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Plasmons in bounded systems: 2D disk and 1D needle

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The whole spectrum of two-dimensional plasma oscillations in a 2D disk with a nonuniform distribution of the surface electron density is calculated. Multipole modes of oscillations connected with the nonuniform distribution of the surface electron density near the edge of the disk are shown to arise. The influence of the magnetic field on this spectrum is considered. Also the spectrum of plasmons in a thin needle is found. The absence of localized modes in this case is shown.

1. Introduction

Electromagnetic radiation or moving charged particles can excite plasma oscillations in bounded conducting objects. Plasma frequencies generally depend on the sizes and shapes of these systems. The quantitative evaluation of the resonance frequencies is of considerable interest because the excitement of them can result in such effects as resonant scattering, absorption of light and enhancement of the field. Plasma oscillations can also amplify the intensity of different physical phenomena, as for instance second harmonic generation [1] and surface enhanced Raman scattering (SERS) [2].

The best-known examples of plasmons in inhomogeneous systems are surface plasmons [3] and plasmons in small metal particles [4]. Plasma oscillations are also considered on cylindrical surfaces with different profiles [5,6], in particles with ellipsoidal shape [7], in sphere-surface [8] and cylinder-surface [9,10] systems, in a system of two spherical particles [11], etc.

Plasma oscillations in bounded low-dimensional systems are also of sufficient interest. The spectra of plasmons in both a 2D disk [12] and a 2D ring [13], in a semi-infinite half plane in a magnetic field [14,15] have been investigated. Volkov and Mikhailov [16] have obtained the exact solution and the exact edge magnetoplasmon dispersion relation for a sharp profile of the surface electron density at the border of the semi-infinite half plane. The electric field and the surface excited density have been found to demonstrate a singular behavior near the edge. Leavitt and Little [17] have only investigated the dipolar plasma oscillation mode (see also ref. [16], and references therein) for a disk with a nonuniform distribution of the surface electron density.

In the present paper the whole spectrum of plasma oscillations in both a 2D disk and a 1D needle has been calculated. The connection of the disk plasma eigenstates with both the edge plasmon and the oscillations in a uniform infinite 2D electron gas is shown. The electron density tends to zero at both the disk edge and the needle spike. The field and the excited surface electron density have no singularity near the edge or the spike. New types of multipole edge plasma oscillations are found to arise in this case. These oscillations are similar to the multipole surface plasma oscillations [18]. The influence of the magnetic field on the spectrum of the disk plasmons is explored. We also discuss the transition from needle plasma eigenstates to oscillations in the uniform infinite 1D electron gas. The impossibility of the existence of plasma modes localized near the spike

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(or the needle edge) is proved below in this case with a special distribution of the linear electron density.

2. The spectrum of plasma oscillations in a two-dimensional disk

Let us consider a conducting oblate spheroid characterized by a dielectric function $\epsilon(\omega)$. The spheroid is placed in an insulator medium with dielectric constant ϵ_0. It is convenient to make use of the oblate spheroid coordinates [19], shown in fig. 1, in which

$$x^2=a^2(\sigma^2+1)(1-\tau^2)\cos^2\varphi, \qquad 0\leqslant\sigma\leqslant\infty, \tag{1a}$$

$$y^2=a^2(\sigma^2+1)(1-\tau^2)\sin^2\varphi, \qquad -1\leqslant\tau\leqslant 1, \tag{1b}$$

$$z=a\sigma\tau, \qquad 0\leqslant\varphi\leqslant 2\pi. \tag{1c}$$

Here a is a parameter. The dielectric spheroid occupies the region $0\leqslant\sigma\leqslant\sigma_0$ (see fig. 1). We work in the electrostatic limit, in which effects due to the finiteness of the speed of light are ignored. The field potential $\Phi(\sigma,\tau,\varphi;t)=\Phi(\sigma,\tau,\varphi)e^{-i\omega t}$ satisfies the Laplace equation $\nabla^2\Phi=0$. The spectrum of eigenfrequencies is determined by matching at the boundary. In the above-mentioned coordinates, the Laplace equation becomes

$$\frac{\partial}{\partial\sigma}\left((1+\sigma^2)\frac{\partial\Phi}{\partial\sigma}\right)+\frac{\partial}{\partial\tau}\left((1-\tau^2)\frac{\partial\Phi}{\partial\tau}\right)+\frac{\sigma^2+\tau^2}{(1+\sigma^2)(1-\tau^2)}\frac{\partial^2\Phi}{\partial\varphi^2}=0. \tag{2}$$

It is well known that the Laplace equation is separable in this coordinate system, so we write

$$\Phi(\sigma,\tau,\varphi)=F(\sigma)G(\tau)\,e^{im\varphi}, \tag{3}$$

where $m=0, \pm1, \pm2, \ldots$. The solutions of eq. (2) are associated Legendre polynomials so that

$$\begin{aligned}\Phi=\Phi_{lm}&=A_{lm}P_l^{|m|}(i\sigma)P_l^{|m|}(\tau)\,e^{im\varphi}, \qquad 0\leqslant\sigma\leqslant\sigma_0,\\ &=B_{lm}Q_l^{|m|}(i\sigma)P_l^{|m|}(\tau)\,e^{im\varphi}, \qquad \sigma\geqslant\sigma_0, \qquad |m|\leqslant l=0,1,2,\ldots.\end{aligned}$$

Here $l(l+1)$ is the separation constant, A_{lm} and B_{lm} are invariable nonzero coefficients. From the condition of the continuity of $\Phi(\sigma,\tau,\varphi)$ and $\epsilon(\omega)\partial\Phi(\sigma,\tau,\varphi)/\partial\sigma$ at $\sigma=\sigma_0$, we obtain the dispersion relation for the electrostatic plasma oscillations in the oblate spheroid [17],

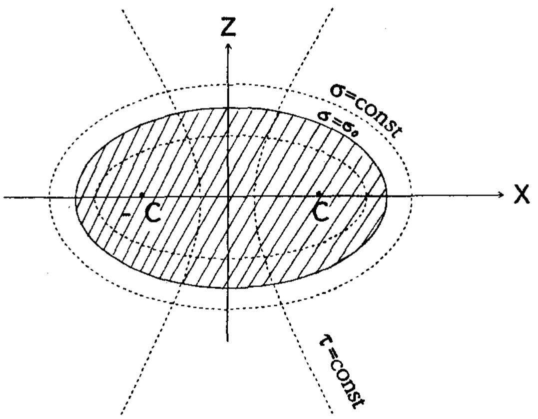

Fig. 1. Coordinates σ, τ of the oblate spheroid. The metal particle occupies the shaded region $0\leqslant\sigma\leqslant\sigma_0$.

$$\frac{\epsilon(\omega)}{\epsilon_0}=\frac{P_l^{|m|}(\mathrm{i}\sigma)}{\mathrm{d}P_l^{|m|}(\mathrm{i}\sigma)/\mathrm{d}\sigma}\frac{\mathrm{d}Q_l^{|m|}(\mathrm{i}\sigma)/\mathrm{d}\sigma}{Q_l^{|m|}(\mathrm{i}\sigma)}\Bigg|_{\sigma=\sigma_0}. \tag{4}$$

The spectrum of eigenfrequencies for a sphere can be found from eq. (4) if we assume that $\sigma_0\to\infty$ and $\sigma_0 a\to R$ simultaneously. Here R is the radius of the sphere.

As mentioned above, the material forming the oblate spheroid is a free-electron metal, then $\epsilon(\omega)=1-\omega_p^2/\omega^2$, where $\omega_p=4\pi n_V e^2/m^*$ is the bulk plasma frequency of the metal. Here $n_V=N/V$ is the bulk electron density; N and V are, respectively, the total number of electrons and the volume of the spheroid; m^* is the effective mass, e is the charge of the electron.

We make a limiting transition with $\sigma_0\to 0$, in which the oblate spheroid changes to a disk of radius a. We also make the total number of electrons N invariable for this transition. The limiting transformation gives us the whole spectrum of disk plasma oscillations,

$$\omega_{l|m|}^2=\frac{3\pi^2\bar{n}_S e^2}{m^*\epsilon_0 a}\frac{l^2-m^2+l(l-|m|)!(l+|m|)!}{2^{2l+1}\{[\frac{1}{2}(l-|m|)]!\}^2\{[\frac{1}{2}(l+|m|)]!\}^2}, \tag{5}$$

where $l+|m|=2, 4, 6, \ldots$. Here $\bar{n}_S=N/S=(N/V)V/S\to n_V\times 4a\sigma_0/3$ (as $\sigma_0\to 0$) is the average surface electron density in the disk and $S=\pi a^2$ is the surface area of the disk. As a result, the local surface electron density is equal to

$$n_S(\rho)=\tfrac{3}{2}\bar{n}_S\sqrt{1-(\rho/a)^2}\,, \tag{6}$$

where ρ is the distance from the center of the disk. The frequency of the optically active dipolar mode in the disk [17] is equal to

$$\omega_{\rm dip}^2=\omega_{11}^2=\frac{3\pi^2\bar{n}_S e^2}{4m^*\epsilon_0 a}. \tag{7}$$

The frequencies of the antisymmetric modes ($l+|m|$) is odd) tend to the frequency of the bulk plasmon ω_p and are much higher than the frequencies of the symmetric modes ($l+|m|$ is even).

Let us investigate the higher types of oscillations in the disk. The asymptotic behavior of eq. (5) as $l\to\infty$ can be found by using the Stirling formula,

$$\omega_{l|m|}^2\xrightarrow[l\to\infty]{}\frac{3\pi\bar{n}_S e^2}{m^*\epsilon_0 a}\frac{l^2-m^2+l}{\sqrt{l^2-m^2}},\qquad l>|m|\,. \tag{8}$$

In fig. 2 we present the curves for $\omega_{l|m|}$ as a function of both l and $|m|$. For fixed values of $|m|=0, 1, 2, \ldots$ we obtain, using eq. (6), $\omega_{l>|m|}^2\to\omega^2(q)=(2\pi n_S^* e^2/m^*\epsilon_0)q$, $n_S^*=\frac{3}{2}\bar{n}_S$, when both l and a tend to infinity but the ratio l/a tends to q which is constant. The frequency $\omega(q)$ corresponds to the frequency of the plasma oscillations of an infinite uniform two-dimensional electron gas. The indicated limiting transition is possible because the eigenfunctions in the limit $l\gg 1$ have the same form as those of a cylindrical wave with wavevector $q=1/a$: $P_l^{|m|}(\tau)\propto\cos[\vartheta(l+\frac{1}{2})+\gamma]$, $\tau=\cos\vartheta$, $\rho=a\sin\vartheta$. The phase γ does not depend on ϑ.

When l tends to infinity, one may show that $\omega_{ll}^2\to(3\pi^{3/2}e^2\bar{n}_S/2m^*\epsilon_0 a)l^{1/2}$. The oscillations of the surface electron density are localized near the edge of the disk at a distance $\Delta\rho\propto a/l$ because $P_l^l\propto(\rho/a)^l\simeq\exp[-(l/a)(a-\rho)]=\exp[-(l/a)\Delta\rho]$. Therefore, the eigenstates with $|m|=l\to\infty$ correspond to the edge plasmon [16] whose eigenfunction has the following form: $\Phi_{l,\pm l}\propto e^{\pm il\varphi}P_l^l(\tau)\propto e^{\pm iq\xi}e^{-q\eta}$, $q=l/a$, where ξ and η are, respectively, the coordinates along and from the border of the semi-infinite half-plane ($-\infty<\xi<\infty$, $\eta\geqslant 0$). Since the local electron density $n(\rho)\to 0$ near the edge of the disk (see eq. (6)), the expression for ω_{ll}^2 ($l\to\infty$), mentioned above, needs to be renormalized in the following manner,

$$\omega_{ll}^2\xrightarrow[l\to\infty]{l/a=q}(\tfrac{1}{8}\pi)^{1/2}\frac{2\pi e^2 n_S^{**}}{m^*\epsilon_0}q\,, \tag{9}$$

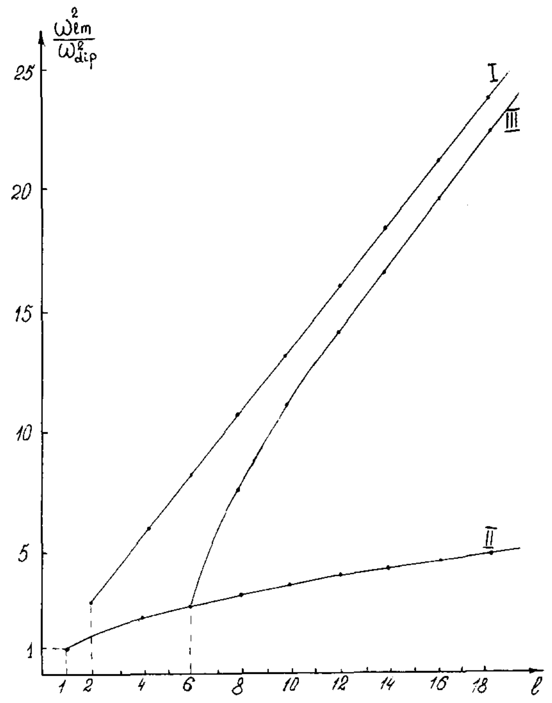

Fig. 2. Spectrum of the plasma oscillations in a thin 2D disk. (I) $m=0$, (II) $|m|=l$, (III) $|m|=6$.

because at a distance $\Delta\rho \ll a$ we have $n_S(\Delta\rho) \approx (3/\sqrt{2})(\Delta\rho/a)^{1/2}\bar{n}_S$. Here n_S^{**} is the surface electron density at the localization length $q^{-1}=a/l$.

The dispersion relation for the edge plasmon is given by eq. (9). It is easy to see from eq. (8) that there exist other types of eigenstates with $l-|m|=2k=2, 4, 6, ...$, whose eigenfunctions have nodes. Their frequencies are higher than that of the ground edge mode with $l=|m|$. Such higher types of edge plasmons are similar to the multipole plasma modes [18] arising in 3D systems with a nonuniform profile of the electron density when spatial dispersion is taken into account. The frequencies of these modes for large k tend to the two-dimensional plasmon frequency in a homogeneous system. The multipole types of oscillations are likely to arise for other surface electron density distributions ($n_S(\Delta\rho)\to 0$, if $\Delta\rho\to 0$).

3. Disk plasma oscillations in a magnetic field

Let us introduce a magnetic field H along the z axis. Then the potential Φ satisfies the equation

$$\mathrm{div}[\epsilon_{ij}(\omega, H)\nabla_j\Phi(r, \omega, H)]=0\,, \tag{10}$$

where ϵ_{ij} are the components of the dielectric function tensor $(i, j,=x, y, z)$. In Euclidian coordinates we have

$$\epsilon_{xx}=\epsilon_{yy}=\epsilon_{\perp}=1-\frac{\omega_p^2}{\omega^2-\omega_H^2}, \qquad \epsilon_{zz}=\epsilon_{\parallel}=1-\frac{\omega_p^2}{\omega^2}, \qquad \epsilon_{xy}=-\epsilon_{yx}=ig, \qquad g=\frac{\omega_H^2\omega_p^2}{\omega(\omega_H^2-\omega^2)}, \qquad \epsilon_{xz}=\epsilon_{yz}=0\,,$$

where $\omega_H=eH/m^*c$ is the cyclotron frequency (c is the speed of light). Thus eq. (10) becomes

$$\epsilon_{\perp}\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)+\epsilon_{\parallel}\frac{\partial^2\Phi}{\partial z^2}=0\,. \tag{11}$$

It is easy to show that all the solutions of the Laplace equation (2) which depend only on the two coordinates x and y are also solutions of eq. (11). Imposing boundary conditions on these solutions of eq. (2) with the electric induction component

$$D_\sigma = -\frac{1}{h_\sigma}\left[\left(\epsilon_\perp + \frac{\omega_H}{\omega} g \frac{\tau^2(\sigma^2+1)}{\sigma^2+\tau^2}\right)\frac{\partial \Phi}{\partial \sigma} + \frac{\omega_H}{\omega} g \frac{\sigma\tau(1-\tau^2)}{\sigma^2+\tau^2}\frac{\partial \Phi}{\partial \tau} + \mathrm{i}g\frac{\sigma}{\sigma^2+1}\frac{\partial \Phi}{\partial \varphi}\right], \qquad h_\sigma^2 = \frac{a^2(\sigma^2+\tau^2)}{\sigma^2+1}, \tag{12}$$

we obtain the plasma frequencies in the presence of a magnetic field perpendicular to the plane of the disk when $l=|m|=1, 2, ...,$

$$\omega_{m=\pm l} = \frac{\sqrt{\omega_H^2+4\omega_{ll}^2} \pm \omega_H}{2}. \tag{13}$$

Here

$$\omega_{ll}^2 = \frac{3\pi^2 \bar{n}_S e^2}{m^* \epsilon_0 a}\frac{l^2(2l)!}{2^{2l+1}(l!)^2}$$

(see eq. (5)). Because $\Phi_{l,\pm l} \mp A_{l,\pm l}\,(\sigma^2+1)^{l/2}(1-\tau^2)^{l/2}\mathrm{e}^{\pm\mathrm{i}l\varphi} = (A_{l,\pm l}/a^l)(x\pm\mathrm{i}y)$, eq. (12) is simplified and becomes equal to

$$D_\sigma = -\frac{1}{h_\sigma}(\epsilon_\perp \mp g)\frac{\partial \Phi}{\partial \sigma}, \qquad \text{if } m=\pm l. \tag{14}$$

If the magnitude of H tends to infinity in eq. (13), then $\omega_{m=l} \to \omega_H$ and $\omega_{m=-l} \to \omega_{ll}^2/\omega_H$. One should emphasize that the magnetic field does not affect the analytical expression for the eigenfunctions, such as $\Phi_{l,\pm l}$, but changes the corresponding resonance frequencies. However, the magnetic field affects the analytical expression of the other eigenfunctions in the oblate spheroid depending on the three coordinates x, y and z because eq. (11) is not separable in this case.

It is worth indicating that in the presence of a magnetic field the frequencies of the optically active dipolar modes in the disk are equal to

$$\omega_{\mathrm{dip}} = \omega_{m=\pm 1} = \frac{\sqrt{\omega_H^2+4\omega_{11}^2} \pm \omega_H}{2}. \tag{15}$$

We also find an edge magnetoplasmon localized at the border of the semi-infinite half plane when $|m|=l\to\infty$ and $a\to\infty$ (see section 2). The size of the edge magnetoplasmon localization does not depend on the value of the magnetic field in this case and is approximately equal to $q^{-1}=a/l$. The edge magnetoplasmon has the following eigenfrequencies,

$$\omega_{\mathrm{edge}}(q, H) = \frac{\sqrt{\omega_H^2+4\omega^2(q)} \pm \omega_H}{2}, \tag{16}$$

where $\omega^2(q) \approx (\frac{1}{8}\pi)^{1/2}(2\pi e^2 \bar{n}_S/m^*\epsilon_0)q$ (see section 2), and the eigenfunctions,

$$\Phi_{\pm q}(\xi, \eta, H) \propto \mathrm{e}^{\pm\mathrm{i}q\xi}\mathrm{e}^{-q\eta}. \tag{17}$$

4. Plasma oscillations in the one-dimensional needle

We discuss a conducting prolate spheroid with dielectric function $\epsilon(\omega)$ and surrounded by an insulator with dielectric constant ϵ_0. In the prolate spheroid coordinates [19] solutions of the Laplace equation are sought in the form (3) and eigenfunctions of a boundary problem are written by the following way,

$$F(\sigma)=P_l^{|m|}(\sigma), \qquad 1\leqslant\sigma\leqslant\sigma_0\,,$$
$$\quad =Q_l^{|m|}(\sigma), \qquad \sigma\geqslant\sigma_0\,,$$

$$G(\tau)=P_l^{|m|}, \qquad -1\leqslant\tau\leqslant 1, \qquad |m|\leqslant l, \qquad l=0, 1, 2, \ldots .$$

Imposing the boundary conditions on these solutions leads to the dispersion relation for the eigenfrequencies of the plasma oscillations in the prolate spheroid,

$$\frac{\epsilon(\omega)}{\epsilon_0}=\left.\frac{P_l^{|m|}(\sigma)}{\mathrm{d}P_l^{|m|}(\sigma)/\mathrm{d}\sigma}\,\frac{\mathrm{d}Q_l^{|m|}(\sigma)/\mathrm{d}\sigma}{Q_l^{|m|}(\sigma)}\right|_{\sigma=\sigma_0}. \tag{18}$$

As $\sigma\to 1$ the prolate spheroid degenerates to a needle of length $L=2a$. Let us make this limiting transition in eq. (18) introducing an average linear electron density in the needle: $\bar{n}_L=N/L=(N/V)V/L\to\frac{1}{3}n_V\,\pi L^2(\sigma_0-1)$ as $\sigma_0\to 1$. As a result of the limiting transformation we obtain

$$\omega^2_{l>0,m=0}=\omega_l^2=\frac{6e^2\bar{n}_L}{m^*\epsilon_0 L^2}\,l(l+1)\ln\frac{2}{\exp[2\sum_{t=1}^{l}(1/t)](\sigma_0-1)}\,. \tag{19}$$

Similar to the procedure described in section 2, the total number of electrons N in the prolate spheroid is also kept constant. Analogously we find the corresponding local electron density,

$$n_L(\rho)=\tfrac{3}{2}\bar{n}_L[1-(2\rho/L)^2]\,, \tag{20}$$

where ρ is the distance from the center of the needle.

Calculating the corrections to eq. (19) one may understand that they have the following form, $\Delta\omega_l/\omega_l\propto l^2(\sigma_0-1)$. Therefore, eq. (19) is only adequate for oscillations with $l\ll(\sigma_0-1)^{-1/2}$. The frequencies of the oscillations with $m\neq 0$ (as $\sigma_0\to 1$) tend to the frequency of the Ritchie surface plasmon [3] and are much higher than $\omega_{l,m=0}$. The frequency of the optically active dipolar mode is equal to

$$\omega^2_{\mathrm{dip}}=\omega_1^2=\frac{12\bar{n}_L e^2}{m^*\epsilon_0 L^2}\ln\frac{2}{(\sigma_0-1)\,\mathrm{e}^2}\,. \tag{21}$$

The logarithm in eq. (21) contains the cutoff constant $(\sigma_0-1)^{-1}\propto L^2 b^{-2}$, where b is the radius of the cross section of the needle. The example of the dipolar mode of the oscillations permits us to observe the deviation of the frequency as the thickness of the needle decreases (fig. 3).

For oscillations with $l\gg 1$ one may obtain, from eq. (19), the following formula,

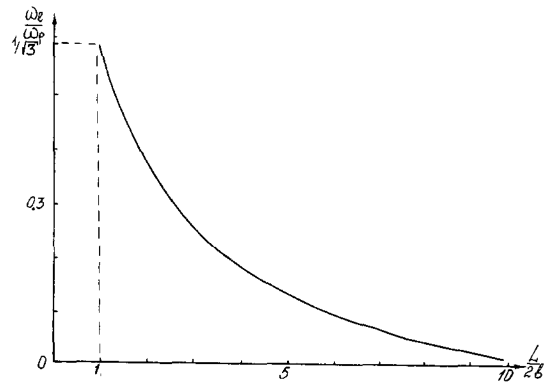

Fig. 3. Dipole frequency of the plasma oscillations in the prolate spheroid as a function of the parameter $L/2b$.

$$\omega_l^2 = \frac{12\bar{n}_L e}{m^* \epsilon_0} \frac{l^2}{L^2} \ln \frac{e^{-C}}{(l/L)b}, \tag{22}$$

where C is the Euler constant. When $l \gg 1$ the eigenfunctions have the form of a plane wave with wavevector $q=2l/L$. After renormalization of the linear electron density using eq. (20), where $L \to \infty$, the eigenfrequency $\omega_l = \omega(q)$ can be represented in the form $\omega^2(q) = (2n_L^* e^2 q^2/m^* \epsilon_0) \ln(1/qb)$, $n_L^* = \frac{3}{2}\bar{n}_L$. The expression for $\omega^2(q)$ coincides with the frequency of the plasma oscillations in a uniform one-dimensional electron gas [9,10,20]. Thus, the edges do not affect the frequencies of higher types of oscillations. Let us also note that the modes of oscillations localized near the spike (or the needle edge) are absent in the system under consideration.

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