose frequency is Ω while the total coordinate of the complete linear system with which $f(t)$ is interacting is

---

[13] Eq. (4.18) is obtained from (4.17) using the identity

$$\lim_{\epsilon\to 0} [(\nu - i\epsilon)^2 - \Omega^2]^{-1} = (\nu^2 - \Omega^2)^{-1} + (i\pi/2\Omega)[\delta(\nu - \Omega) - \delta(\nu + \Omega)]$$

$\int_0^\infty G(\Omega)X(\Omega, t)\, d\Omega = X(t)$. It is the relationship between $f(t)$ and $X(t)$ in which we are interested in this classical case:

$$L[\dot{X}(\Omega), X(\Omega), t] = \int_0^\infty G(\Omega)\, d\Omega[\tfrac{1}{2}\dot{X}(\Omega)^2 - \tfrac{1}{2}\Omega^2 X(\Omega)^2] + f(t)\int_0^\infty G(\Omega)X(\Omega)\, d\Omega \tag{4.19}$$

The equations of motion are the infinite set represented by

$$\ddot{X}(\Omega) + \Omega^2 X(\Omega) = f(t) \tag{4.20}$$

They result from varying L with respect to the independent variables $X(\Omega)$. For quiescent initial conditions and for $f(t)$ applied at $t = 0$, this solution is expressed

$$X_\nu(\Omega)/f_\nu = -[(\nu - i\epsilon)^2 - \Omega^2]^{-1} = [i\nu Z_\nu(\Omega)]^{-1}$$

The relation of the total coordinate X_ν to f_ν is obtained simply,

$$\begin{aligned} X_\nu/f_\nu &= [\int_0^\infty X_\nu(\Omega)G(\Omega)\, d\Omega](f_\nu)^{-1} \\ &= -\int_0^\infty G(\Omega)[(\nu - i\epsilon)^2 - \Omega^2]^{-1}\, d\Omega = (i\nu Z_\nu)^{-1} \end{aligned} \tag{4.21}$$

Referring to Eq. (4.17) it is seen that the same expression for Z_ν is obtained in the quantum and classical cases. In addition, since Z_ν is now identified with a classical impedance, the real part represents resistance while the imaginary part corresponds to reactance. Therefore, at least for the case that loss is represented by distributions of oscillators, its effect can be included in the influence functional by using the appropriate impedance expression. The spontaneous emission of a particle in free space represents a good example of such a loss mechanism. A demonstration of this point is included in Appendix II where the oscillator distribution is related to the probability of spontaneous emission starting from the influence functional representing the effect of free space.

The relationship

$$(i\nu Z_\nu)^{-1} = \int_0^\infty B(t)e^{-i\nu t}dt$$

has already been established during the course of the derivation of the influence phase for the single oscillator. Now the inverse relation between $F(t)$ and $1/i\nu Z_\nu$ can be written for the zero temperature case. In the time domain the influence phase for the distribution of oscillators is

$$\begin{aligned} i\Phi(Q, Q') = -(2\hbar)^{-1}\int_0^\infty G(\Omega)\Omega^{-1}\, d\Omega\int_{-\infty}^{\infty}\int_{-\infty}^{t} (Q_t - Q'_t) \\ \cdot (Q_s e^{-i\Omega(t-s)} - Q'_s e^{i\Omega(t-s)})\, ds\, dt \end{aligned}$$

Comparing this with Eq. (4.4) it is evident that

$$F(t) = \int_0^\infty G(\Omega)\Omega^{-1}e^{i\Omega t}\, d\Omega$$

But from Eq. (4.18),

$$\mathrm{Im}(i\nu Z_\nu)^{-1} = -\pi G(\nu)/2\nu$$

Therefore, it can be immediately written that

$$F(t) = -(2/\pi)\int_0^\infty \mathrm{Im}(i\nu Z_\nu)^{-1} e^{i\nu t}\, d\nu \qquad (4.22)$$

as was given in Eq. (4.5b).

The results above can now be extended by a simple argument to include all linear systems composed entirely of distributions of oscillators. To do this it need only be shown that the general system can be reduced to a distribution of oscillators independently coupled to the test system, which was the situation just considered. Explicitly, suppose there exists a test system Q, coupled to an assemblage of oscillators which are also interconnected with each other. For instance, the situation might be as in Fig. 4, where each of the X_n components of the total interaction system could also represent a system of oscillators. However, it is well known[14] that such a linear system may be represented by an equivalent set of oscillators (the normal modes of the total system) independently coupled to Q.[15] Or, stated another way, the classical representation of the Lagrangian in normal modes finds new linear combinations of the X_n which makes the total Lagrangian, except for the coupling, a sum of individual quadratic forms with no cross terms. But, this same transformation of variables can be made on the expression for $\mathfrak{F}(Q, Q')$ (see Eq. (2.7)). The effect of this transformation is to change the $\mathfrak{D}X(t)$ volume by a numerical factor, since the transformation is linear.[16] Thus, in effect, we get the sum of independent systems in the quantum mechanical case also. From this argument it is concluded that the results above regarding a distribution of independent oscillators coupled to a test system, apply to any linear interaction system. Therefore, it has been found that the influence functional for all linear systems has exactly the same form $\exp[i\Phi_0(Q, Q')]$ where $\Phi_0(Q, Q')$ is a quadratic functional of the Q and Q'. $\Phi_0(Q, Q')$ is adapted to a particular linear system only through the classical response of that linear system to a force. Thus, the procedure for finding the influence functional for a linear system has been reduced to a classical problem. The fact that eliminating the coordinate of an oscillator always yields an in-

---

[14] This point is considered more fully in Section IV,B on classical forces.

[15] The fact that one or more of the X_n might represent continuous distributions of oscillators need not be bothersome since in principle they represent the behavior of the total system in terms of its infinite set of normal modes.

[16] The only result of such a numerical factor would be to change the normalization of $\mathfrak{F}(Q, Q')$. However, we already know that for the case that the final states of the interaction system are summed over $\mathfrak{F}(Q, Q') = 1$. Therefore the normalization of $\mathfrak{F}(Q, Q')$ is not changed by the transformation and thus is not dependent upon the coordinates chosen to represent the interaction system.

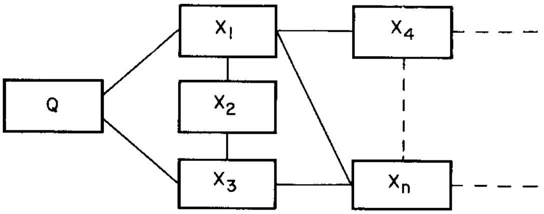

FIG. 4. Test system coupled to an arbitrary assemblage of oscillators

fluence functional which is quadratic in the potential applied to that oscillator, is a basic property of linear systems. For example, where the coupling Lagrangian is linear between an oscillator of coordinate X and another system of coordinate Q, the elimination of the X coordinate yields an influence phase which is quadratic in Q as has already been shown. If Q were the coordinate of another oscillator coupled to P, then elimination of the Q coordinates would yield an influence phase quadratic in P, etc. This can be understood mathematically by observing that the Lagrangian for all the oscillators with linear coupling is always quadratic. Doing the path integral to eliminate a coordinate is basically a process of completing the square and performing Gaussian integrals. This process of completing the square also yields quadratic terms. It is therefore not surprising that the influence phase for any linear system should be always of the same quadratic form.

It is to be emphasized that the analysis so far presented has been concerned entirely with systems whose complete behavior can be described by combinations of lossless oscillators at zero temperature. The only example of such a system is the electromagnetic field in free space. In all other physical situations linear behavior is an approximation to the actual behavior. However, this approximation may be very good over a wide range of operating conditions. In Section V the problem of approximately linear systems will be considered in detail. The results will be found to be the same as for perfect oscillators to the extent that linear behavior is realized.

Form of Influence Functionals for Linear Systems and Classical Forces as Deduced from Properties of Influence Functionals

So far, we have found the influence phase for classical potentials, uncertain classical potentials, and linear systems at zero temperature. By studying Eqs. (3.1), (3.6), and (4.4), we see that the general form for the influence functional in which all three of these were acting on Q is

$$\begin{aligned}\mathfrak{F}(Q, Q') = \exp\{&\textstyle\int_\tau^T iC_1(t)(Q_t - Q'_t)\,dt - \int_\tau^T \int_\tau^t A_1(t-s)(Q_t - Q'_t)\\ &\cdot(Q_s - Q'_s)\,ds\,dt - \textstyle\int_\tau^T \int_\tau^t iB_1(t-s)(Q_t - Q'_t)(Q_s + Q'_s)\,ds\,dt\}\end{aligned} \tag{4.23}$$

The exponent is written solely in terms of Q for simplicity although when the potentials are not linear in Q (as $XV(Q)$), the same general form exists, except that it is written in terms of $V(Q)$. We now observe that there are other possible combinations of the Q, Q' variables not represented here such as terms in $(Q_t + Q'_t)$, $(Q_t + Q'_t)(Q_s + Q'_s)$. To see if such terms are possible, let us form a hypothetical functional containing all possible forms up to second order in Q.

$$\begin{aligned}\mathfrak{F}(Q, Q') = \exp\{&\textstyle\int_\tau^T [iC_1(t)(Q_t - Q'_t) + D_1(t)(Q_t + Q'_t)]\, dt \\ &- \textstyle\int_\tau^T \int_\tau^t [A_1(t - s)(Q_t - Q'_t)(Q_s - Q'_s) + iB_1(t - s)(Q_t - Q'_t)(Q_s + Q'_s) \\ &+ iD_2(t - s)(Q_t + Q'_t)(Q_s - Q'_s) + D_3(t - s)(Q_t + Q'_t)(Q_s + Q'_s)]\, ds\, dt\}\end{aligned} \tag{4.24}$$

That the coefficients of the Q's inside the double integrals should be functions of $(t - s)$ is evident since the functional should not depend on the absolute time. We now will try to eliminate terms in the exponent by using the general properties of $\mathfrak{F}(Q, Q')$ given in Section II. First, we know $\mathfrak{F}(Q, Q') = \mathfrak{F}^*(Q', Q)$. This implies that all the functions A_1, B_1, C_1, D_1, D_2, and D_3 are real. Next, we know that $\mathfrak{F}(Q, Q') = 1$ if $Q'(t) = Q(t)$. Hence, D_1, D_3 are zero. This leaves only one term which we did not have before, that of $D_2(t - s)$. Now we apply the property of these functionals requiring that if $Q_t = Q'_t$ for $t > t_0$ then $\mathfrak{F}(Q, Q')$ is independent of Q for $t > t_0$. This statement is obviously true for the $C_1(t)$ and $A_1(t - s)$ terms. Consider now the $B_1(t - s)$ term. For $t > t_0$, $Q_t - Q'_t = 0$ and therefore this term is also legitimate. As for the $D_2(t - s)$ term, let us consider $t > t_0$, but $s < t_0$. Then $Q_s - Q'_s \neq 0$. Furthermore, $Q_t + Q'_t = 2Q_t \neq 0$. Therefore, $D_2(t - s)$ must also be zero. The fact that $D_2(t - s) = 0$ is actually a statement of causality, i.e., that the effect due to an applied force cannot precede the time the force was applied. To see this, let us change the limits of integration on this term

$$\begin{aligned}&\textstyle\int_\tau^T \int_\tau^t D_2(t - s)(Q_t + Q'_t)(Q_s - Q'_s)\, ds\, dt \\ &\qquad = \textstyle\int_\tau^T \int_t^T D_2(s - t)(Q_t - Q'_t)(Q_s + Q'_s)\, ds\, dt\end{aligned}$$

Now the integrand is of the same form as that of the $B(t - s)$ term. However, for a fixed t the integration over s is over the range of $s > t$. This amounts to a sum over the future rather than a sum over past histories of the variable Q.

The conclusion to be drawn is that there are three possible types of terms up to second order in Q and Q' when definite classical forces, indefinite classical forces, and linear systems act on Q. Terms of this type have already been derived during the course of our analysis. Therefore, there are no major types of phenomena which have not been noticed. In the light of the above discussion we would expect the effects of additional phenomena, if they are described by terms of second order or less in Q and Q', to be contained in one or more of the three forms of exponents shown in Eq. (4.23). For instance, in the case of a linear system at finite temperature, it will be found that the effect of temperature is to change the effective value of $A_1(t - s)$ in the exponent of Eq. (4.23) from its minimum value which occurs at zero temperature. It should be pointed out that although $A_0(t - s)$ (see Eq. (4.5b)) occurs in a term which has the form of an uncertain classical potential acting on the test system, at zero temperature one must be careful about this interpretation, for the existence of a random classical potential implies a random fluctuation of the variables of the interaction system which could induce transitions in the test system either upwards or downwards in energy. However, if the interaction system is already in its lowest state it can only induce downward transitions in the test system so that the term in $A_0(t - s)$ by itself is not sufficient. Thus, as has already been found, the exponent of the zero temperature influence functional contains two terms, one in $A_0(t - s)$ and the other in $B(t - s)$, which are related through Eq. (4.5c). Together they give the whole picture, i.e., that there is a zero point, random fluctuation of the variables of the interaction system but that this fluctuation can induce only those transitions in the test system which, through spontaneous emission, give up energy to the interaction system.

B. INFLUENCE FUNCTIONALS FOR DRIVEN LINEAR SYSTEMS

It is to be expected that if a classical force is applied to a linear interaction system which in turn is coupled to a test system, the effect of the interaction system is to modify the character of the force applied to the test system. In this section we will find the exact form for the influence functional of this effective force. *If a linear system is coupled to $Q(t)$ through one of its coordinates $X(t)$ and if a classical force $C(t)$ is coupled to another coordinate $Y(t)$, then $\Phi(Q, Q')$ representing the effect of both the linear system and the force is*

$$\Phi(Q, Q') = \Phi_0(Q, Q') + (2\pi\hbar)^{-1} \int_0^\infty \left[\frac{C_\nu(Q_{-\nu} - Q'_{-\nu})}{i\nu z_\nu} + \frac{C_{-\nu}(Q_\nu - Q'_\nu)}{-i\nu z_{-\nu}}\right] d\nu \tag{4.25}$$

where Φ_0 is the influence phase of the linear system is the absence of a classical force $C(t)$, and z_ν is a transfer impedance function which modifies the effect of C_ν on Q. The impedance z_ν is found by computing the classical response of the coordinate X to the force C with all other potentials acting on the linear system (including those due to coordinates of external systems such as Q) set equal to zero. The result of the calculation yields $i\nu z_\nu = C_\nu / X_\nu$. Alternatively, in the time domain,

$$\Phi(Q, Q') = \Phi_0(Q, Q') + \hbar^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{t} (Q_t - Q'_t) b(t - s) C_s \, ds \, dt \tag{4.26}$$

where[17]

$$(i\nu z_\nu)^{-1} = \int_0^\infty b(t)e^{-i\nu t}\,dt$$

The theorem can be stated in the form of a diagram as shown in Fig. 5. In this figure $f(t) = \int_{-\infty}^{t} F(s)b(t - s)\,ds$. It will be convenient to work in the frequency domain.

First we recall the influence phase for a classical force acting directly on Q. From this expression we will be able to identify the character of the force acting on Q in more complicated expressions. If the potential is of the form $-C(t)Q(t)$, we have, from Section III,

$$\Phi(Q, Q') = (2\pi\hbar)^{-1} \int_0^\infty [C_\nu(Q_{-\nu} - Q'_{-\nu}) + C_{-\nu}(Q_\nu - Q'_\nu)]\,d\nu \qquad (4.27)$$

Classical Potential and Q Coupled to the Same Coordinate

Before developing the general situation we consider the simpler situation where Q is coupled to a linear system through the potential $-QX$, and a force $F(t)$ is applied to the same system through the potential $-FX$. The Lagrangian for the complete system is

$$L\text{ (system)} = L_0(\dot{Q}, Q, t) + (F + Q)X + L(\dot{X}, \dot{Y}, \cdots X, Y, t) \qquad (4.28)$$

where $X, Y, \cdots$ represent all the coordinates of the linear system. If $F = 0$,

$$\Phi(Q, Q') \equiv \Phi_0(Q, Q')$$
$$= (2\pi\hbar)^{-1} \int_0^\infty \left[\frac{Q'_\nu(Q_{-\nu} - Q'_{-\nu})}{(i\nu Z_\nu)} + \frac{Q_{-\nu}(Q_\nu - Q'_\nu)}{(-i\nu Z_{-\nu})}\right] d\nu \qquad (4.29)$$

If $F \neq 0$ it is evident from Eq. (4.28) that the required influence phase can be found by replacing Q by $Q + F$ and Q' by $Q' + F$. Notice that F does not carry the prime notation since it is not a coordinate. If this substitution is made in Eq. (4.29) we have

$$\Phi(Q, Q') = (2\pi\hbar)^{-1} \int_0^\infty \left[\frac{(Q'_\nu + F_\nu)(Q_{-\nu} - Q'_{-\nu})}{i\nu Z_\nu}\right.$$
$$\left. + \frac{(Q_{-\nu} + F_{-\nu})(Q_\nu - Q'_\nu)}{(-i\nu Z_{-\nu})}\right] d\nu = \Phi_0(Q, Q') + (2\pi\hbar)^{-1} \qquad (4.30)$$
$$\cdot \int_0^\infty \left[\left(\frac{F_\nu}{i\nu Z_\nu}\right)(Q_{-\nu} - Q'_{-\nu}) + \left(\frac{F_{-\nu}}{-i\nu Z_{-\nu}}\right)(Q_\nu - Q'_\nu)\right] d\nu$$

---

[17] The notation z_ν, $b(t - s)$ was chosen to avoid confusion with Z_ν and $B(t - s)$ which are the impedance and response function respectively of the linear system as seen by the test system.

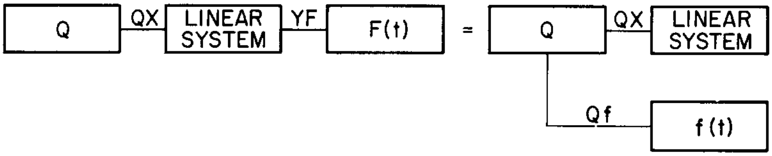

FIG. 5. Equivalent influences of a linear system and a force, $F(t)$, acting on a test system, Q.

As might be expected the total effect of the linear system and the driving force consists of two separate terms, one describing the effect of the linear system alone, and the other describing the effect of the driving force. Comparison of Eqs. (4.26) and (4.30) shows that the effective force applied to Q is, in transform language, $F_\nu/i\nu Z_\nu$ and, further, shows that F_ν is modified by $1/i\nu Z_\nu$, the classical impedance function of the interaction system. In this special example where F and Q are both coupled to the same coordinate, Z_ν is both the correct impedance to be used in $\Phi_0(Q, Q')$, i.e., that impedance seen by the test system, and is the transfer impedance z_ν which modifies F_ν. This is not true generally as we shall see in the next section. In addition, it is interesting to observe that no unexpected quantum effects appear because of the addition of a force to the interaction system. The only effect of the interaction system is to modify the characteristics of $F(t)$ in an entirely classical way.

Classical Forces Acting Through a General Linear System

Having obtained an idea of the type of results to expect in the above simplified analysis we now proceed to the more general case. Let the N coordinates of the interaction system be represented by X_i, $i = 1 \cdots N$. Its coupling to the test system, $Q(t)$, and to the driving force $C(t)$ is given by the potentials $-X_nQ$ and $-X_kC$, respectively. Thus we are assuming, for simplification in writing, that Q is coupled only to the variable X_n and the force $C(t)$ is applied just to the variable X_k. Again the interaction system is assumed to be composed entirely of harmonic oscillators. The Lagrangian is

$$L\ (\text{system}) = L(\dot{Q}, Q, t) + \sum\nolimits_{i,j} [\tfrac{1}{2}(T_{ij}\dot{X}_i\dot{X}_j - V_{ij}X_iX_j)] + X_nQ + X_kC \tag{4.31}$$

It is well known in the theory of linear systems that new coordinates may be defined by means of a linear transformation of the X_i. These new coordinates will be chosen as the eigenvectors, Y_l, of the interaction system (*10*). Thus,

$$X_i = \sum\nolimits_{l=1}^{n} a_{il} Y_l \qquad i = 1, 2, \cdots n$$

Assuming the a_{il} to be properly normalized, the Lagrangian may be rewritten as follows

$$L(\dot{Q}, Q, \dot{Y}_l, Y_l, t) = L_0(\dot{Q}, Q, t) + \sum_l [\tfrac{1}{2}(\dot{Y}_l^2 - \omega_l^2 Y_l^2) + Y_l(a_{nl}Q + a_{kl}C)] \tag{4.32}$$

Since these are now independent oscillators coupled to Q the influence phase can be written down immediately,

$$\begin{aligned}\Phi(Q, Q') = \sum_l \Phi_l(Q, Q') &= \sum_l \frac{1}{2\pi\hbar} \int_0^\infty \left[Q'_\nu(Q_{-\nu} - Q'_{-\nu}) \left(\frac{a_{nl}^2}{i\nu Z_l(\nu)} \right) \right. \\ &\quad \left. + Q_{-\nu}(Q_\nu - Q'_\nu) \left(\frac{a_{nl}^2}{-i\nu Z_l(-\nu)} \right) \right] d\nu + \sum_l \frac{1}{2\pi\hbar} \\ &\quad \cdot \int_0^\infty \left[\frac{a_{nl}\, a_{kl}\, C_\nu}{i\nu Z_l(\nu)} (Q_{-\nu} - Q'_{-\nu}) + \frac{a_{nl}\, a_{kl} C_{-\nu}}{-i\nu Z_l(-\nu)} (Q_\nu - Q'_\nu) \right] d\nu\end{aligned} \tag{4.33}$$

where

$$\frac{1}{i\nu Z_l(\nu)} = \left. \frac{Y_l(\nu)}{a_{nl}\, Q_\nu} \right|_{C_\nu = 0} \equiv \left. \frac{Y_l(\nu)}{a_{kl}\, C_\nu} \right|_{Q_\nu = 0} \tag{4.34}$$

calculated classically.

This can be written in the form of Eq. (4.25) if we make the correspondence

$$\frac{1}{i\nu Z_\nu} = \sum_l \frac{a_{nl}^2}{i\nu Z_l(\nu)}$$

and

$$\frac{1}{i\nu z_\nu} = \sum_l \frac{a_{nl}\, a_{kl}}{i\nu Z_l(\nu)} \tag{4.35}$$

Using Eqs. (4.34) and (4.35) we can now show that $1/i\nu Z_\nu$ and $1/i\nu z_\nu$ are equivalent to $X_n(\nu)/Q_\nu$ and $X_n(\nu)/C_\nu$ respectively. Thus

$$\begin{aligned}\frac{1}{i\nu Z_\nu} &= \sum_l \frac{a_{nl}^2}{i\nu Z_l(\nu)} = \sum_l a_{nl}^2 \left. \left(\frac{Y_l(\nu)}{a_{nl}\, Q_\nu} \right) \right|_{C_\nu = 0} \\ &= \sum_l \left. \frac{a_{nl}\, Y_l(\nu)}{Q_\nu} \right|_{C_\nu = 0} = \left. \frac{X_n(\nu)}{Q_\nu} \right|_{C_\nu = 0}\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}\frac{1}{i\nu z_\nu} &= \sum_l \frac{a_{nl}\, a_{kl}}{i\nu Z_l(\nu)} = \sum_l a_{nl}\, a_{kl} \left. \frac{Y_l(\nu)}{a_{kl}\, C_\nu} \right|_{Q_\nu = 0} \\ &= \left. \frac{\sum_l a_{nl}\, Y_l(\nu)}{C_\nu} \right|_{Q_\nu = 0} = \left. \frac{X_n(\nu)}{C_\nu} \right|_{Q_\nu = 0}\end{aligned} \tag{4.37}$$

Equation (4.36) is a mathematical expression of the argument used earlier to find the appropriate impedance function to be used in the influence functional for a linear system acting on Q. The additional information obtained here in this regard is that when other forces are present they are to be set equal to zero when this computation is made. Equation (4.37) states the new result that the transfer impedance which modifies C_ν in its effect on the test system is to be found by computing the ratio of C_ν to the coordinate $X_n(\nu)$ to which the test system is coupled. The total force acting on the test system when several forces are acting on the interaction system is simply the sum of these forces each modified by the appropriate transfer impedance determined in the above described manner.

C. LINEAR SYSTEMS AT FINITE TEMPERATURES

The forms of influence functionals which are possible for linear systems have been established by an argument which utilized the general properties discussed in Section II. Each of these forms has already occurred in the analyses of classical potentials, random potentials, and zero temperature linear systems. Therefore, the results to be expected here are one or more of the forms already obtained.

The discussion is begun again with a single oscillator as our linear system, for simplicity. From this, the extension to distributions of oscillators is immediate, as it was in Section IV.A. The complete problem is set up in the same way as for zero temperature except that the initial state of the oscillator is not simply the ground state or any definite eigenstate. The effect of temperature is to make the initial state uncertain and it is properly represented by a sum over all states weighted by the Boltzmann factor $e^{-\beta E_n}$ where $\beta = 1/kT$, T being the temperature in this case. The final state is again arbitrary: therefore, a formal expression for the influence functional is

$$\begin{aligned}\mathfrak{F}(Q, Q') = \int \delta\,(X_T - X'_T)K_Q(X_T\,, X_\tau)K^*_{Q'}(X'_T, X'_\tau)\\ \cdot \textstyle\sum_n N^{-1}e^{-\beta E_n}\phi_n(X_\tau)\phi_n{}^*(X'_\tau)\, dX_\tau \cdots dX'_T\end{aligned} \tag{4.38}$$

where N, the normalization constant, is $\sum_n e^{-\beta E_n}$. The ϕ_n represent the energy eigenfunctions of the oscillator unperturbed by external forces. The first problem is to find a closed form for the expression $\sum_n \phi_n(X)\phi_n(X')e^{-\beta E_n}$. This can be done by noticing that its form is identical with the kernel which takes a wave function from one time to another if we make the correspondence that β represents an imaginary time interval. If the times involved are t_2 and t_1, this kernel is

$$\begin{aligned}K_0(X_2\,, X_1) &= \textstyle\sum_n \phi_n(X_2)\phi_n{}^*(X_1) \exp\left[-(i/\hbar)E_n(t_2 - t_1)\right]\\ &= \exp\left[(i/\hbar)S_{\mathrm{cl}}\right]\end{aligned} \tag{4.39}$$

for the harmonic oscillator, and where the subscript 0 indicates the absence of external forces. For the harmonic oscillator the expression for S is easily obtained in terms of the initial and final positions X_1 and X_2 (*2*). Thus,

$$S_{cl} = m\omega[2 \sin \omega(t_2 - t_1)]^{-1}[(X_1^2 + X_2^2) \cos \omega(t_2 - t_1) - 2X_1X_2] \quad (4.40)$$

Utilizing Eqs. (4.42) and (4.43) and making the correspondence $\beta = i(t_2 - t_1)/\hbar$, $X_1 = X_\tau$, and $X_2 = X_\tau'$, we find that

$$\begin{aligned} &\sum \phi_n(X_\tau)\phi_n^*(X'_\tau) \exp(-\beta E_n) \\ &\quad = \exp\{-m\omega[2\hbar \sinh(\beta\hbar\omega)]^{-1}[(X_\tau^2 + X_\tau'^2) \cosh(\beta\hbar\omega) - 2X_\tau X'_\tau]\} \end{aligned} \quad (4.41)$$

Using this closed expression for the average initial state of the oscillator and the kernel for the driven harmonic oscillator (Eq. (4.7b)) the influence functional can be evaluated by evaluation of a series of Gaussian integrations just as in the zero temperature case. The result expressed in the frequency domain is,

$$\begin{aligned} i\Phi(Q, Q') &= i\Phi_0(Q, Q') \\ &\quad - (\pi\hbar^2)^{-1} \int_0^\infty \pi\hbar[2m\omega(e^{\beta\hbar\omega} - 1)]^{-1}\delta(\nu - \omega)|Q_\nu - Q'_\nu|^2 \, d\nu \end{aligned} \quad (4.42)$$

Thus, the influence phase is made up of two terms, the first of which is the effect of the oscillator at zero temperature. The second is recognized as having the same form which was found for an uncertain classical potential with a Guassian distribution. Therefore, the effect of finite temperature is to introduce a noisy potential acting on Q at the frequency of the original oscillator. The power spectrum of the noise produced by the finite temperature is

$$\phi(\nu) = \pi\hbar[2m\nu(e^{\beta\hbar\nu} - 1)]^{-1}\delta(\nu - \omega) \quad (4.43)$$

To indicate more clearly the relationship of $\phi(\nu)$ to the characteristics of the linear system it is instructive to extend this expression to the case of a distribution of oscillators $G(\Omega)$ all at the same finite temperature. The resulting influence phase is

$$\begin{aligned} i\Phi(Q, Q') &= i \int_0^\infty \Phi_0(Q, Q')G(\Omega) \, d\Omega \\ &\quad + (\pi\hbar^2)^{-1} \int_0^\infty \pi\hbar G(\nu)[2\nu(e^{\beta\hbar\nu} - 1)]^{-1} |Q_\nu - Q'_\nu|^2 \, d\nu \end{aligned} \quad (4.44)$$

where in the distribution m has been set equal to unity. The first term is again the influence phase for zero temperature, while the second term again has the form of a noisy potential whose power spectrum

$$\phi(\nu) = (\hbar\pi/2)G(\nu)[\nu(e^{\beta\hbar\nu} - 1)]^{-1}$$

Recalling the analysis of the distribution of oscillators, it is found from Eq. (4.18) that $\pi G(\nu)/2 = \mathrm{Re}(1/Z_\nu)$. Therefore, the power spectrum can be written

$$\phi(\nu) = \hbar \,\mathrm{Re}\,(1/Z_\nu)[\nu(e^{\beta\hbar\nu} - 1)]^{-1} \quad (4.45)$$

In the time domain the influence phase is

$$\begin{aligned} i\Phi = i\Phi_0 - (\pi\hbar^2)^{-1} \textstyle\int_0^\infty d\nu G(\nu)\hbar[\nu(e^{\beta\hbar\nu} - 1)] \\ \times \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{t} (Q_t - Q'_t)(Q_s - Q'_s) \cos \nu(t - s)\, ds\, dt \end{aligned} \tag{4.46}$$

Comparing this with Eq. (3.6) for random classical forces we see that the correlation function of the noise due to the finite temperature is

$$\begin{aligned} R(t - s) &= \textstyle\int_0^\infty \hbar G(\nu)[\nu(e^{\beta\hbar\nu} - 1)]^{-1} \cos \nu(t - s)\, d\nu \\ &= -(2/\pi) \textstyle\int_0^\infty \hbar \operatorname{Im} (i\nu Z_\nu)^{-1}(e^{\beta\hbar\nu} - 1)^{-1} \cos \nu(t - s)\, d\nu \end{aligned} \tag{4.47}$$

Finally, if we write Eq. (4.46) in terms of $F(t - s)$ as, for instance, in Eq. (4.4), we find that

$$F(t) = \textstyle\int_0^\infty [G(\nu)/\nu] \coth (\beta\hbar\nu/2) \cos \nu t\, d\nu + i \int_0^\infty [G(\nu)/\nu] \sin \nu t\, d\nu$$

Thus,

$$\begin{aligned} A(t) &= -(2/\pi) \textstyle\int_0^\infty \operatorname{Im} (i\nu Z_\nu)^{-1} \coth (\beta\hbar\nu/2) \cos \nu t\, d\nu \\ &= -(2/\pi) \textstyle\int_0^\infty \operatorname{Im} (i\nu Z_\nu)^{-1}[1 + 2(e^{\beta\hbar\nu} - 1)^{-1}] \cos \nu t\, d\nu \end{aligned} \tag{4.48}$$

and

$$B(t) = -(2/\pi) \textstyle\int_0^\infty \operatorname{Im} (i\nu Z_\nu)^{-1} \sin \nu t\, d\nu$$

which are the more general counterparts of Eq. (4.5b). Notice, however, that only the relation for $A(t)$ changes with temperature.

Thus, if a linear interaction system is initially at a finite temperature, the resulting effect is the same, as far as the test system is concerned, as if the linear system were at zero temperature and, in addition, a random classical potential were connected independently to the test system. The power spectrum of the random potential is given by Eq. (4.45) and is related both to the temperature and to the dissipative part of the impedance of the linear system. The theorem is stated in terms of a diagram in Fig. 6 where the power spectrum $\phi(\nu)$ of the random force C_ν is defined by

$$\phi(\nu) = 4\pi\langle C_\nu C_{-\nu'}\rangle\delta(\nu + \nu') \tag{4.49}$$

This fluctuation dissipation theorem has a content which is different from those stated by Callen and Welton (*11*), Kubo (*12*), and others. It represents still another generalization of the Nyquist theorem which relates noise and resistance in electric circuits (*13*). These previously stated fluctuation dissipation theorems related the fluctuations of some variable in an isolated system, which is initially at thermal equilibrium, to the dissipative part of the impedance of the isolated system. This would be equivalent in our case to relating $\langle Q^2\rangle$ when the test system

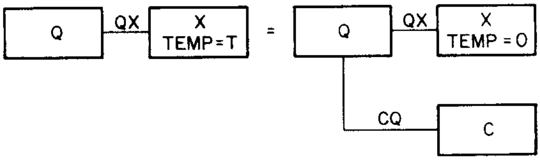

FIG. 6. Equivalent influences of a linear system at a finite temperature acting on a test system.

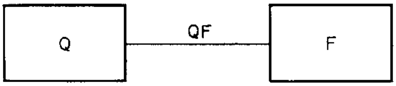

FIG. 7. Linear classical force F acting on a test system

is at equilibrium ($F = 0$) to the dissipative part of F_ν/Q_ν where F is a classical force acting on Q through a linear potential as shown in Fig. 7. However, we have shown that the effect of an external quantum system at thermal equilibrium on a test system can be separated into two effects, a zero point quantum term, which cannot be classified as pure noise, and a random potential term. Using this influence functional approach we can find $\langle Q^2\rangle$ due not only to the internal fluctuations of Q but also due to the effect of X.

The fact that all the derivations so far have been exact, which is a consequence of dealing only with systems made up of distributions of oscillators, brings up two interesting aspects of the theory. The first one is that Z_ν does not depend on the temperature, only on the distribution of oscillators. Yet in any real finite system, it is to be expected that the temperature of a system does affect its impedance. The second aspect is that when a force $f(t)$, of any magnitude whatever, is applied to the linear system, its temperature does not change although it is obvious that if z_ν has a finite real part the linear system must absorb energy. For example, we have shown that the effect on a test system of a linear system at a finite temperature acted on by a force $f(t)$ is the same as the effect of a linear system at zero temperature, a force dependent only on the temperature of the linear system, and the force $f(t)$ modified by the transfer characteristics of the linear system. Figure 8 shows the situation where $f_\nu = F_\nu/i\nu z_\nu$ and $C(t)$ is the random force with a power spectrum given in Eq. (4.49). The influence phase acting on Q is

$$i\Phi = i\Phi_0 + i(2\pi\hbar)^{-1} \int_{-\infty}^{\infty} (f_\nu/i\nu z_\nu)(Q_{-\nu} - Q'_{-\nu})\, d\nu$$
$$- (\pi\hbar^2)^{-1} \int_0^\infty \phi(\nu)|Q_\nu - Q'_\nu|^2\, d\nu$$

As can be seen, the addition of the driving force $f(t)$ (which is denoted in the above expression by its Fourier transform f_ν) does not change either the imped-

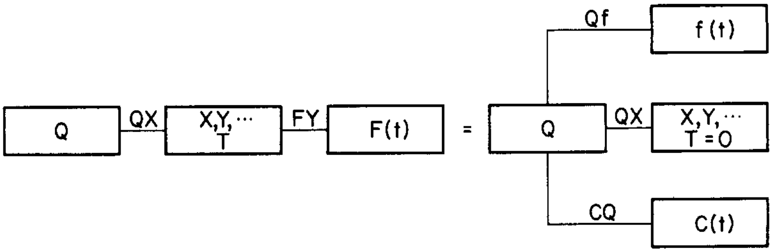

FIG. 8. Equivalent influences of a linear system at a finite temperature and a force acting on Q.

ance characteristics or the temperature of the interaction system (which would be reflected as changes in $\Phi_0(Q, Q')$, z_ν, and $\phi(\nu)$).

The fact that the impedance characteristics of the interaction system are not temperature dependent is a direct result of linearity. The incremental response of a perfectly linear system due to a driving force is independent of its initial state of motion or of that induced by other linearly coupled forces. Therefore, the random motion of the coordinate of the interaction system implied by a finite temperature does not affect its response to a driving force.

To see the reason that temperature of the linear interaction system has no dependence on the applied force $F(t)$, let us consider a linear system represented by a very large box whose dimensions we will allow to become infinite. In addition, we assume that a test system is located in the box and is receiving portions of a classical (very large amplitude) electromagnetic wave which is being transmitted by an antenna also within the box. As long as the box is of finite size and has lossless walls, any signal transmitted by the antenna will find its way either to the test system or back to the antenna. Transmission of the signal will be effected through the modes of the cavity which have a discrete frequency distribution. This system exhibits no loss since each mode represents a lossless harmonic oscillator. For the rest of the discussion it is convenient to separate the energy transmitted by the antenna into parts which reach the test system directly or through reflection from the walls. As the dimensions of the box are allowed to become infinite (i.e., the distribution of oscillators describing the electromagnetic behavior of the box becomes continuous), the time required for the energy to reach the test system by reflection from the walls also becomes infinite. Therefore, since part of the energy is lost from the antenna and test systems, the box has, in effect, dissipation. However, because of the volume of the box is infinite, its temperature is not changed by this lost energy. In other words, the box has an infinite specific heat. In addition, the energy transmitted by the antenna will generally have different characteristics from that of the Gaussian noise associated with temperature and even if the average background energy content of the box were changed it could not be properly described by a temperature parameter.

All the results so far suggest that any linear system can be handled by the same rules that have been developed for systems of oscillators. This will be developed fully in the next section. However, we will assume this to be true now and conclude this section by applying the theorem just derived to obtain Nyquist's result for noise from a resistor. Take as an example an arbitrary circuit as the test system connected to a resistor $R_T(\nu)$ at a temperature T as shown in Fig. 9. The resistor comprises the interaction system. The interaction between the test system and the resistor is characterized by a charge $Q(t)$ flowing through the test system and resistor and a voltage $V(t)$ across the terminals. Let us associate $Q(t)$ with the coordinate of the test system and $V(t)$ with the coordinate of $R(\nu)$. The interaction part of the Lagrangian is symbolically represented by $-Q(t)V(t)$ since the current voltage relationship in $R(\nu)$ is opposite to that of a generator. The quantity $i\nu Z_\nu$ appearing in the influence functional is given by

$$-[Q_\nu/V_\nu] = -(i\nu R_\nu)^{-1} = i\nu Z_\nu$$

Thus, it follows that

$$\text{Re}\,(Z_\nu)^{-1} = \nu^2 R_\nu$$

Then the results of this section tell us that this situation, as shown in Fig. 9, may be replaced by a resistor at zero temperature (i.e., a resistor with thermal fluctuations appropriate to zero temperature but with the same magnitude of resistance it has at temperature T) and a random classical voltage whose power spectrum is

$$\phi(\nu) = \hbar\nu R_\nu(e^{\beta\hbar\nu} - 1)^{-1} \tag{4.50}$$

as shown in Fig. 10.

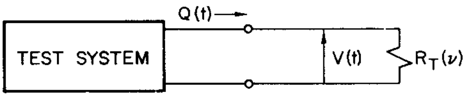

FIG. 9. Test system acted on by a linear system represented by an electrical resistance R at a finite temperature.

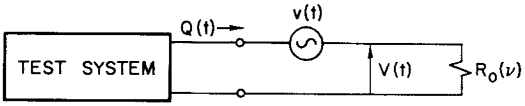

FIG. 10. Equivalent interaction system of a resistor at finite temperature acting on the test system Q.

The mean value of this voltage is

$$\begin{aligned}\langle v^2(t)\rangle &= (2/\pi)\int_0^\infty \phi(\nu)\,d\nu \\ &= \int_0^\infty 2\hbar\nu R_\nu[\pi(e^{\beta\hbar\nu}-1)]^{-1}\,d\nu\end{aligned} \tag{4.51}$$

For high temperature $\beta \ll 1$, and we find that over the frequency range where $\beta\hbar\nu \ll 1$, $\phi(\nu) = kTR_\nu$. If R_ν is constant over the frequency range of interest whose limits are ν_2 and ν_1, then as a result of the noise power in this range

$$\langle v^2(t)\rangle = 4kTR(f_2 - f_1) \tag{4.52}$$

where f is the circular frequency. This is the famous Nyquist result for noise from a resistor. Notice that the noise voltage generator must be placed in series with the resistor as a consequence of interacting directly with the coordinate of the test system, which in this case is $Q(t)$, the charge flowing in the circuit. If the coordinates were chosen such that $Q(t)$ were the coordinate of the resistor and $V(t)$ that of the test system, then the noise generator would become a current source interacting with the voltage $V(t)$ so that the situation would be as shown in Fig. 11, where $\langle i^2\rangle = (4kT/R)(f_2 - f_1)$ in the high temperature limit.

It is worth mentioning, but obvious from the derivation, that if there were many sources of dissipation coupled to the test system, each at different temperatures, then there would be a fluctuating potential associated with each source of dissipation with a power spectrum characteristic to the temperature involved. The case of different temperatures represents a nonequilibrium condition in that the hot resistors are always giving up energy to the cold ones. However, when the resistors are represented by continuous distributions of oscillators as in the case of an infinite box (free space), the temperatures do not change because of the infinite specific heat of the ensemble of oscillators.

V. WEAKLY COUPLED SYSTEMS

We are now faced with the problem of finding influence functionals whose behavior is in some sense linear but whose total behavior is not representable by systems of perfect oscillators. There are many examples of this. The concepts of resistance, electric and magnetic polarizations, etc. are basic quantities which characterize the classical electrical behavior of matter. However, for an accurate description of this behavior, these quantities can be constant, independent of the applied electromagnetic field only in the range of approximation that the

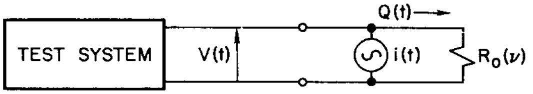

FIG. 11. Equivalent interaction system of a resistor at finite temperature when the coordinate associated with the resistor is the charge flowing through it.

magnitude of this applied field does not become too large. We will consider here the analysis of such systems and the approximations which make linear analysis valid. Insofar as linear behavior is obtained, the results of this section are basically the same as those obtained in previous sections with regard to finding influence functionals. However, it is interesting to see the same principles come out of the analysis another way. In addition, it will be found that expressions corresponding to Z_ν which appear in the influence functional are actually closed forms which can be used to compute such quantities as the conductivity from a knowledge of the unperturbed quantum characteristics of a system. These expressions have been derived before by several authors but it is interesting to find that they also appear in the influence functional quite naturally. The results will then be applied to the case of a beam of non-interacting particles passing near a test system such as a cavity. This analysis naturally lends itself to a discussion of noise in beam-type maser amplifiers. In Appendix III, the results of this and Section IV are used to compute the spontaneous emission of a particle in a cavity.

A. INTERACTION SYSTEMS WITH COUPLING POTENTIALS OF THE FORM $-V(Q)U(P)$

The specific result to be shown here can be stated as follows: If a general interaction system, P, is coupled to a test system Q so that the interaction potential is small and of the form $-V(Q)U(P)$, then the effect of the test system is that of a sum of oscillators whose frequencies correspond to the possible transitions of the interaction system. Therefore, to the extent that second order perturbation theory yields sufficient accuracy, the effect of an interaction system is that of a linear system.

To show this, we shall first assume the interaction system to be in an eigenstate $\phi_a(P_\tau) \exp\{-(i/\hbar)E_a\tau\}$ at the beginning of the interaction and in an arbitrary state at the end of the interaction consistent with the usual procedure we have followed. Also, for convenience in writing, the interaction potential will be assumed $-U(P)Q$. The influence functional is then

$$\begin{aligned}\mathfrak{F}(Q, Q') = \int \delta(P_T - P'_T) \exp\{(i/\hbar)[S(P) - S(P') \\ + \int_\tau^T (QU - Q'U')\, dt]\} \phi_a^*(P'_\tau)\phi_a(P_\tau)\, dP_\tau \cdots \mathfrak{D}P'(t)\end{aligned} \tag{5.1}$$

where in the above we have written U for $U(P)$ and U' for $U(P')$. Since the magnitude of the interaction is assumed to be small, the perturbation approach can be used to good advantage. Thus, expanding the interaction part of the exponent and keeping terms to second order in Q only,

$$\begin{aligned}\mathfrak{F}(Q, Q') = \int \delta(P_T - P'_T) \exp\{(i/\hbar)[S(P) - S(P')]\} \\ \times \{1 + (i/\hbar) \int_\tau^T (QU - Q'U')\, dt + (i/\hbar)^2 \int_\tau^T \int_\tau^t (Q_tU_t - Q'_tU'_t) \\ \times (Q_sU_s - Q'_sU'_s)\, ds\, dt\} \phi_a^*(P'_\tau)\phi_a(P_\tau)\, dP_\tau \cdots \mathfrak{D}P'(t)\end{aligned} \tag{5.2}$$

The evaluation of this is done in an entirely straightforward manner and the following is obtained,[18]

$$\mathfrak{F}(Q, Q') = 1 + (iU_{aa}/\hbar) \int_\tau^T (Q_t - Q'_t)\, dt - \sum_b |U_{ab}/\hbar|^2 \times \int_\tau^T \int_\tau^t (Q_t - Q'_t)\{Q_s \exp[-i\omega_{ba}(t - s)] - Q'_s \exp[i\omega_{ba}(t - s)]\}\, ds\, dt \tag{5.3}$$

where

$$U_{ab} = \int \phi_a^*(P) U \phi_b(P)\, dP \quad \text{and} \quad \omega_{ba} = (E_b - E_a)/\hbar$$

All terms which involve U_{aa} will be disregarded. This is because U_{aa} represents the average value of the operator $U(P)$ in an eigenstate of the interaction system alone. Even if it is not zero, it will be noted from Eq. (5.2) that terms involving U_{aa} can be written

$$1 + (iU_{ao}/\hbar) \int_\tau^T (Q_t - Q'_t)\, dt + (1/2!)[(iU_{aa}/\hbar) \int_\tau^T (Q_t - Q'_t)\, dt]^2 + \cdots = \exp\{(i/\hbar) \int_\tau^T U_{aa}(Q_t - Q'_t)\, dt\}$$

When $\mathfrak{F}(Q, Q')$ is used to make a calculation on the test system, this term has the effect of adding a constant potential $V(Q(t)) = -U_{aa}Q(t)$ to the unperturbed test system. Disregarding U_{aa}, the influence functional then becomes

$$\mathfrak{F}(Q, Q') = 1 + \alpha_a \tag{5.4}$$

where

$$\alpha_a = \left(\frac{1}{\hbar}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{t} \gamma_t \gamma_s (Q_t - Q'_t)[Q_s F_a^*(t - s) - Q'_s F_a(t - s)]\, ds\, dt$$

and

$$F_a(t - s) = \sum_b (|U_{ba}|^2/\hbar) \exp[i\omega_{ba}(t - s)]$$

where the limits have been extended and the factors $\gamma_t\gamma_s$ inserted to allow finite

---

[18] A typical term in Eq. (5.2) is as follows:

$$\int \delta(P_T - P'_T) \exp\{(i/\hbar)[S(P) - S(P')]\} (i/\hbar)^2 \int_\tau^T \int_\tau^t Q_t U_t Q_s U_s\, ds\, dt \times \phi_a^*(P'_\tau)\phi_a(P_\tau)\, dP_\tau \cdots \mathfrak{D}P(t)$$

Taking the time integrations outside the path integral and replacing the path integrations by propagating kernels, this becomes

$$(i/\hbar)^2 \int_\tau^T \int_\tau^t Q_t Q_s\, ds\, dt \int \delta(P_T - P'_T) K^*(P'_T, P'_\tau) K(P_T, P_t) \cdot U_t K(P_t, P_s) U_s K(P_s, P_\tau) \phi_a^*(P'_\tau)\phi_a(P_\tau) dP_\tau \cdots dP'_T$$

Remembering that $K(P_T, P_t) = \sum \phi_n(P_T)\phi_n^*(P_t) \exp\{-iE_n(T - t)/\hbar\}$, this expression becomes simply,

$$-\sum_b |U_{ab}/\hbar|^2 \int_\tau^T \int_\tau^t Q_t Q_s \exp[-i\omega_{ba}(t - s)]\, ds\, dt$$

coupling time if necessary. If the strength of coupling is sufficiently weak then Eq. (5.4) can be rewritten

$$\mathfrak{F} \approx \exp\,(\alpha_c) \tag{5.5}$$

In this form we recognize $\mathfrak{F}(Q, Q')$ as that describing the effect of a sum of harmonic oscillators independently connected to the test system each of whose "weights" is $U_{ba}U_{ab}/\hbar^2$. The complete response function for the system of oscillators is

$$B_a(t - s) = \operatorname{Im} F_a(t - s) = \textstyle\sum_b \,(2|U_{ba}|^2/\hbar)\sin\omega_{ba}(t - s)$$

where the subscript a on B_a refers to the initial eigenstate. According to previous definition, the mass of each individual oscillator is identified by $m = \hbar(2|U_{ba}|^2\omega_{ba})^{-1}$ and its characteristic frequency by ω_{ba}. Therefore, to the extent that second order perturbation theory yields satisfactory accuracy, any system may be considered as a sum of harmonic oscillators. This is equivalent in classical mechanics to the theory of the motion of a particle having small displacements around an equilibrium position. Its motion, to a first approximation, is also that of a harmonic oscillator, if the first effective term in a power series expansion of the potential around that equilibrium position is quadratic in the displacement.

In this part of the discussion we should again point out the motivation for writing the approximate influence function, Eq. (5.4), in terms of the approximate exponential of Eq. (5.5), apart from the obvious advantage of making the form agree with that of exactly linear systems. Frequently, we deal with a test system which is influenced by another system which is actually made up of a large number of very small systems. Examples of such an interaction system would be a beam of atoms or the electrons in a metal. Although the expression for the influence functional for any one of the subsystems is only good to second order, their individual effects are so small that this accuracy is very good and Eqs. (5.4) or (5.5) is equally valid. However, when the sum of the effects of the subsystems is not small, then the two forms above do not describe the situation equally insofar as the composite effect of the interaction system is concerned. We know that when these subsystems are dynamically and statistically independent, the total influence functional is simply a product of the individual ones. In such a case the influence functional obtained by using Eq. (5.5) as follows

$$\mathfrak{F}_k(Q, Q') \approx \exp\,\{i\textstyle\sum_k\Phi_k(Q, Q')\}$$

yields much greater accuracy than that obtained from Eq. (5.4) where we would find

$$\mathfrak{F}(Q, Q') \approx 1 + i\textstyle\sum_k \Phi_k(Q, Q')$$

$\Phi_k(Q, Q')$ being the influence phase for the kth subsystem (see Appendix IV).

The expression given by Eq. (5.5) has additional implications which are not immediately apparent from the analogy with the harmonic oscillator. No assumption was made as to whether the initial state was necessarily the lowest of the system. Therefore, ω_{ba} could be either positive or negative. Suppose for a moment that the interaction system has only two states ($\phi_a(P)$ and $\phi_b(P)$) and that the initial state, a, is the lower one ($\omega_{ba} > 0$). It is obvious that the only effect it can have on the test system Q is to absorb energy from Q. However, if the initial state, a, is the upper state (corresponding to $\omega_{ba} < 0$) then the interaction system can only give up energy to the test system. It can do this in two ways, through spontaneous emission or through coherent emission due to some coherent driving force exerted on it by the test system. So for the case $\omega_{ba} > 0$ we expect the influence functional to show that the interaction system has the effect of a cold system characterized by a dissipative impedance (or positive resistance). Conversely, for $\omega_{ba} < 0$ it is expected that $\mathfrak{F}(Q, Q')$ will be characterized by a negative resistance and a random potential due to the spontaneous emission transitions. This situation is made more obvious if we translate Eq. (5.5) into transform notation. Thus,

$$\begin{aligned}\Phi(Q, Q') = (2\pi\hbar)^{-1} \int_0^\infty \left[\frac{Q'_\nu(Q_{-\nu} - Q'_{-\nu})}{(i\nu Z_{ba,\nu})} + \frac{Q_{-\nu}(Q_\nu - Q'_\nu)}{(-i\nu Z_{ba,-\nu})}\right] d\nu \\ - (\pi\hbar^2)^{-1} \int_0^\infty |Q_\nu - Q'_\nu|^2 \pi\, |U_{ba}|^2 \delta(\nu + \omega_{ba})\, d\nu\end{aligned} \tag{5.6}$$

where

$$i\nu Z_{ba,\nu} = -\hbar(2\omega_{ba}|U_{ba}|^2)^{-1}[(\nu - i\epsilon)^2 - \omega_{ba}^2] \tag{5.7}$$

First of all we notice that the sign of $Z_{ba,\nu}$ changes with that of ω_{ba} and therefore its dissipative part can be positive or negative as was argued above. Secondly, for $\omega_{ba} < 0$ there is a random potential acting on Q whose power spectrum is given by $\phi(\nu) = \pi|U_{ba}|^2\delta(\nu + \omega_{ba})$. Of course when $\omega_{ba} > 0$, the integral involving this term disappears indicating the noise potential does not exist and the effect is the same as that of a harmonic oscillator initially in the ground state with $1/2m\omega$ identified with $|U_{ba}|^2/\hbar$.

In a real physical situation it is not likely that the interaction system will be in a definite state initially. So to extend the above results to a more general case we assume that the initial state is described as a sum over states weighted by a density matrix $\rho(\tau)$ which is diagonal in the energy eigenstates of the system. For example, if the system is initially in temperature equilibrium, $\rho = e^{-\beta H}/T_r(e^{-\beta H})$ where H is the Hamiltonian operator such that $\rho_{mn} = \delta_{mn} \exp(-\beta E_n)/\sum_n \exp(-\beta E_n)$. The influence functional becomes

$$\begin{aligned}\mathfrak{F}(Q, Q') = \int \delta(P_T - P'_T) \exp\{(i/\hbar)[S(P) - S(P') \\ + \int_{\cdot}^{T} (UQ - U'Q')\, dt]\} \sum_a \rho_{aa}\phi_a^*(P'_\tau)\phi_a(P_\tau)\, dP_\tau \cdots \mathfrak{D}P'(t)\end{aligned} \tag{5.8}$$

Within the limit of small coupling, then we can simply extend the influence phase of Eq. (5.5) by summing over all initial states weighted by the initial density matrix ρ_{aa}. If this is done, we obtain the usual form of the influence phase, Eq. (5.4), with a response function $B(t-s)$ given by

$$B(t-s) = \sum_{a,b} (2\rho_{aa} |U_{ab}|^2/\hbar) \sin \omega_{ba}(t-s) \tag{5.9}$$

Again $\Phi(Q, Q')$ is the phase for a sum of oscillators, each of whose weights is $\rho_{aa}|U_{ab}|^2/\hbar^2$.[19] In Section IV it was shown that $B(t-s)$ was the classical response of the linear system to an impulse of force applied to the coordinate $U(P)$. However, the expression above is in a form which is familiar to us only when we think of the interaction system, P, as consisting of a sum of oscillators. In this connection $B(t-s)$ is the total classical response of the oscillators describing the system P to an impulse of force applied to the "coordinate" $U(P)$.[20] To obtain a more direct interpretation of Eq. (5.9) we now calculate the linear classical response $B(t-s)$ of the interaction system to an applied impulse of force in terms of its unperturbed quantum characteristics. In so doing we will show that Eq. (5.9) is indeed this expression. Therefore, we will again have the result that the influence functional for a general, linear interaction system is formed simply from a knowledge of its classical characteristics just as in the case of systems of perfect oscillators. By the "classical characteristics" of the quantum mechanical system we mean the expected value of $U(P)$ as a function of time after a potential $-f(t)U(P)$ is applied starting at $t = 0$. Thus

$$\langle U(P_t)\rangle = \int \psi^*(P_t) U(P_t) \psi(P_t)\, dP_t \tag{5.10}$$

where $\psi(P_t)$ represents the state of system P at t. Using the path integral representation for the development of a wave function with time, as outlined in Section II, this can be written

$$\langle U(P_t)\rangle = \int U_t \delta(P_t - P'_t) \exp\{(i/\hbar)[S_0(P) - S_0(P') \\ + \textstyle\int_0^t f(s)(U_s - U'_s)\, ds]\}\psi^*(P'_0)\psi(P_0)\, dP_0 \cdots \mathfrak{D}P'(t)$$

Alternatively, this expression can evidently be written

$$\langle U(P_t)\rangle = -i\hbar \frac{\partial}{\partial f} \mathfrak{F}(f_t\,; f'_t)\,|_{f=f'}$$

---

[19] It is interesting to notice that the relative populations of any two levels may be described by an effective temperature $T_e = 1/k\beta_e$. For instance if the probabilities of occupation of states a and b are ρ_{aa} and ρ_{bb} respectively, we use the definition $\rho_{aa}/\rho_{bb} = \exp\beta_e(E_b - E_a)$. If $\rho_{aa} = 0$ this is described by setting $T_e = 0+$, meaning to approach zero from the positive side. Similarly, if the two states were inverted $\rho_{bb} = 0$ and $T_e = 0-$. This device has been used widely in the description of such situations.

[20] $U(P)$ may be regarded as a coordinate which is a function of other coordinates P in terms of which we choose to describe the interaction system.

The initial definite state $\psi_0^*(P')\psi_0(P)$ will be replaced by an average state described by the following density matrix diagonal in the energy representation, $\sum \rho_{aa}\phi_a^*(P'_0)\phi_a(P_0)$ so that

$$\psi^*(P'_0)\psi(P_0) = \textstyle\sum_a \rho_{aa}\phi_a^*(P'_0)\phi_a(P_0)$$

Assuming $f(s)$ to be small in magnitude, Eq. (5.10) can be written to first order

$$\begin{aligned}\langle U_t\rangle = {} & \textstyle\sum_a \rho_{aa} \int U_t\delta(P_t - P'_t) \exp\{(i/\hbar)[S_0(P) - S_0(P')]\} \\ & \qquad \cdot\{1 + (i/\hbar) \int_0^t f_s U_s\, ds - (i/\hbar) \int_0^t f_s U'_s\, ds\} \\ & \qquad \cdot\phi_a^*(P'_0)\phi_a(P_0)\, dP_0 \cdots \mathfrak{D}P'(t) \\ = {} & \textstyle\sum_a \rho_{aa}\{U_{aa} + \sum_b (i/\hbar)|\, U_{ab}\,|^2 \int_0^t f_s \exp[-i\omega_{ba}(t-s)]\, ds \\ & \qquad - \textstyle\sum_b (i/\hbar)|\, U_{ab}\,|^2 \int_0^t f_s \exp[i\omega_{ba}(t-s)]\, ds\}\end{aligned} \tag{5.11}$$

Again assuming $U_{aa} = 0$, Eq. (5.11) becomes

$$\langle U_t\rangle = \textstyle\sum_{a,b} (2\rho_{aa}|\, U_{ab}\,|^2/\hbar) \int_0^t f_s \sin\omega_{ba}(t-s)\, ds \tag{5.12}$$

For $f(s) = \delta(s)$, then, the classical response function is[21]

$$B(t) = \textstyle\sum_{a,b} (2\rho_{aa}|\, U_{ab}\,|^2/\hbar) \sin\omega_{ba}t \tag{5.13}$$

which is identical with the response function found in the influence functional.

Since we have found expressions identifiable as classical response functions and impedances, it remains to show that associated with the dissipative part of the impedance is a noise potential. The impedance is simply obtained from[22]

$$\begin{aligned}(i\nu Z_\nu)^{-1} &= \textstyle\int_0^\infty B(t)e^{-i\nu t}\, dt \\ &= \textstyle\sum_{a,b} (-2\rho_{aa}\,\omega_{ba}\,|\, U_{ba}\,|^2/\hbar)[(\nu - i\epsilon)^2 - \omega_{ba}^2]^{-1}\end{aligned} \tag{5.14}$$

To obtain the power spectrum it is only necessary to sum the influence phase of Eq. (5.6) over all initial states weighted by ρ_a . Thus we find

$$\phi(\nu) = \textstyle\sum_{a,b} \pi\rho_{aa}|\, U_{ab}\,|^2\, \delta(\nu + \omega_{ba}) \tag{5.15}$$

and we now wish to relate this to the real part of $1/Z_\nu$. From Eq. (5.14) it is found that

$$\mathrm{Re}\,(Z_\nu)^{-1} = \textstyle\sum_{a,b} (\pi\nu\,|\, U_{ba}\,|^2/\hbar)(\rho_{bb} - \rho_{aa})\delta(\nu + \omega_{ba}) \tag{5.16}$$

---

[21] It should be noted that implicit in the use of first order perturbation theory to obtain Eq. (5.13) is the fact that for this relation for the response function to hold as a steady-state description of the linear system, the initial distribution must not be significantly disturbed by the application of the driving force.

[22] Expressions of this kind have been used by several authors to compute quantities such as the conductivity of materials. See, for instance, ref. *14*.

Rewriting $\phi(\nu)$

$$\phi(\nu) = \sum_{a,b} \pi \mid U_{ab} \mid^2 (\rho_{bb} - \rho_{aa})[(\rho_{bb}/\rho_{aa}) - 1]^{-1} \delta(\nu + \omega_{ba}) \tag{5.17}$$

If the average initial state of the interaction system is one of temperature equilibrium, then $\rho_{aa} = e^{-\beta E_a} / \sum_n e^{-\beta E_n}$ and $\rho_{bb}/\rho_{aa} = e^{\beta(E_a - E_b)} \equiv e^{\beta \hbar \omega_{ab}}$. Taking advantage of the characteristics of $\delta(\nu + \omega_{ba})$ so that ν can replace ω_{ab}, from Eqs. (5.16) and (5.17)[23]

$$\begin{aligned} \phi(\nu) &= \sum_{a,b} (\pi/\hbar) \mid U_{ba} \mid^2 \nu(\rho_{bb} - \rho_{aa}) \delta(\nu + \omega_{ba}) \hbar [\nu(e^{\beta\hbar\nu} - 1)]^{-1} \\ &= \hbar \operatorname{Re} (Z_\nu)^{-1} [\nu(e^{\beta\hbar\nu} - 1)]^{-1} \end{aligned} \tag{5.18}$$

Therefore, again we find that the power spectrum is related to the dissipative parts of the impedance when the initial state is one of temperature equilibrium.[24] The power spectrum given in the form of Eq. (5.17) again illustrates the origin of thermal noise and identifies it as being just another aspect of spontaneous emission. Pound has also discussed this (*15*). The only contribution to the noise power spectrum is through the possible downward transitions of each possible state, a, weighted by the statistical factor, ρ_{aa}.

B. Beam of Particles Interacting with a Cavity

In the cases just considered the interaction system provided a steady-state environment for the test system. In contrast, let us now examine the situation where the interaction system is made up of a large number of independent particles coupled to the test system at different times. As an example consider a beam of noninteracting, identical particles which interact weakly with a resonant cavity as might occur, for instance, in a gas maser. We assume that the beam is not necessarily in temperature equilibrium but that the initial state of the particles entering the cavity would be properly represented by a density matrix diagonal in the energy representation. Such a situation would occur if the beam were prepared by passing it through a beam separator whose function would be to eliminate certain particles from the beam depending on their energy levels. For the purposes of simplifying the analysis we assume the molecules to be two-level quantum systems and that before they enter the cavity all of them are in

---

[23] Notice that if P were a two-level system initially in the lower state, then $\rho_{bb} = 0$ and $T_e = 0+$. In this case $\phi(\nu) = 0$ which agrees with the required result for $\nu > 0$ (Eq. (5.6) for $\omega_{ba} > 0$). If initially P were in the upper state then $\rho_{aa} = 0$ and $T_e = 0-$ yielding $\phi(\nu) = -(\hbar/\nu)\operatorname{Re}(1/Z_\nu)$ agreeing with Eq. (5.6) for $\omega_{ba} < 0$. This is the power spectrum of the so-called spontaneous emission noise from an inverted two-level system.

[24] It may be disturbing that $\operatorname{Re}(1/Z_\nu)$ contains singular forms such as $\delta(\nu + \omega_{ba})$. However, the infinite sums over the distribution of states of which it is a coefficient can be replaced by integrals over densities of states in most practical situations and as part of an integrand $\delta(\nu + \omega_{ba})$ is not unrealistic.

the lower state or all in the upper state. It is easy to extend these results to the case where the beam is mixed with a certain fraction in the upper state and a certain fraction in the lower state initially. Since the beam is assumed to be composed of noninteracting particles, we can consider the total beam as composed of two independent beams appropriate to the two possible initial states of the constituent particles. The influence phase for the complete beam is simply the sum of the influence phases for the two beams. In addition, we assume that the beam is characterized by a spatial density such that the number of particles passing a given point along the beam in a time dt is Ndt and if t_0 is the time a molecule passes a reference point in the cavity, $\gamma(t - t_0)$ describes the coupling between the molecule and cavity. Thus, the beam is a univelocity beam. Again, in a real case where the beam is characterized by a distribution of velocities, the total beam may be split up into many univelocity beams. The total influence phase is simply a sum of those for each component beam.

Let us call the coordinates of the cavity Q, representing the test system and the coordinates of a particle in the beam P. The interaction between beam and particles is given by $L_I(Q, P) = \gamma(t - t_0)QP$. Under these circumstances the influence functional for the effect of the beam on the cavity can be written down immediately:

$$\begin{aligned} i\Phi_B(Q, Q') = -N \,|\, P_{ab}/\hbar \,|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{t} ds\, dt \left[\int_{-\infty}^{\infty} \gamma(t - t_0)\gamma(s - t_0)\, dt_0 \right] \\ \times (Q_t - Q'_t)(Q_s e^{-i\omega_{ba}(t-s)} - Q'_s e^{i\omega_{ba}(t-s)}) \end{aligned} \tag{5.19}$$

The integral involving the coupling parameters can be used to define a new function Γ,

$$\begin{aligned} \int_{-\infty}^{\infty} \gamma(t - t_0)\gamma(s - t_0)\, dt_0 &= \int_{-\infty}^{\infty} \gamma(\xi)\gamma(\xi - t + s)\, d\xi \\ &\equiv \Gamma(t - s) \end{aligned} \tag{5.20}$$

Therefore,

$$\begin{aligned} i\Phi(Q, Q') = -N \,|\, P_{ab}/\hbar \,|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{t} \Gamma(t - s) \\ \times (Q_t - Q'_t)(Q_s e^{-i\omega_{ba}(t-s)} - Q'_s e^{i\omega_{ba}(t-s)})\, ds\, dt \end{aligned} \tag{5.21}$$

From this we can identify the response function of the beam as

$$B(t - s) = (2/\hbar)N \,|\, P_{ab} \,|^2 \Gamma(t - s) \sin \omega_{ba}(t - s) \tag{5.22}$$

In transform notation the influence functional for the effect of the beam has the same form as has been previously derived with

$$(i\nu Z_\nu)^{-1} = (i/\hbar)N \,|\, P_{ab} \,|^2 (\Gamma_{\nu-\omega_{ba}} - \Gamma_{\nu+\omega_{ba}}) \tag{5.23}$$

where

$$\Gamma_{\nu+\omega_{ba}} = \int_{-\infty}^{\infty} 1(r)\Gamma(r) e^{-i(\nu+\omega_{ba})r}\, dr \tag{5.24}$$

and with a power spectrum

$$\phi(\nu) = \tfrac{1}{2}N \mid P_{ab} \mid^2[\Gamma_{\nu+\omega_{ba}} + \Gamma_{-\nu-\omega_{ba}}] \tag{5.25}$$

Previously, we have found that for a linear system initially in the ground state the power spectrum is zero over the range of positive ν thus indicating a zero noise potential due to the linear system. However, ϕ_ν is not necessarily zero for $\nu > 0$ in Eq. (5.25) in the case that the beam is initially in its lowest state ($\omega_{ba} > 0$). This is because no restrictions were placed on the time variation of $\gamma(t - t_0)$. However, in practical situations the coupling between a cavity and a particle in a beam passing through the cavity varies adiabatically so that for all practical purposes $\phi(\nu)$ is really zero for $\nu > 0$. To point this out more clearly, let us assume that $\gamma(t - t_0) = 1(t - t_0)1(t_0 + \tau_0 - t)$, that is, the coupling is turned on at t_0 and off at $t_0 + \tau_0$, the time of transit in the cavity being τ_0. We find by evaluating Eq. (5.20) that

$$\Gamma(t - s) = (s - t + \tau_0)1(s - t + \tau_0)$$

and from this we can find

$$\begin{aligned}\Gamma_{\nu+\omega_{ba}} &= \int_{-\infty}^{\infty} (\tau_0 - r)1(\tau_0 - r)1(r) \exp[-i(\nu + \omega_{ba})r]\, dr \\ &= \{1 - i\tau_0(\nu + \omega_{ba}) - \exp[-i(\nu + \omega_{ba})\tau_0]\}(\nu + \omega_{ba})^{-2}\end{aligned}$$

Therefore, from Eq. (5.25),

$$\phi(\nu) = (\tfrac{1}{2})N \mid P_{ab} \mid^2 \tau_0^2(\sin^2\theta/\theta^2) \tag{5.26}$$

where

$$\theta = \tfrac{1}{2}(\nu + \omega_{ba})\tau_0$$

To find the effect of suddenly turning the coupling on and off we find the ratio of the total noise power to the noise power for $\nu > 0$. This is given by

$$\frac{\int_{-\infty}^{\infty} \phi(\nu)\, d\nu}{\int_0^{\infty} \phi(\nu)\, d\nu} > \frac{\int_{-\infty}^{\infty} (\sin^2\theta/\theta^2)\, d\theta}{\int_{\omega_{ba}\tau_0}^{\infty} d\theta/\theta^2} = \pi\omega_{ba}\tau_0$$

For ammonia molecules at a temperature of $T = 290°\mathrm{K}$, the average velocity $v \approx 6 \times 10^4$ cm/sec. For a microwave cavity of 10 cm length $\tau_0 = 2 \times 10^4$ sec. For the 3–3 line of ammonia $\omega_{ba} \approx 1.5 \times 10^{11}$ rad/sec. For our case then $(\pi\omega_{ba}\tau_0) \approx 10^8$. From this, we can conclude that even in this unfavorable case of coupling time variation, the $\phi(\nu)$ is negligible for $\nu > 0$.

Examination of $\mathrm{Re}(1/Z_\nu)$, derived from Eq. (5.23), reveals terms of the same form as those just discussed,

$$\begin{aligned}\mathrm{Re}(1/Z_\nu) &= \tfrac{1}{2}[(1/Z_\nu) + (1/Z_\nu^*)] \\ &= (\nu N/2\hbar) \mid P_{ab} \mid^2[\Gamma_{\nu+\omega_{ba}} - \Gamma_{\nu-\omega_{ba}} - \Gamma_{-\nu+\omega_{ba}} + \Gamma_{-\nu-\omega_{ba}}]\end{aligned} \tag{5.27}$$

By the same type of argument as above, the terms in $\Gamma_{\nu-\omega_{ba}}$ can be neglected when $\nu > 0$, $\omega_{ba} < 0$ and conversely, when $\nu > 0$, $\omega_{ba} > 0$ the terms in $\Gamma_{\nu+\omega_{ba}}$ can be neglected. Since this is the case, we can write

$$\phi(\nu) \approx \hbar \operatorname{Re}(1/Z_\nu)[\nu(e^{-\beta_e \hbar \omega_{ba}} - 1)]^{-1} \tag{5.28}$$

where $\beta_e = 1/kT_e$, describes the relative initial populations of the two states. Therefore, we conclude that in most practical cases the power spectrum can be written in the form of Eq. (5.28). In cases where the transients cannot be neglected for very low ω_{ba} and very short transit times, however, $\phi(\nu)$ is not so simply related to $\operatorname{Re}(1/Z_\nu)$ and must be written in the form given by Eq. (5.25).

It is to be noted that $\phi(\nu)$ of Eq. (5.28) is not precisely of the form for the Nyquist relation because of the appearance of $e^{\beta\hbar\omega_{ba}}$ in the denominator rather than $e^{\beta\hbar\nu}$. This is a consequence of the finite coupling time between each part of the beam and the cavity which results in a nonequilibrium condition. If the coupling times were infinite, the expression for $\operatorname{Re}(1/Z_\nu)$ would contain forms such as $\delta(\nu + \omega_{ba})$, a situation discussed earlier, so that the Nyquist form then results. However, when the coupling time is long as in masers,

$$\Gamma_{\nu+\omega_{ba}} + \Gamma_{-\nu-\omega_{ba}} \approx 4\pi\delta(\nu + \omega_{ba})$$

so that the true Nyquist relation may be used with negligible error.

VI. SOURCES OF NOISE IN MASERS

Having developed the theory of linear systems in detail we are now in a position to discuss the sources of noise in linear quantum-mechanical devices such as maser amplifiers. The subject of maser noise has been explored by many authors (*15–19*). The details of the treatment given the subject differ, but the principles are essentially the same. The amplifiers are considered as operating at signal levels high enough (classical) that a signal entering a maser may be considered as a group of photons whose number is large enough that the amplification process increases the signal in a continuous fashion. The sources of noise were found to be those derived from the thermal noise arising from the sources of dissipation, and those derived from spontaneous emission from the "active" quantum material. They go further and define an effective temperature of the active quantum system so that the noise it produces is related to the negative resistance of the active materials. Our analysis of linear systems has also shown that these same sources of noise exist. However, if the signal level entering the maser is very small such that its strength can be characterized by a few quanta per second, a serious question arises as to the nature of the signal out of the maser (here assumed to be at a classical level). An additional fluctuation of the output signal or "quantum" noise might be expected due solely to what might be termed a "shot noise" effect created by each individual photon entering the cavity. It has been shown that no such signal exists by other authors (*20, 21*). We wish to show how the same result follows from the theory developed here in an extremely transparent way. We will show that the only fluctuations in output signal which are to be expected are those noise sources computed classically. No additional "shot noise" does, in fact, appear.

Let us suppose that we have a beam-type maser amplifier in which all participating systems used meet the requirement of linearity. There may be one or more beams interacting with various electromagnetic resonators which can be coupled together in any way desired. The output of the maser is connected to a detector of some sort which perhaps consists of a resistor in which the current is to be measured. To the input of the maser system we now apply an incoming classical signal of large magnitude and of frequency ω through an attenuator whose value of attenuation may be varied at will as shown in Fig. 12. In practice such a situation could arise if the classical wave originates from a distant antenna with a very large magnitude of output, so large that all quantum effects in the wave are effectively obscured. The long distance would then play the role of the attenuator.

Now, if the classical wave were attenuated by a large amount so that only a few photons/sec were entering the maser, the only uncertainty in the signal in the output of the maser caused by the maser itself arises from those sources of noise which can be arrived at by a classical calculation of the characteristics of the maser. There is no extra quantum fluctuation introduced by the maser into the output signal due to the small number of quanta entering the maser.

It is true that the amplitude of the signal output from the maser might itself be so small that it is still on a quantum level. In this case the detector output would be uncertain due to the inherently small magnitude of the signal from the maser. However, if this is the case, we may put as many amplifiers in series as necessary to bring the output signal back to a classical level. When this is done the signal applied to the detector consists of the original signal modified by the transfer characteristics of the maser system (and attenuator) and noise signals which arise from all the possible sources computed classically. The proof of this assertion is not difficult. We divide up the total system into a test system, here the detector, and an interaction system which consists of the maser, attenuator, and classical signal, $C(t)$. Then, to find the effect of the interaction system on the

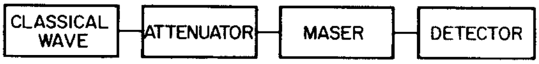

FIG. 12. System in which a classical wave is attenuated to a very low level (a few photons/sec), then amplified by a maser and detected.

detector we need only to look at the influence functional. However, we already know that it can be written as follows:

$$\mathfrak{F}(Q, Q') = \exp\left\{\left(\frac{i}{2\pi\hbar}\right)\int_0^\infty \left[\frac{Q'_\nu(Q_{-\nu} - Q'_{-\nu})}{(i\nu Z_\nu)} + \frac{Q_{-\nu}(Q_\nu - Q'_\nu)}{(-i\nu Z_{-\nu})}\right] d\nu \right.$$
$$\left. + \left(\frac{i}{2\pi\hbar}\right)\int_{-\infty}^\infty \left(\frac{C_\nu}{i\nu z_\nu}\right)(Q_{-\nu} - Q'_{-\nu})\, d\nu - \left(\frac{1}{\pi\hbar^2}\right)\int_0^\infty |Q_\nu - Q'_\nu|^2 \textstyle\sum_i \phi_i(\nu)\, d\nu\right\}$$

where $\phi_i(\nu)$ represents the power spectrum of the noise from the ith source, z_ν is the classical transfer characteristic of the complete interaction system, and Z_ν is the impedance of the maser system as seen by the detector. All the terms in the influence functional are familiar in view of the derivations which have been presented previously. The first term in the exponent of the influence functional is recognized as describing a linear system at zero temperature (Section IV), the linear system in this case being the maser. This term describes the spontaneous emission of the detector back into the maser (see Appendix I). Furthermore, it can be deduced that this spontaneous emission can be thought of as resulting from a noise generator created by the detector (test system) acting on the interaction system in the usual classical way, i.e., whose power spectrum was related to the dissipative part of the detector impedance and to the temperature by the Nyquist relation

$$\phi(\nu) = \hbar\nu R_\nu[\exp(\hbar\nu/kT) - 1]^{-1}$$

where R_ν is the detector resistance and T its temperature. The second term in the above composite influence functional is easily interpreted and is simply the effect of a classical voltage, related to the input voltage by the classical transfer characteristic of the maser, acting on the detector (see Section IV.B). The last term represents the effect on the detector of random noise voltages associated with the various classical noise sources in the maser (Sections IV.C and V). Both positive and negative resistances are such noise sources. In either case the power spectrum of the noise from a particular resistance is computed from the same relation as given above.

If R_ν is negative the effective temperature of R_ν will also be negative always giving a positive power spectrum. Therefore, if we were to compute the current in the detector due to the interaction system (maser) using the influence functional we would find components of current due to: (1) the noise voltage generated by the detector itself, the power spectrum of which is related to the resistance of the detector by the generalized Nyquist relation given above; (2) a classical voltage related to the input voltage $C(t)$ by the classical transfer characteristic of the maser; and (3) random noise voltages associated with the various classical noise sources (resistances) in the maser. Therefore, the maser simply acts as a classical amplifier with sources of noise which can be predicted from considerations of its classical characteristics.

APPENDIX I

In Section II.E the problem of making perturbation calculations using influence functionals was outlined. Here we will calculate in detail the probability, to second order in the coupling potentials, that a test system which is in a definite state at $t = \tau$ and finds itself in another state at $t = T$ when interacting with a zero temperature interaction system. Let us call the initial and final states $\phi_n(Q_\tau)$ and $\phi_m(Q_T)$ respectively and, for simplicity, assume that these are eigenstates. The formal expression for the transition probability is given by Eq. (4.24) and for this case,

$$\begin{aligned}\mathfrak{F}(Q, Q') = \exp \{ -(2\hbar)^{-1} \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{t} \gamma_t \gamma_s (Q_t - Q'_t) \\ \times [Q_s F^*(t-s) - Q'_s F(t-s)]\, ds\, dt \}\end{aligned} \tag{I.1}$$

Then to second order

$$\begin{aligned}P_{nm} \approx \textstyle\int \phi_m^*(Q_T) \phi_m(Q'_T) \exp \{ (i/\hbar)[S_0(Q) - S_0(Q')] \} \\ \times \{ 1 - (2\hbar)^{-1} \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{t} \gamma_t \gamma_s (Q_t - Q'_t)[Q_s F^*(t-s) - Q'_s F(t-s)]\, ds\, dt \} \\ \times \phi_n^*(Q'_\tau) \phi_n(Q_\tau)\, dQ_\tau \cdots \mathfrak{D}Q'(t)\end{aligned} \tag{I.2}$$

Making use of the fact that

$$\begin{aligned}\textstyle\int \exp [(i/\hbar) S_0(Q)] \mathfrak{D}Q(t) &= K(Q_T, Q_\tau) \\ &= \textstyle\int K(Q_T, Q_t) K(Q_t, Q_\tau)\, dQ_t \quad (\text{where } T < t < \tau)\end{aligned}$$

and writing (I.2) as a sum of integrals

$$\begin{aligned}P_{nm} \approx \textstyle\int \phi_m^*(Q_T) \phi_m(Q'_T) \exp \{ (i/\hbar)[S_0(Q) - S_0(Q') \} \\ \cdot \phi_n^*(Q'_\tau) \phi_n(Q_\tau)\, dQ_\tau \cdots \mathfrak{D}Q'(t) \\ - (2\hbar)^{-1} \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{t} ds\, dt \gamma_s \gamma_t \int \phi_m^*(Q_T) \phi_m(Q'_T) \\ \cdot \exp \{ (i/\hbar)[S_0(Q) - S_0(Q')] \} \\ \times (Q_t - Q'_t)[Q_s F^*(t-s) - Q'_s F(t-s)] \\ \cdot \phi_n^*(Q'_\tau) \phi_n(Q_\tau)\, dQ_\tau \cdots \mathfrak{D}Q'(t)\end{aligned} \tag{I.3}$$

Replacing $K(Q_T, Q_t)$ by $\sum_k \phi_k(Q_T) \phi_k^*(Q_t) \exp [-iE_k(T-t)/\hbar]$ and taking matrix elements we have

$$P_{nm} = \delta_{nm}[1 - \textstyle\sum_k (2\hbar)^{-1} | Q_{nk} |^2 f(\nu_{nk})] + (2\hbar)^{-1} | Q_{nm} |^2 f(\nu_{nm}) \tag{I.4}$$

where

$$f(\nu_{nm}) = \int_{-\infty}^{\infty} \int_{-\infty}^{t} \gamma_t \gamma_s [F^*(t-s)e^{i\nu_{nm}(t-s)} + F(t-s)e^{-i\nu_{nm}(t-s)}]\, ds\, dt \quad (\text{I.5})$$

$f(\nu)$ can be simplified by restating the integral over t and s in terms of frequency. To do this we replace the upper limit t by $+\infty$ and multiply the integrand by a step function $1(t-s)$. Then utilizing the convolution theorem in the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M(t)N(s)R(t-s)\, ds\, dt = (2\pi)^{-1} \int M_{-\nu} N_\nu R_\nu \, d\nu$$

where

$$M(\nu) = \int_{-\infty}^{\infty} M(t) e^{-i\nu t}\, dt$$

$f(\nu)$ becomes

$$f(\nu) = (2\pi)^{-1} \int_{-\infty}^{\infty} |\gamma_{\nu-\omega}|^2 (F_\nu + F_\nu^*)\, d\nu$$

where

$$F_\nu = \int_{-\infty}^{\infty} 1(t) F(t) e^{-i\nu t}\, dt$$

From Eqs. (4.5a) and (4.5b) we have that

$$F(t) = -(2/\pi) \int_0^\infty \operatorname{Im}(1/i\eta Z_\eta) e^{i\eta t}\, d\eta$$

Therefore,

$$\begin{aligned} F_\nu + F_\nu^* &= -(2/\pi) \int_0^\infty d\eta \operatorname{Im}(1/i\eta Z_\eta) \int_0^\infty dt (e^{i(\eta-\nu)t} + e^{-i(\eta-\nu)t}) \\ &= -4 \operatorname{Im}(1/i\nu Z_\nu) = (4/\nu) \operatorname{Re}(1/Z_\nu) \end{aligned}$$

and

$$f(\nu) = (2/\pi) \int_{-\infty}^{\infty} |\gamma_{\eta-\nu}|^2 \eta^{-1} \operatorname{Re}(1/Z_\eta)\, d\eta \quad (\text{I.6})$$

Thus, to second order

$$\begin{aligned} P_{nm} = \delta_{nm}\{1 - \textstyle\sum_k (\pi\hbar)^{-1} |Q_{nk}|^2 \int_0^\infty \nu^{-1} |\gamma_{\nu-\nu_{nk}}|^2 \operatorname{Re}(Z_\nu)^{-1} d\nu\} \\ + (\pi\hbar)^{-1} |Q_{nm}|^2 \int_0^\infty \nu^{-1} |\gamma_{\nu-\nu_{nm}}|^2 \operatorname{Re}(Z_\nu)^{-1} d\nu \end{aligned} \quad (\text{I.7})$$

For the special case that the coupling γ is

$$\gamma = 0 \quad \text{for} \quad t > T/2 \quad \text{and} \quad t < -T/2$$

$$\gamma = 1 \quad \text{for} \quad -T/2 < t < T/2$$

$$\begin{aligned} |\gamma_{\nu-\nu_{nk}}|^2 = \left| \int_{-T/2}^{T/2} e^{-i(\nu-\nu_{nk})t} dt \right|^2 &= 4(\nu-\nu_{nk})^{-2} \sin^2 (\nu-\nu_{nk})T/2 \\ &\rightarrow 2\pi T \delta(\nu-\nu_{nk}) \quad \text{for large } T \end{aligned}$$

Then,

$$\begin{aligned} P_{nm} &= \delta_{nm}\{1 - \textstyle\sum_{k,\nu_{nk}>0} 2T(\hbar\nu_{nk})^{-1} \mid Q_{nk} \mid^2 \mathrm{Re}(Z_{\nu_{nk}})^{-1}\} \\ &\quad + 2T(\hbar\nu_{nm})^{-1} \mid Q_{nm} \mid^2 \mathrm{Re}(Z_{\nu_{nm}})^{-1} \quad \text{for } \nu_{nm} > 0 \\ &= 0 \text{ for } \nu_{nm} < 0 \end{aligned} \tag{1.8}$$

Thus, it is seen that the P_{nm} now is proportional to the matrix element and to the dissipative part of the impedance $Z_{\nu_{nm}}$. Appropriately enough, no transition is possible to energy states such that $\nu_{nm} < 0$ since the interaction system, being at zero temperature initially, can give up no energy to the test system.

APPENDIX II

A. INFLUENCE PHASE FOR EFFECT OF FREE SPACE ON AN ATOM

As an illustration, the influence phase for the effect of free space on an atom will be calculated. This problem is more complicated than the idealized systems considered in deriving the formalism since the interaction here is of the form $\mathbf{Q}\cdot\mathbf{X}$, $\mathbf{Q}$ and $\mathbf{X}$ being vectors rather than QX where Q and X are scalars. This difficulty could be overcome by writing the influence phase in tensor notation or by recasting the problem so that the interaction is of the form QX. The latter will be done to adhere more closely to the point of view of the derivations. Since a linear system is being dealt with, it is only necessary to determine a suitable coordinate for the atom and find the impedance function Z_ν for the effect of free space. It is assumed that the atom is made up of a system of particles of mass m_n, charge e_n, and position $\mathbf{r}_A + x_n$ where $\mathbf{r}_A$ is the position of the center of charge of the atom. If the transverse part of the radiation field in the box is expanded into a series of plane waves each representing independent harmonic oscillations (*22*), then the nonrelativistic Lagrangian for the complete system consisting of the atom and the field in the box can be written (*2*)

$$\begin{aligned} L(\dot{X}, X, \dot{q}_{\mathbf{k}}, q_{\mathbf{k}}, t) &= L_A + c^{-1}\textstyle\sum_n e_n\dot{\mathbf{X}}_n\cdot\mathbf{A}^{\mathrm{tr}}(\mathbf{r}_A + \mathbf{r}_n) \\ &\quad + \tfrac{1}{2}\textstyle\sum_{\mathbf{k}} \sum_{r=1}^{4} [(\dot{q}_{\mathbf{k}}^{(r)}) - c^2k^2(q_{\mathbf{k}}^{(r)})^2] \end{aligned} \tag{II.1}$$

where L_A is the Lagrangian of the atom unperturbed by outside forces and

$$\begin{aligned} \mathbf{A}^{\mathrm{tr}}(\mathbf{X}) &= (8\pi c^2)^{1/2}\textstyle\sum_{\mathbf{k}} [\mathbf{e}_\lambda(q_{\mathbf{k}}^{(1)}\cos(\mathbf{k}\cdot\mathbf{X}) + q_{\mathbf{k}}^{(3)}\sin(\mathbf{k}\cdot\mathbf{X})) \\ &\quad + \mathbf{e}_l(q_{\mathbf{k}}^{(2)}\cos(\mathbf{k}\cdot\mathbf{X}) + q_{\mathbf{k}}^{(4)}\sin(\mathbf{k}\cdot\mathbf{X}))] \end{aligned} \tag{II.2}$$

Here $\mathbf{e}_\lambda$ and $\mathbf{e}_l$ are two mutually orthogonal polarization vectors, each orthogonal to the propagation vector $\mathbf{k}$. Now we assume that the radiation field of the box is constant over the particle, i.e., that $\mathbf{A}(\mathbf{r}_A + \mathbf{r}_n) \approx \mathbf{A}(\mathbf{r}_A)$, the dipole approximation.[25] This permits one to replace $\sum_n e_n\dot{\mathbf{X}}_n$ by $\mathbf{j}$, the current operator for

---

[25] This is equivalent to taking $\mathbf{A}(\mathbf{k}_A + \mathbf{x}_n)$, expanding it in a series of $\mathbf{k}\cdot\mathbf{x}_n$ since this is assumed small, and keeping only those terms which keep the interaction term of the Lagrangian linear. Since the interaction is of the form $e_n\mathbf{x}_n\cdot\mathbf{A}(\mathbf{k}_A + \mathbf{x}_n)$ for the nth particle, then $\mathbf{A}$ can only contain constant terms.

the atom. In addition, even though $\mathbf{e}_\lambda \cdot \mathbf{e}_l = 0$, which fixes their relative orientations, their absolute directions in a plane perpendicular to $\mathbf{k}$ are still arbitrary. Choosing $\mathbf{e}_l$ so that

$$\mathbf{e}_l \cdot \mathbf{j} = 0 \tag{II.3}$$

We assume the box to be very large so that $\sum_{\mathbf{k}} \to (16\pi^3)^{-1} \int d^3\mathbf{k}$, because the mode corresponding to $\mathbf{k}$ and $-\mathbf{k}$ is the same. Combining (II.1), (II.2), and (II.3) the following total Lagrangian is obtained.

$$\begin{aligned} L_T = L_A + \sum_{r=1,3} \int \frac{d^3\mathbf{k}}{16\pi^3} \left[\frac{1}{2} (\dot{q}_{\mathbf{k}}^{(r)})^2 - \frac{1}{2} k^2 c^2 (q_k^{(r)})^2 \right] \\ + (8\pi)^{1/2} \int \frac{d^3\mathbf{k}}{16\pi^3} (\mathbf{j} \cdot \mathbf{e}_\lambda) q_{\mathbf{k}}^{(1)} \cos(\mathbf{k} \cdot \mathbf{r}_A) \\ + (8\pi)^{1/2} \int \frac{d^3\mathbf{k}}{16\pi^3} (\mathbf{j} \cdot \mathbf{e}_\lambda) q_{\mathbf{k}}^{(3)} \sin(\mathbf{k} \cdot \mathbf{r}_A) \end{aligned} \tag{II.4}$$

Thus, the number of each of the two sets of oscillators ($q_{\mathbf{k}}^{(1)}$ and $q_{\mathbf{k}}^{(3)}$) in a volume of k space $d^3\mathbf{k}$ is $d^3\mathbf{k}/16\pi^3$. The coupling strength of the $q_{\mathbf{k}}^{(1)}$ and $q_{\mathbf{k}}^{(3)}$ oscillators with the atom is $(8\pi)^{1/2}(\mathbf{j} \cdot \mathbf{e}_\lambda) q_{\mathbf{k}}^{(1)} \cos(\mathbf{k} \cdot \mathbf{r}_A)$ and $(8\pi)^{1/2}(\mathbf{j} \cdot \mathbf{e}_\lambda) q_{\mathbf{k}}^{(3)} \cos(\mathbf{k} \cdot \mathbf{r}_A)$ respectively. If $\mathbf{j}$ is oriented along the $\theta = 0$ axis in polar coordinate representation, $\mathbf{j} \cdot \mathbf{e}_\lambda = j \sin\theta$. Then choosing j as the atom coordinate, the impedance for the two oscillators of frequency kc can be found from the rule found in Section IV to be

$$\begin{aligned} [i\nu Z_\nu(\mathbf{k})]^{-1} &= (1/j_\nu)[(8\pi)^{1/2} \sin\theta \cos(\mathbf{k} \cdot \mathbf{r}_A) q_{\mathbf{k}}^{(1)} + (8\pi)^{1/2} \sin\theta \sin(\mathbf{k} \cdot \mathbf{r}_A) q_{\mathbf{k}}^{(3)}]_\nu \\ &= -\frac{8\pi \sin^2\theta \cos^2(\mathbf{k} \cdot \mathbf{r}_A)}{(\nu - i\epsilon)^2 - k^2c^2} - \frac{8\pi \sin^2\theta \sin^2(\mathbf{k} \cdot \mathbf{r}_A)}{(\nu - i\epsilon)^2 - k^2c^2} \\ &= -8\pi \sin^2\theta [(\nu - i\epsilon)^2 - k^2c^2]^{-1} \big|_{\epsilon \to 0} \end{aligned} \tag{II.5}$$

The total effect of all the oscillators is

$$\begin{aligned} (Z_\nu)^{-1} &= (16\pi^3)^{-1} \int Z_\nu^{-1}(\mathbf{k})\, d^3\mathbf{k} = (16\pi^3)^{-1} \int k^2 \sin\theta\, d\theta\, d\phi\, dk (Z_\nu^{-1}) \\ &= -i4\nu(3\pi c^3)^{-1} \int_0^\infty \Omega^2\, d\Omega [(\nu - i\epsilon)^2 - \Omega^2]^{-1} \big|_{\epsilon \to 0} \\ &= (2\nu^2/3c^3) - i(4\nu/3\pi c^3) \int_0^\infty \Omega^2\, d\Omega (\nu^2 - \Omega^2)^{-1} \end{aligned} \tag{II.6}$$

where the substitution $\Omega = kc$ has been made. Thus, the effect of free space is characterized by $(i\nu Z_\nu)^{-1}$. The equivalent distribution of oscillators coupled to j_ν is

$$G(\Omega) = (4\Omega^2/3\pi c^3) \tag{II.7}$$

B. Spontaneous Emission Probability of an Atom in Free Space

To compute the transition probability for this atom, we use second order perturbation theory developed in Appendix I for a system initially in state $\phi_n(X_\tau)$ and finally in state $\phi_m(X_T)$ when acted on by an influence functional for a linear system at zero temperature. The expression is

$$\begin{aligned} P_{nm} &= 2T \mid j_{nm} \mid^2 (\hbar\nu_{nm})^{-1} \operatorname{Re}(Z_{\nu_{nm}})^{-1}, \quad \nu_{nm} > 0, \; n \neq m \\ &= 0, \; \nu_{nm} < 0 \end{aligned} \tag{II.8}$$

From Eq. (II.7) we find

$$\operatorname{Re}(Z_{\nu_{nm}})^{-1} = (\pi/2)G(\nu_{nm}) = 2\nu_{nm}^2/3c^3 \tag{II.9}$$

Using this in Eq. (II.8)

$$\begin{aligned} P_{nm} &= 4\nu_{nm} \mid j_{nm} \mid^2 T/3\hbar c^3 \\ &= 4\nu_{nm}^3 e^2 \mid X_{nm} \mid^2 T/3\hbar c^3 \end{aligned} \tag{II.10}$$

where to obtain the last, more familiar form, the substitution $j_{nm} = \nu_{nm} X_{mn}$ has been made. This is the first order spontaneous emission probability for an atom in free space.

Now we can form an expression for the intensity of radiation per unit time. The power radiated from the dipole is

$$\begin{aligned} \hbar\nu_{nm} P_{nm}/T &= 2\nu_{nm}^2 e^2 \mid X_{nm} \mid^2 \operatorname{Re}(Z\nu_{nm})^{-1} \\ &= 4e^2 \mid X_{nm} \mid^2 \nu_{nm}^4/3c^2 \end{aligned} \tag{II.11}$$

an expression which is almost the same as that for power radiated from a classical dipole. The expression becomes exactly the same if we apply the correspondence principle by replacing the matrix element of the time average of the coordinate of the oscillator by its corresponding classical quantity. Thus, if X is the coordinate of the corresponding classical oscillator (X_0 is its maximum value) then[26]

$$2 \mid X_{mn} \mid^2 \rightarrow \langle X^2 \rangle = \tfrac{1}{2} X_0^2$$

---

[26] That this correspondence is true can be seen easily as follows. Consider the dipole, a harmonic oscillator as above, to be in a high quantum state, ϕ_n. Classically the motion of the dipole can be described as $X = X_0 \sin \omega t$. We wish to relate the classical value of $\langle X^2 \rangle$ to its matrix element. In a quantum mechanical sense,

$$\langle X^2 \rangle = \int \phi_n^* (X) X^2 \phi_n(X) \, dX = \iint \phi_n^* (X) X \textstyle\sum_k \phi_k(X) \phi_k^*(X') X' \phi_n(X') \, dX \, dX'$$

Since matrix elements exist, in the case of a harmonic oscillator only for $k = n - 1$, $k = n + 1$, we have

$$\langle X^2 \rangle = \mid X_{n,n-1} \mid^2 + \mid X_{n,n+1} \mid^2$$

For very high quantum numbers these two terms become nearly equal since

$$\mid X_{n,n-1} \mid^2 = n\hbar/2m\omega, \quad \mid X_{n,n+1} \mid^2 = (n+1)\hbar/2m\omega$$

Thus, as $n \rightarrow \infty$ $\langle X^2 \rangle = 2 \mid X_{n,n-1} \mid^2$. But in the classical case, $\bar{X}^2 = \tfrac{1}{2} X_0^2$. Therefore, $\mid X_{n,n-1} \mid^2 \Rightarrow \tfrac{1}{4} X_0^2$.

If $|X_{mn}|^2$ is replaced by $X_0^2/4$ then Eq. (II.11) becomes the expression for the power radiated from a classical dipole. Our purpose, however, in doing this example, was to show for a specific problem that the effect of a distribution of oscillators interacting on a system is the same as the effect of loss on the system.[27] This has been done by relating the energy lost from the radiating dipole to the distribution. It is not surprising that a sea of oscillators should give this effect. If the dimensions of the box are allowed to be finite then energy emitted from the system under observation is reflected from the walls and eventually finds its way back to be absorbed again. This is equivalent to saying that the number of oscillators comprising the electromagnetic field in the box is infinite with a finite frequency spacing between the modes. Since the oscillators are independent there is no coupling between them and energy coupled into one of the oscillators from the test system must eventually return to it. If the dimensions of the box are allowed to get infinitely large, energy emitted from the test system never gets reflected and thus never returns. In oscillator language this means that the frequency spacing between oscillators has become infinitesimal, so close that a little of the energy absorbed by each one gradually leaks into nearby modes and eventually is completely gone.

APPENDIX III. SPONTANEOUS EMISSION OF AN ATOM IN A CAVITY

In this calculation as in the free space calculation the dipole approximation will be used in computing the spontaneous emission probability. The linear coordinates inside the cavity will be represented by the vector **Q** while the time varying coordinates of the single cavity mode being considered will be $X(t)$. The Lagrangian of the system may be written

$$L_T \equiv L(\dot{Q}_n, Q_n, t) + c^{-1}\textstyle\sum_n e_n \dot{\mathbf{Q}}_n \cdot \mathbf{A}(\mathbf{Q}_t + \mathbf{Q}_n, t) + L_{\text{cavity}} \qquad \text{(III.1)}$$

where $\mathbf{Q}_p$ is the atom coordinate, $\mathbf{Q}_n + \mathbf{Q}_p$ is the particle coordinate in the atom, and **A** is the vector potential of the cavity field. The interaction term is the one of interest, since from it we find the terms that we wish to solve for classically. This term will be put into more convenient form. Let us write

$$\mathbf{A}(\mathbf{Q}, t) = \mathbf{a}(\mathbf{Q})X(t) \qquad \text{(III.2)}$$

---

[27] If the power radiated from the oscillator is related to the classical expression $\frac{1}{2}I^2R$ then from Eq. (II.11) it can be seen that R is proportional to $\text{Re}[1/Z(\nu)]$ which in turn is related to the distribution of oscillators. One might expect $\text{Im}[1/Z(\nu)]$ to be replaced to the reactance seen by an oscillating dipole, a quantity which is known to be infinite classically. From Eq. (II.7),

$$\text{Im}(1/Z_\nu) = 4\nu/3\pi c^3 \int_0^\infty \Omega^2(\nu^2 - \Omega^2)^{-1}\,d\Omega = 4\nu\Omega/3\pi c^3 \,|_{\Omega\to\infty}$$

The integral is linearly divergent. This factor is also related to the infinite self energy of a point charge which occurs both classically and in quantum electrodynamics. Here this divergence does not bother us since it never enters into the calculation.

where

$$\int \mathbf{a}(\mathbf{Q})\cdot\mathbf{a}(\mathbf{Q})\, d^3\mathbf{Q} = 4\pi c^2 \tag{III.3}$$

If **A** does not vary much over the atom, then $\mathbf{A}(\mathbf{Q}_p) \approx \mathbf{A}(\mathbf{Q}_p + \mathbf{Q}_n)$ and the interaction term is written

$$c^{-1}\textstyle\sum_n e_n\dot{\mathbf{Q}}_n\cdot\mathbf{a}(\mathbf{Q}_p)X(t) = (j/c)\,|\,\mathbf{a}(\mathbf{Q}_p)\,|\,X(t) \tag{III.4}$$

where $j = (\sum e_n\dot{\mathbf{Q}}_n)\cdot\mathbf{a}(\mathbf{Q}_p)/\,|\,\mathbf{a}(\mathbf{Q}_p)\,|$, the component of the atom current in the direction of the cavity field.

Let us now determine the ratio

$$j_\nu[|\,\mathbf{a}(\mathbf{Q}_p)\,|\,X_\nu c^{-1}]^{-1} = i\nu Z_\nu \tag{III.5}$$

classically. The wave equation appropriate for this calculation (high Q)[28] is

$$\nabla^2\mathbf{A} - (1/c^2)\ddot{\mathbf{A}} + (\omega/c^2Q)\dot{\mathbf{A}} - (4\pi/c)\dot{\mathbf{P}} = 0 \tag{III.6}$$

The atom is located at $\mathbf{Q}_p$ and, since the dipole moment is induced, its direction on the average is the same as that of the field **A** in the cavity. We have then

$$\dot{\mathbf{P}} = j\delta(\mathbf{Q} - \mathbf{Q}_p)\mathbf{a}(\mathbf{Q}_p)/\,|\,\mathbf{a}(\mathbf{Q}_p)\,| \tag{III.7}$$

Substituting (III.7) and (III.2) into (III.6) we obtain

$$\begin{aligned}&[-\omega^2X + (\omega/Q)\dot{X} - \ddot{X}]c^{-2}\mathbf{a}(\mathbf{Q})\\ &\qquad - (4\pi/c)j\delta(\mathbf{Q}-\mathbf{Q}_p)\mathbf{a}(\mathbf{Q}_p)/|\,\mathbf{a}(\mathbf{Q}_p)\,| = 0\end{aligned} \tag{III.8}$$

where ω is the resonant frequency of the cavity. Multiplying by $\mathbf{a}(Q)$, integrating over Q, and taking Fourier transforms, (III.8) becomes

$$(\nu^2 - \omega^2 + i\nu\omega/Q)X_\nu - c^{-1}\,|\,\mathbf{a}(\mathbf{Q}_p)|\,j_\nu = 0. \tag{III.9}$$

We find that the ratio (III.5)

$$i\nu Z_\nu = [\nu^2 - \omega^2 + (i\nu\omega/Q)]c^2/|\,\mathbf{a}(\mathbf{Q}_p)^2$$

The influence phase for this is (although it is unnecessary to write it)

$$\Phi(j, j') = \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{j'_\nu(j_{-\nu} - j'_{-\nu})}{(i\nu Z_\nu)} + \frac{j_{-\nu}(j_\nu - j'_\nu)}{(-i\nu Z_{-\nu})}\right] d\nu$$

From second order perturbation theory we know

$$P_{nm} = 2T\,|\,j_{nm}\,|^2\,(\hbar\nu_{nm})^{-1}\,\mathrm{Re}(Z_{nm})^{-1} \quad \text{for} \quad \nu_{nm} > 0$$

[28] Q is used here as the dissipation factor of the cavity, ω_0L/R, while $\mathbf{Q}$ is a vector representing the linear coordinates inside the cavity.

Noting that

$$\mathrm{Re}(Z\nu_{nm})^{-1} = \omega\nu_{nm}^2 \,|\, \mathbf{a}(\mathbf{Q}_p)|^2 Q^{-1}c^{-2}[(\nu_{nm}^2 - \omega^2)^2 + \omega^2\nu_{nm}^2Q^{-2}]^{-1}$$

and defining a cavity from factor $f^2 = V \,|\, \mathbf{a}(\mathbf{Q}_p)|^2/4\pi c^2$

$$P_{nm} = (8\pi \,|\, j_{nm} \,|^2 f^2\omega\nu_{nm}T)(\hbar VQ)^{-1}[(\nu_{nm}^2 - \omega^2) + \omega^2\nu_{nm}^2Q^{-2}] \quad \text{(III.10)}$$

At resonance this expression reduces to

$$P_{nm} = 8\pi \,|\, j_{nm} \,|^2 f^2QT/\hbar V\nu_{nm}^2 \quad \text{(III.11)}$$

The quantity usually computed is the ratio of the transition probability in the cavity at resonance to that in free space. This ratio is

$$\frac{P_{nm}\,(\text{cavity})}{P_{nm}\,(\text{free space})} = \frac{[8\pi \,|\, j_{nm} \,|^2Qf^2/\nu_{nm}^2\, V\hbar]T}{[4 \,|\, j_{nm} \,|^2\nu_{nm}/3\hbar c^3]T} = \frac{6\pi c^3Qf^2}{V\nu_{nm}^3}$$

At resonance, the ratio increases with respect to Q as one might expect and decreases with respect to the cavity volume V and ν_{nm}^3. This expression agrees with the one given by E. M. Purcell (*23*) although the form factor in his calculation was left out. This does not matter since for a particle located near the maximum field point in a cavity the magnitude of f is of the order of unity.

APPENDIX IV

It is to be demonstrated that

$$G(X) = \lim_{N \text{ large}} \textstyle\prod_{k=1}^N (1 + X_k) \rightarrow \exp[\textstyle\sum_k X_k] \quad \text{(IV.1)}$$

where the X_k are small but not necessarily equal to each other, and where the total sum $\sum_k X_k$ is finite. Rewriting the expression for $G(X)$ we have (where the summations on all indices go from 1 to N)

$$\begin{aligned} G(X) &= 1 + \textstyle\sum_k X_k + \tfrac{1}{2}\textstyle\sum_{j\neq k} X_jX_k + (1/6)\textstyle\sum_{j\neq k\neq l} X_jX_kX_l + \cdots \\ &= 1 + \textstyle\sum_k X_k + \tfrac{1}{2!}\textstyle\sum_{j,k} X_jX_k(1 - \delta_{jk}) + (\tfrac{1}{3!})\textstyle\sum_{j,k,l} X_jX_kX_l \\ &\quad \times (1 - \delta_{jk} - \delta_{kl} - \delta_{jl} + 2\delta_{jk}\delta_{kl}) \end{aligned} \quad \text{(IV.2)}$$

As N is allowed to get very large the contribution of the terms involving quantities such as δ_{jk} becomes less significant. For instance, in the third term

$$\textstyle\sum_{j,k} (X_jX_k)(1 - \delta_{jk}) \sim (NX)^2 - (NX)^2/N \quad \text{(IV.3)}$$

and for very large N only the leading term in this sum is important. Thus, we have the result that

$$\begin{aligned} G(X)_{\text{large } N} &\approx 1 + \textstyle\sum_k X_k + \tfrac{1}{2!}(\textstyle\sum_k X_k)^2 + \tfrac{1}{3!}(\textstyle\sum_k X_k)^3 \\ &\approx \exp[\textstyle\sum_k X_k] \end{aligned} \quad \text{(IV.4)}$$

RECEIVED: April 5, 1963

REFERENCES

1. R. P. FEYNMAN, *Rev. Mod. Phys.* **20,** 367 (1948).
2. R. P. FEYNMAN, *Phys. Rev.* **80,** 440 (1950).
3. R. P. FEYNMAN, *Phys. Rev.* **84,** 108 (1951).
4. V. FANO, *Rev. Mod. Phys.* **29,** 74 (1957).
5. R. W. HELLWARTH (private discussion).
6. R. MCWEENY, *Rev. Mod. Phys.* **32,** 335 (1960).
7. R. C. TOLMAN, "The Principles of Statistical Mechanics," Chapter 9. Oxford Univ. Press, New York, 1930.
8. D. MIDDLETON, "An Introduction to Statistical Communication Theory." McGraw-Hill, New York, 1960.
9. D. MIDDLETON, "An Introduction to Statistical Communication Theory," pp. 362–364. McGraw-Hill, New York, 1960.
10. H. GOLDSTEIN, "Classical Mechanics," Chap. 10. Addison Wesley, Reading, Mass, 1953.
11. H. B. CALLEN AND J. H. WELTON, *Phys. Rev.* **83,** 34 (1951).
12. R. KUBO, *J. Phys. Soc. Japan* **12,** 570 (1957).
13. H. NYQUIST, *Phys. Rev.* **32,** 110 (1928).
14. H. NAKANO, *Progr. Theoret. Phys. (Kyoto)* **17,** 145 (1957).
15. R. V. POUND, *Ann. Phys. (N.Y.)*, **1,** 24 (1957).
16. M. W. MUELLER, *Phys. Rev.* **106,** 8 (1957).
17. K. SHINODA, T. C. WANG, AND C. H. TOWNES, *Phys. Rev.* **102,** 1308 (1956).
18. M. W. P. STRANBERG, *Phys. Rev.* **106,** 617 (1957).
19. K. SHINOLA, H. TAKAHASI, AND C. H. TOWNES, *J. Phys. Soc. Japan* **12,** 686 (1957).
20. W. H. WELLS, *Ann. Phys. (N.Y.)*, **12,** 1 (1961).
21. J. P. GORDON, *Proc. IRE* **50,** 1898 (1962).
22. E. FERMI, *Rev. Mod. Phys.* **4,** 87 (1932).
23. E. M. PURCELL, *Phys. Rev.* **69,** 681 (1946).