Resonant frequencies of cylindrical Helmholtz resonators

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A simple analysis predicts the Helmholtz and higher modes of cylindrical resonators. Formulas for the quarter-wave tube, the closed tube, and the classic Helmholtz resonator are special cases. It is found that the classic Helmholtz formula begins to lose accuracy for resonator lengths $L > \lambda/16$. A corrected Helmholtz equation is given as an approximation to the exact result. The analysis requires values of inside and outside orifice end corrections. Rayleigh's model of a piston in an infinite wall and Ingard's calculations for a piston radiating into a tube give satisfactory results when the theory is compared to experiments. Resonator experiments were performed with k_0L in the range $\pi/8$ to $3\pi/8$ and orifice to cavity diameter ratios from 0.1 to 0.8.

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INTRODUCTION

The classical analysis of a Helmholtz resonator predicts that the resonant frequency is independent of the shape of the cavity. This result arises from an assumption that all dimensions are small compared to the wavelength. We will present a simple, but very useful, analysis which is valid for cylindrical resonators of arbitary length.

The analysis predicts the higher modes as well as the Helmholtz or first mode. Special cases of the theory give the resonant frequencies of the quarter-wave tube and the completely closed tube. The results show that the classical analysis is valid only when the length dimension is less than $\frac{1}{16}$ of the wavelength, significantly shorter than the often quoted $\frac{1}{4}$-wavelength criterion. An approximate form of our analysis is a closed form expression for the Helmholtz frequency. It contains a length correction term and is valid over a much larger region of wavelength than the classic formula.

Experiments were conducted on three groups of resonators with nominal frequencies of 250, 500, and 1000 Hz. Each group had three resonators of different dimensions, so in fact nine different resonators were tested. The experimental results confirm the usefulness of the analysis.

I. ANALYSIS

A Helmholtz resonator is shown in Fig. 1. The resonator consists of a cylindrical tube and an orifice. The tube area A is closed at $x=0$, and is driven by a mass of air in the orifice at $x=L$. The orifice area is S and the effective length l' includes end corrections on both sides. The acoustic transmission tube equations are solved for the cavity with boundary conditions of infinite impedance at the solid end and a reactive mass loading from the orifice. The left-moving plane wave is denoted as

$$p_- = \alpha \exp[-j(\omega t + kx)] ,$$

and the right-moving wave is represented by

$$p_+ = \beta \exp[-j(\omega t - kx)] .$$

The volume velocities of these two waves are $U_+ = p_+/(\rho c/A)$ and $U_- = -p_-/(\rho c/A)$. An expression for the acoustic impedance at any point in the cavity is

$$Z = \frac{p_- + p_+}{U_- + U_+} = \frac{\rho c}{A} \cdot \frac{\alpha e^{jkx} + \beta e^{-jkx}}{\alpha e^{jkx} + \beta e^{-jkx}} . \quad (1)$$

This equation is evaluated at $x=0$ and $x=L$, then α and β are eliminated in favor of Z_0 and Z_L. A rigid wall at $x=0$ means that Z_0 is infinite and results in

$$Z_L = j\frac{\rho c}{A} \cot kL , \quad (2)$$

a purely reactive impedance.

The orifice at $x=L$ has a reactive impedance modeled by the mass m of fluid which oscillates in the orifice

$$Z_L = j\omega \frac{m}{S^2} = jkc\frac{\rho l'}{S} = jkL\frac{c\rho l'}{LS} . \quad (3)$$

Equating Eqs. 2 and 3 yields a transcendental equation for all resonant wavenumbers:

$$\frac{l'A}{LS} kL = \cot kL . \quad (4)$$

The solutions are depicted graphically in Fig. 2, where $[(l'/L)\cdot(A/S)]kL$ and $\cot kL$ are plotted. The intersections of these functions designate the resonant frequencies since $\omega L/c = kL$. The first intersection is the "Helmholtz" frequency, and succeeding intersections determine the higher modes which are sometimes called "standing wave" frequencies. Thus, Eq. 4 theoretically produces the Helmholtz and the nonharmonic standing wave modes of a cylindrical Helmholtz resonator. The theory accounts for the fact that cavity length may be comparable to or longer than wavelength. The transverse dimensions and orifice length must still be small compared to wavelength. Extension of the theory to cavities whose area varies with x could be made using the appropriate extension of wave tube theory.

Special cases of Eq. 4 give the resonant frequencies of the quarter-wave tube and a closed tube. If the orifice area S equals the tube area and l' is taken as the value for a radiating piston, then the quarter-wave tube formula corrected for an end effect is obtained. The formula for a closed tube is found by noting that the end correction $l' \sim r$ (the orifice radius), while $S \sim r^2$. The

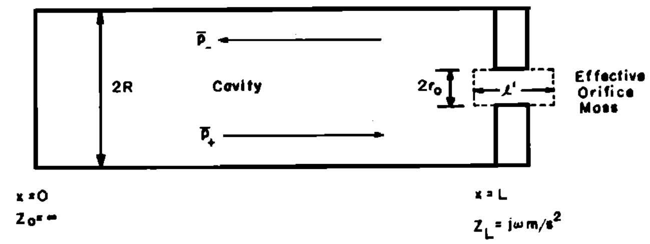

FIG. 1. Cylindrical Helmholtz resonator.

limit $r \to 0$ is a solid wall, and Eq. 4 becomes $\cot kL = \infty$. This produces the customary closed tube resonances.

Closed-form approximate results are obtained by expanding the right side of Eq. 4:

$$\cot(kL) = \frac{1}{kL} - \frac{1}{3}kL - \frac{1}{45}(kL)^2 - \dots . \tag{5}$$

The classical formula for a Helmholtz resonator is obtained by retaining only the first term in the series. The first term is also plotted on Fig. 2 where one can see that it begins to deviate from the cot curve around $kL = \pi/8$; $\frac{1}{16}$ of a wavelength. This point marks the limit of validity for the classical formula.

An improved Helmholtz resonator formula is found by retaining two terms in Eq. 5 and simplifying to

$$\omega_0 = c\left(\frac{S}{l'V + \frac{1}{3}L^2S}\right)^{1/2} . \tag{6}$$

The term preceded by $\frac{1}{3}$ is the only difference between Eq. 6 and the classical formula. The accuracy of Eq. 6, as an approximation to Eq. 4, is good almost to the quarter-wavetube case. For example, at $kL = 15\pi/32$ the error is 6.6% while for a quarter-wave tube $kL = \pi/2$ the error is 10%. Most cases of practical interest are covered by this approximate formula.

Previous articles which are related to the current paper are by Alster[1] and by Tang and Sirignano.[2] The first article gives a semiempirical formula for resonators of various shapes. The second article is a sophisticated analysis of cylindrical resonators and acoustic dampers beginning with nonlinear wave propagation equations. They include an orifice whose length is long compared to a wavelength. Equation 4 with l' taken as the actual orifice length is a special case of their analysis.

TABLE I. Helmholtz resonators dimensions. All orifice thickness $t = 0.318$ cm.

Group	Orifice radius r_0 (cm)	Orifice area S (cm^2)	Cavity radius R (cm)	Cavity length L (cm)	Cavity volume V (cm^3)
	0.254	0.203	1.91	10.82	123
250 Hz	0.381	0.456	1.91	18.90	215
	0.508	0.811	1.91	27.46	313
	0.254	0.203	1.27	6.10	31.0
500 Hz	0.381	0.456	1.27	10.64	53.9
	0.508	0.811	1.27	10.64	53.9
	0.254	0.203	0.635	6.10	7.70
1000 Hz	0.381	0.456	0.635	6.10	7.70
	0.508	0.811	0.635	6.10	7.70

TABLE II. Experimental and theoretical Helmholtz frequencies.

Group	r_0/R	$f_{0_{exp}}$ Hz	$f_{0_{cl}}$[a] Hz	σ %	$f_{0_{In}}$[b] Hz	σ %	$f_{0_{Eq.4}}$[c] Hz	σ %
	0.133	252	259	2.78	266	5.56	254	0.79
I 250 Hz	0.200	234	259	10.68	271	15.81	238	1.71
	0.267	211	259	22.75	276	30.81	213	0.95
	0.200	498	517	3.82	537	7.83	508	2.01
II 500 Hz	0.300	472	517	9.53	554	17.37	467	−1.06
	0.400	538	623	15.80	691	28.44	540	0.37

[a]Rayleigh's end correction in classical formula.
[b]Ingard's end correction in classical formula.
[c]Ingard's end correction in Eq. 4.

II. EXPERIMENTS

A series of nine resonators was constructed by machining cavities from solid aluminum bar stock. Three groups with nominal Helmholtz frequencies of 250, 500, and 1000 Hz contained three resonators each. The dimensions are given in Table I. The wavenumber range for the resonators was $\pi/8 < k_0L < 3\pi/8$. This means all resonators were outside the classic theory. A $\frac{1}{4}$-in.-diameter Brüel & Kjaer microphone was mounted through the cavity wall.

During a test the resonator was excited with a loudspeaker. In each case, the orifice directly faced the loudspeaker which was several feet distant. The frequency response of the resonator was monitored on an oscilloscope. Care was taken to separate actual cavity resonances from testing environment resonances. Each resonator was tested inside a small sound attenuation booth with one open side. A Brüel & Kjaer type 1022 beat frequency oscillator was used as a power source. The loudspeaker was driven by the oscillator, while the output from the oscillator was regulated by means of a compressor circuit. A regulating microphone was placed in the sound field of the loudspeaker. This microphone's output voltage was used as a control voltage to insure constant sound pressure. The oscillator frequency scale was manually scanned between 0 and 5000

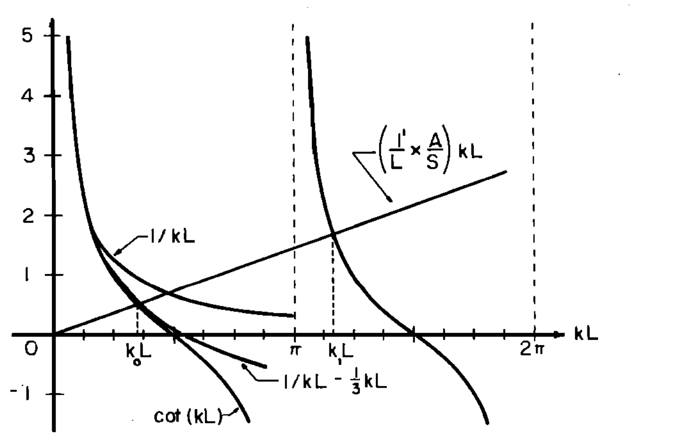

FIG. 2. Analysis of resonant frequencies.

TABLE III. Measured and predicted resonant frequencies.

Group	r_0/R	f_{exp} (Hz)	$f_{Eq.}$[a] (Hz)	σ (%)
		252	254	0.79
	0.133	1648	1646	−0.12
		3238	3228	−0.31
		4845	4824	−0.43
		234	238	1.71
		980	990	1.02
	0.200	1855	1875	1.08
		2775	2780	0.18
		3666	3691	0.68
I. 250 Hz		211	213	0.95
		728	728	0.00
		1319	1320	0.08
	0.267	1929	1934	0.26
		2552	2557	0.20
		3183	3183	0.00
		3813	3810	−0.08
		4434	4439	0.11
		498	508	2.01
	0.200	2968	2944	−0.81
		472	467	−1.06
II. 500 Hz	0.300	1818	1795	−1.27
		3423	3350	−2.13
		538	540	0.37
	0.400	1899	1867	−1.69
		3454	3397	−1.65
		×	899	×
	0.400	3060	3213	5.00
		×	1081	×
III. 1000 Hz	0.600	3346	3451	3.14
		×	1188	×
	0.800	3393	3649	7.54

[a]Ingard's end correction in Eq. 4.

Hz. When a resonance was encountered, the amplfied cavity response was observed on the oscilloscope. The frequency was digitally displayed by a Hewlett Packard 5221A Electronic Counter.

III. RESULTS

Measurements of the Helmholtz frequencies are presented in Table II and the higher modes in Table III. In order to compare with theory one must evaluate the effective length of the orifice ($l' = l + \Delta_0 + \Delta_i$) including an inside and outside end correction. Rayleigh proposed the well-known correction of $\Delta_0 = 8r_0/3\pi$ which models a piston radiating into half-space. This is also frequently used for the inside correction. However, Ingard[3] proposed that the inside end correction should model a piston radiating into a tube of radius R. This correction is given approximately by

$$\Delta_i = \frac{8r_0}{3\pi}\left(1 - 1.24\,\frac{r_0}{R}\right) \quad \text{for } \frac{r_0}{R} < 0.4 .$$

For cases where $r_0/R > 0.4$ one should consult Fig. 3 of Ref. 3.

The measurements in Table II actually validate the use of Ingard's inside end correction and also possibly explain why it has not received wider acceptance. Three calculations of the Helmholtz frequency were made. The classical formula with Rayleigh's correction on both sides is in error by 3 to 20%. This error doubles when Ingard's correction is used for the inside. However, when the wave tube analysis equation, Eq. 4, is used with Ingard's correction, the error is 2% or less. Since Ingard's correction and the cavity length correction are in opposite directions, it has been more accurate to neglect both corrections than include only Ingard's correction. Recall that classical theory was valid only up to about $L = \lambda/16$ and only very-low-frequency resonators can meet this criterion.

The higher modes listed in Table III are compared with calculated values using Eq. 4 with Rayleigh's Δ_0 and Ingard's Δ_i correction. The comparison is very good for the 250- and-500-Hz group and moderate for the 1000-Hz group. We were not able to pin down a Helmholtz frequency for the 1000-Hz group. This is an inadequacy in the test setup as several large responses were observed near the expected Helmholtz frequency.

[1]M. Alster, J. Sound Vib. 24, 63 (1972).
[2]P. K. Tang and W. A. Sirignano, J. Sound Vib. 26, 247 (1973).
[3]U. Ingard, J. Acoust. Soc. Am. 26, 1037 (1953).