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NOTE

ANALYTICAL APPROXIMATION TO THE CAPACITANCE OF THE MICROSTRIP DISK CAPACITOR

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1. INTRODUCTION

In a recent paper, Gelmont *et al.*[1] discuss the capacitance of a structure composed of a metal disk separated from a ground plane by a dielectric layer (the microstrip disk capacitor). The capacitance was calculated by numerically solving an integral equation; this approach, though with different solution techniques, was used in previous studies of the same problem[2,3], and was shown to yield results of great accuracy. Earlier treatments of this problem are quoted in [1,3].

On the basis of their numerical solution, Gelmont *et al.*[1] proposed the following interpolation formula for the capacitance, C, of a disk of radius a on a dielectric sheet of thickness b and relative permittivity ϵ backed by a metal plane (see Fig. 1):

$$C_a = \frac{2\pi(\epsilon+1)\epsilon_0 a}{\tan^{-1}[2b(\epsilon+1)/(a\epsilon)]}, \tag{1}$$

where C_a is the approximate capacitance and ϵ_0 is the vacuum permittivity. This formula reproduces the correct limiting expressions both for $b/a \to 0$ and $b/a \to \infty$, but for intermediate values of the ratio b/a it underestimates the true capacitance by an amount that can reach 25% for $\epsilon = 1$. The value of 33% for the error given in [1] is incorrect since it is based upon the ratio C/C_a. The actual relative error is $C_a/C - 1 = 1/1.33 - 1 = -25\%$.

The purpose of this note is to directly derive a different approximation for C, which has a maximum error of 10% (7.5% for $\epsilon = 1$). This approximation is obtained without introducing integral equations, but only through a theorem of potential theory and known results of electrostatics; because of this derivation, the resulting formulae have a more evident physical interpretation than an interpolation formula, such as eqn (1).

2. APPROXIMATE CAPACITANCE FOR $\epsilon = 1$

We consider first the simpler case where the disk is surrounded by vacuum ($\epsilon = 1$), to better illustrate the principle of the method. The starting point is a vector integral identity, known as Green's reciprocal theorem (see [4], p. 1089), which is an immediate consequence of his second identity; this theorem is applied here to the region Ω external to the disk surface S_1, and bounded by the conducting plane S_2 ($z = 0$) and the sphere at infinity. A disk can be regarded as an oblate spheroid of infinitesimal thickness; this notion clarifies the meaning of "external" and also shows that S_1 is actually composed of two sheets. If ϕ and ϕ' are two solutions of Laplace's equation inside Ω, then Green's reciprocal theorem states that:

$$\sum_{i=1}^{2}\int_{S_i}\phi'\frac{\partial\phi}{\partial n}\,\mathrm{d}S = \sum_{i=1}^{2}\int_{S_i}\phi\frac{\partial\phi'}{\partial n}\,\mathrm{d}S, \tag{2}$$

where $\partial/\partial n$ is the derivative along the outward normal to S_i. We assume that ϕ and ϕ' are due to two different distributions σ_1, and σ_1' on S_1 of the same charge Q, in the presence of the grounded plane S_2. We take for σ_1 the density that is appropriate to S_1 in the absence of the plane S_2, i.e., when S_1 is isolated. We choose σ_1' to be the equilibrium charge density on S_1, and hence ϕ' will have a constant value V', on S_1. Thus eqn (2) becomes:

$$V'\int_{S_1}\frac{\partial\phi}{\partial n}\,\mathrm{d}S = \int_{S_1}\phi\frac{\partial\phi'}{\partial n}\,\mathrm{d}S. \tag{3}$$

The contribution of S_2 vanishes, since $\phi = \phi' = 0$ on S_2. The application of Gauss's theorem to the first integral of eqn (3) shows that its value is $-Q/\epsilon_0$. In the second integral of this equation, $\partial\phi'/\partial n = -\sigma_1'/\epsilon_0$, since ϕ' corresponds to an equilibrium configuration, and the field vanishes "inside" S_1. More precisely, $(\partial\phi'/\partial n)_\pm = -(\sigma_1')_\pm/\epsilon_0$, where $+$ ($-$) refers to the upper (lower) face of S_1, and $\sigma_1' = (\sigma_1')_+ + (\sigma_1')_-$. Since ϕ is continuous across S_1, the second, two-sheet integral of eqn (3) can be written as a single sheet integral involving σ_1' [eqn (4)]. Thus eqn (3) yields:

$$V' = \frac{1}{Q}\int_{S_1}\sigma_1'\phi\,\mathrm{d}S. \tag{4}$$

Therefore, the capacitance $C = Q/V'$ can be written in the form:

$$C = \frac{Q^2}{\int_{S_1}\sigma_1'\phi\,\mathrm{d}S}. \tag{5}$$

The potential ϕ on S_1 can be expressed in a relatively simple form in terms of the isolated-disk potential in oblate spheroidal coordinates (ξ, η), with their origin at the centre of the disk and interfocal distance $2a$; these coordinates are defined through the cylindrical coordinates (ρ, z) with the same origin by[5]:

$$\begin{cases}\rho = a\sqrt{(1+\xi^2)(1-\eta^2)}\\ z = a\xi\eta.\end{cases} \tag{6}$$

In fact, the potential of an isolated conducting disk carrying a charge Q (and hence with surface charge density σ_1) is given by $(2/\pi)V_0\tan^{-1}(1/\xi)$, where $V_0 = Q/(8\epsilon_0 a)$, and the disk itself corresponds to $\xi = 0$ (see [5] or [6] (Vol. 1, p. 254), with the interchange of ξ with η). To obtain the potential in the presence of S_2 at $\phi = 0$, we must add to the potential of S_1 that of its negative mirror image S_1' with respect to the plane S_2 (see [6] Vol. 2, p. 199). The resulting values of ϕ on S_1 is:

$$\phi(\rho) = V_0 - (2/\pi)V_0\tan^{-1}[1/\xi(\rho)], \tag{7}$$

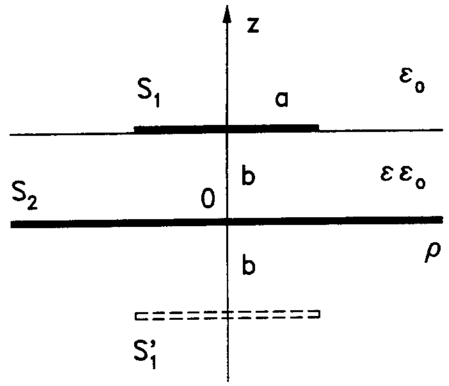

Fig. 1. Geometry of the microstrip disk capacitor.

where $\xi(\rho)$ is obtained from eqn (6) by replacing z with the distance $2b$ of S_1' from S_1 (see Fig. 1). This yields:

$$\xi(\rho)=\left(\sqrt{2}/2\right)\left[-(1-4\beta^2-\rho^2/a^2)+\sqrt{(1-4\beta^2-\rho^2/a^2)^2+16\beta^2}\right]^{1/2}, \quad (8)$$

where $\beta = b/a$. By substituting eqn (7) into eqn (5), making use of the identity $\tan^{-1}(1/x) = \pi/2 - \tan^{-1} x$, and, recalling the definition of V_0, we obtain for C the expression:

$$C=\frac{2\epsilon_0 a}{\int_0^a\left[\frac{\sigma_1'(\rho)}{Q}\right]\tan^{-1}[\xi(\rho)]\rho\,\mathrm{d}\rho}, \quad (9)$$

where $\xi(\rho)$ is given by eqn (8). The expression of eqn (9) is exact, but contains the unknown equilibrium charge distribution $\sigma_1'(\rho)$ on the disk. However, a first approximation to C can be obtained by replacing $\sigma_1'(\rho)$ with the constant density $Q/(\pi a^2)$; this distribution is correct only in the limit of $b/a \to 0$, but it will be shown that it yields acceptable accuracy for any value of the ratio b/a. For constant σ_1', the integral in eqn (9) can be evaluated exactly, by changing the integration variable from ρ to ξ by means of the square of eqn (8). The resulting approximation for C is given by:

$$C_a=\frac{4\pi\epsilon_0 a}{\tan^{-1}\xi_1+(\xi_1-\xi_0)\left[\frac{4\beta^2}{\xi_1\xi_0}-1\right]}, \quad (10)$$

where ξ_0 and ξ_1 are the values of the function $\xi(\rho)$ of eqn (8) for $\rho/a = 0$ and $\rho/a = 1$, respectively. By examining the asymptotic form of ξ_0, ξ_1 for $b/a \to 0$ and $b/a \to \infty$, it can be proved that C_a in eqn (10) correctly approaches the elementary expression $\pi\epsilon_0 a^2/b$ as $b/a \to 0$, and the isolated disk capacitance, $8\epsilon_0 a$, as $b/a \to \infty$.

Figure 2 presents a comparison between the values of the ratio C_a/C according to Gelmont *et al.*[1] and eqn (10) for different values of b/a. The values of C_a/C of Gelmont *et al.* have been obtained from Fig. 2 in their article[1]. The exact values of C were not reported in [1], therefore they have been deduced from those (C_{pd} values) given by Carlson and Illman[7] for the parallel disk capacitor; in fact, an examination of Fig. 1 shows that $C(b/a) = 2C_{pd}(2b/a)$. These values were used to evaluate C_a/C on the basis of eqn (10).

The plots of Fig. 2 indicate that eqn (10) yields a maximum error of about 7%; the corresponding value for eqn (1) is about 25%. Equation (1) gives a somewhat better accuracy than eqn (10) as $b/a \to 0$, whereas this latter equation is much more accurate for intermediate or large values of b/a. A more detailed indication of the accuracy of eqn (10) for $b/a \geqslant 0.2$ is given in Table 1, where the values of $C^* = C/(\pi\epsilon_0 a^2/b)$ are compared with those given in Table 1 in [3].

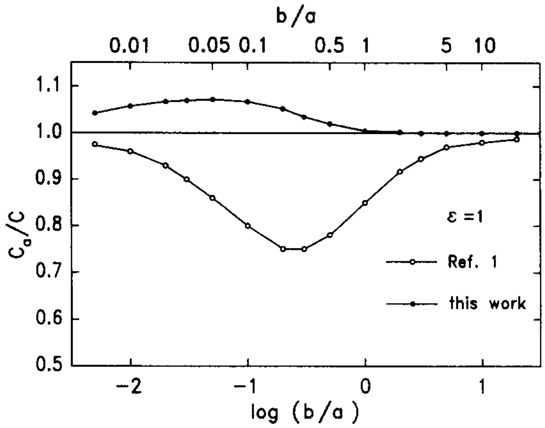

Fig. 2. Values of the ratio C_a/C vs b/a according to eqn (1) for $\epsilon = 1$[1] and eqn (10).

Table 1. The normalized capacitance $C^* = C/(\pi\epsilon_0 a^2/b)$ of the microstrip disk *in vacuo* for selected values of b/a. The values obtained from eqn (10) are compared to those of Table 1 ($N = 2$) in [3].

b/a	C^* [3]	C^* [eqn (10)]	Relative error (%)
0.2	1.5800	1.66	5
0.5	2.3183	2.36	2
1.0	3.5345	3.552	0.5
2.5	7.2684	7.271	0.04
5	13.5920	13.5928	0.006
10	26.3006	26.3008	0.0008

3. APPROXIMATE CAPACITANCE FOR ARBITRARY ϵ

We turn now to the more general case where the metal disk lies on a dielectric layer with relative permittivity ϵ. In the region Ω the permittivity $K\epsilon_0$ is now point dependent, with $K = 1$ for $z > b$ and $K = \epsilon$ for $0 < z < b$. By applying Gauss's divergence theorem to the vector $\phi' K \nabla\phi - \phi K \nabla\phi'$ in this region, we obtain a generalization of Green's reciprocal theorem of eqn (1) in the form:

$$\sum_{i=1}^{2}\int_{S_i}\phi' K\frac{\partial\phi}{\partial n}\,\mathrm{d}S=\sum_{i=1}^{2}\int_{S_i}\phi K\frac{\partial\phi'}{\partial n}\,\mathrm{d}S. \quad (11)$$

Repeating the reasoning of Section 2, in terms of electric displacement instead of electric field, leads again to the expression of eqn (5) for the capacitance. The potential ϕ required in eqn (5) can again be found by applying the method of images, although here an infinite sequence of them is needed, as explained in [6] (Vol. 3, p. 233), or in the more explicit analysis by Kleefstra and Herman[8]. Accordingly, the potential at $z \geqslant b$ of a point charge q resting on a dielectric slab (i.e., at $z = b$ in Fig. 1), which lies on a conducting plane ($z = 0$), can be described by means of the following system of images. The charge q itself is replaced by $(1-k)q$, with $k = (\epsilon - 1)/(\epsilon + 1)$; a sequence of image charges:

$$-(1-k^2)q, k(1-k^2)q, \ldots, -(-1)^n k^n(1-k^2)q, \ldots, \quad (12)$$

is introduced, which are located at $z < 0$ at respective distances from q:

$$2b, 4b, \ldots, 2(n+1)b, \ldots \tag{13}$$

By introducing the analogous system of image disks, we obtain the generalization of eqn (7) in the form:

$$\phi(\rho) = (1-k)V_0 - (2/\pi)V_0(1-k^2)\{\tan^{-1}[1/\xi_0(\rho)] - k\tan^{-1}[1/\xi_1(\rho)] + \cdots + (-1)^n k^n \tan^{-1}[1/\xi_n(\rho)] + \cdots\}, \tag{14}$$

where the functions $\xi_n(\rho)$ are defined through eqn (8), by replacing β with $\beta_n = (n+1)b/a$. Consequently, eqn (9) becomes:

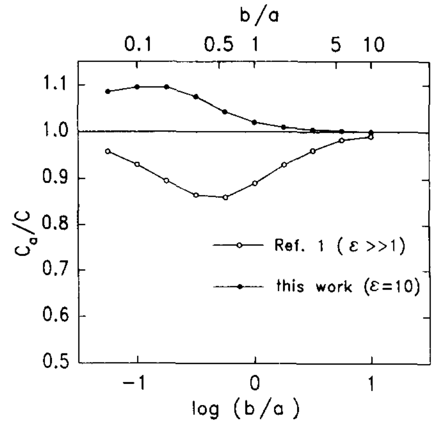

Fig. 3. Values of the ratio C_a/C vs b/a according to [1] ($\epsilon \gg 1$) and eqn (16) for $\epsilon = 10$.

$$C = \frac{2\epsilon_0 a}{(1-k^2)\sum_{n=0}^{\infty}(-1)^n k^n \int_0^a \left[\frac{\sigma_1'(\rho)}{Q}\right]\tan^{-1}[\xi_n(\rho)]\rho\, d\rho}, \tag{15}$$

and the new approximate capacitance, to be compared to eqn (10), is given by:

$$C_a = \frac{4\pi\epsilon_0 a}{(1-k^2)\sum_{n=0}^{\infty}(-1)^n k^n\left\{\tan^{-1}\xi_{n1} + (\xi_{n1} - \xi_{n0})\left[\frac{4\beta_n^2}{\xi_{n1}\xi_{n0}} - 1\right]\right\}}, \tag{16}$$

where ξ_{n0}, ξ_{n1} are the values of $\xi_n(\rho)$ for $\rho/a = 0$ and $\rho/a = 1$, respectively. An examination of the limiting behaviour of eqn (16) shows that this expression correctly approaches $\pi\epsilon\epsilon_0 a^2/b$ as $b/a \to 0$, and $4(\epsilon+1)\epsilon_0 a$ as $b/a \to \infty$. For $b/a > 1$, the term in curly brackets in eqn (16) can be approximated by replacing $\tan^{-1}(\xi_{n1})$ with $\pi/2 - 1/(2\beta_n)$ and neglecting its second addend. The resulting series can be summed with the aid of eqns (0.231.1) and (1.511) in [4], to give:

$$C_a \cong \frac{4(\epsilon+1)\epsilon_0 a}{1 - \frac{a}{\pi b}\frac{1+k}{k}\ln(1+k)}. \tag{17}$$

It has been checked that the difference between this expression and eqn (16) is less than 3% for $b/a > 1.5$. The result of eqn (17) is also useful for accelerating the convergence of the series of eqn (16), following the procedure suggested in [9].

Figure 3 compares the ratio C_a/C according to eqn (1) and eqn (16) as a function of b/a. The values of C_a/C according to Gelmont *et al.*[1] have been obtained from the plot ($\epsilon \gg 1$) of Fig. 2 in their paper. The values of C required to estimate C_a/C according to eqn (16) were obtained from that plot and eqn (1), with the tentative value $\epsilon = 10$. This because neither the actual value of ϵ used to produce the plot nor the exact values of C were given in [1].

The plots of Fig. 3 indicate that eqn (16) yields a maximum error of about 9.5% (this value becomes 8.5% for $\epsilon = 20$); the maximum error of eqn (1) is about 14%. As before, the present approximation performs better than eqn (1) for $b/a > 1$, whereas the converse is true for $b/a \ll 1$.

Approximate formulae for C valid for any ϵ were derived by Coen and Gladwell[3] on the basis of a Legendre expansion of the solution of a Fredholm integral equation for the charge density. Their approximation was shown to have a maximum error of about 6%, which is lower than that of eqn (16). However, the single eqn (16) is valid for any value of b/a, whereas in [3] the capacitance is approximated by two different expressions, depending on whether b/a is smaller or larger than approximately 0.2. Moreover, eqn (16) uses only elementary functions, whereas the approximation in [3] for $b/a \leqslant 0.2$ contains a series of elliptic integrals.

4. CONCLUSION

An approximate analysis has been given for the capacitance of a metal disk separated from a ground plane by a dielectric sheet. On the basis of Green's reciprocal theorem, an exact relation has been derived, which expresses the capacitance in terms of the (unknown) equilibrium charge density on the disk and the (known) potential produced by the disk when it is isolated. By approximating this density with a constant, expressions for the capacitance have been derived and have been compared to exact results and a recently proposed interpolation formula. More accurate estimates of the capacitance can be obtained by using a better approximation to the true charge density on the disk. This improved charge distribution may be found by specific physical considerations or, more generally, by means of the principle of minimum energy of the system (see, e.g., [6], Vol. 2, p. 161), also known as Thomson's theorem[10]. The proposed method of approximate calculation of capacitance may be applied to other configurations, with a special advantage when an exact analysis is difficult to perform.

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