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A note on the almost sure convergence of sums of negatively dependent random variables

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Abstract: We study the almost sure convergence of sums of negatively dependent random variables, in particular, the classical strong law of large numbers for independent and identically distributed random variables is generalized to the case of pairwise negative quadrant dependent random variables. We also extend the three series theorem to the case of negatively associated random variables.

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1. Introduction

Let us consider a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables defined on some probability space $(\Omega, \mathscr{F}, P)$. We start with definitions.

A finite family $\{X_1,\dots,X_n\}$ is said to be associated if $\operatorname{Cov}(f(X_1,\dots,X_n), g(X_1,\dots,X_n)) \geqslant 0$, for any real coordinatewise nondecreasing functions f and g on $\mathbb{R}^n$, such that this covariance exists. It is said to be negatively associated if for any disjoint subsets $A,B\subset\{1,\dots,n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^A$, g on $\mathbb{R}^B$, $\operatorname{Cov}(f(X_k,\ k\in A), g(X_k,\ k\in B)) \leqslant 0$. Infinite family of random variables is associated (negatively associated) if every finite subfamily is associated (negatively associated). Two random variables X and Y are negative quadrant dependent (NQD) if $P[X>x,\ Y>y] \leqslant P[X>x]P[Y>y]$, for all $x,y\in\mathbb{R}$, and positive quadrant dependent (PQD) if $P[X>x,\ Y>y]\geqslant P[X>x]P[Y>y]$ for all $x,y\in\mathbb{R}$. These concepts of dependence were introduced by Lehmann (1966), Essary, Proschan and Walkup (1967), and Joag-Dev and Proshan (1983).

Let us set $S_n=\sum_{k=1}^n X_k$. The problem of almost sure convergence of $(S_n-ES_n)/n$ for stationary sequences of associated and negatively associated random variables was studied by Newman (1984). There are some results on the SLLN for nonstationary sequences of associated and pairwise PQD random variables (see Birkel, 1989, and references therein). For negatively correlated random variables Etemadi (1983) obtained SLLN, but our results cannot be deduced from his.

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2. SLLN for pairwise NQD random variables

Theorem 1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of pairwise NQD random variables, with the same distribution function $F(x)$. $S_n/n \to a$ almost surely, for some constant $a \in \mathbb{R}$, if and only if $E|X_1| < \infty$. If $E|X_1| < \infty$, then $a = EX_1$.*

In order to prove this theorem we shall state some lemmas. Let us recall the following version of the Borel–Cantelli lemma (cf. Petrov, 1987).

Lemma 1. *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(A_n)_{n\in\mathbb{N}}$ a sequence of events. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup A_n) = 0$, if $\sum_{n=1}^{\infty} P(A_n) = \infty$ and $P(A_k \cap A_m) \leqslant P(A_k)P(A_m)$ for $k \neq m$, then $P(\limsup A_n) = 1$.* □

Lemma 2. *If $(X_n)_{n\in\mathbb{N}}$ is a sequence of pairwise NQD random variables, $(f_n)_{n\in\mathbb{N}}$ a sequence of nondecreasing functions $f_n : \mathbb{R} \to \mathbb{R}$, then $(f_n(X_n))_{n\in\mathbb{N}}$ are also pairwise NQD.*

Proof. It suffices to observe that $[f_n(X_n) > x] = [X_n > \inf[t : f_n(t) > x]]$. □

The following lemma generalizes Lemma 10 of Petrov (1987, p. 222).

Lemma 3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of pairwise NQD random variables and $(a_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers. If $S_n/a_n \to 0$ a.s. and $\sup_{n\in\mathbb{N}} a_{n-1}/a_n \leqslant M < \infty$ for some $M \in \mathbb{R}$, then $\sum_{n=1}^{\infty} P[|X_n| \geqslant a_n] < \infty$.*

Proof. If $S_n/a_n \to 0$ a.s., then $X_n/a_n \to 0$ a.s., as $n \to \infty$. Thus putting $X_n^+ = \max(0, X_n)$, $X_n^- = \max(0, -X_n)$, we see that $X_n^+/a_n \to 0$ a.s. and $X_n^-/a_n \to 0$ a.s. It is easy to verify that $(-X_n)_{n\in\mathbb{N}}$ is a sequence of pairwise NQD random variables, thus taking into account Lemma 2 we see that $(X_n^+)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ are NQD. Thus defining the following events,

$$A_n = \left[X_n^+ > \tfrac{1}{3}a_n\right], \quad B_n = \left[X_n^- > \tfrac{1}{3}a_n\right], \qquad n \in \mathbb{N},$$

we get

$$P(A_k \cap A_m) \leqslant P(A_k)P(A_m) \quad \text{for } k \neq m \quad \text{and} \quad P(B_k \cap B_m) \leqslant P(B_k)P(B_m) \quad \text{for } k \neq m.$$

By Lemma 1, if $\sum_{n=1}^{\infty} P[X_n^+ \geqslant \frac{1}{3}a_n]$ diverges then $P[\limsup A_n] = 1$ contrary to the almost sure convergence of X_n^+/a_n to zero. The same argument for X_n^- yields $\sum_{n=1}^{\infty} P[X_n^- \geqslant \frac{1}{3}a_n] < \infty$. Thus

$$\sum_{n=1}^{\infty} P[|X_n| \geqslant a_n] = \sum_{n=1}^{\infty} P[X_n^+ + X_n^- \geqslant a_n] \leqslant \sum_{n=1}^{\infty} P\left[X_n^+ > \tfrac{1}{3}a_n\right] + \sum_{n=1}^{\infty} P\left[X_n^- > \tfrac{1}{3}a_n\right] < \infty,$$

and the proof is complete. □

Proof of Theorem 1. Setting $a_n = n$ in Lemma 3, and taking into account that X_n's are equidistributed we get $\sum_{n=1}^{\infty} P[|X_1| > n] < \infty$, thus $E|X_1| < \infty$, so necessity is proved.

Now, assume $E|X_1| < \infty$, then $(X_n^+)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ are sequences of NQD random variables with the same distribution and finite absolute moments. Since $X_n = X_n^+ - X_n^-$, it suffices to prove the theorem for $(X_n^+)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$, thus we may and do assume that $X_n \geqslant 0$, $n \in \mathbb{N}$. Let $F(x)$ denotes common distribution of X_n's. Let us put $Y_n = X_n \wedge n$ for $n \in \mathbb{N}$, $S_n^* = \sum_{k=1}^{n} Y_k$, and remark that Y_n's are pairwise

NQD and thus negatively correlated. For arbitrary $\alpha > 1$ set $k_n = [\alpha^n]$ ($[\cdot]$ stands for integer part of a number). Then, for any $\varepsilon > 0$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} P\left[k_n^{-1} | S_{k_n}^* - ES_{k_n}^* | > \varepsilon\right] &\leqslant K \sum_{n=1}^{\infty} n^{-2} \operatorname{Var} Y_n \\ &\leqslant K \sum_{n=1}^{\infty} n^{-2} EX_n^2 I[X_n \leqslant n] + K \sum_{n=1}^{\infty} P[X_n > n] \\ &\leqslant KEX_1 + K \sum_{n=1}^{\infty} n^{-2} \sum_{k=0}^{n-1} \int_k^{k+1} x^2 \, \mathrm{d}F(x) \\ &\leqslant KEX_1 + K \sum_{n=0}^{\infty} \int_n^{n+1} x \, \mathrm{d}F(x) \\ &\leqslant KEX_1 < \infty,\end{aligned}$$

where K is a positive constant, which may be different in the consecutive inequalities. From the above inequalities and the Borel–Cantelli lemma we see that $k_n^{-1}(S_{k_n}^* - ES_{k_n}^*) \to 0$ almost surely as n goes to infinity.

The existence of absolute moment of X_1 yields $k_n P[X_1 > k_n] \to 0$, as $n \to \infty$, thus we get $k_n^{-1} ES_{k_n}^* = E(X_1 \wedge k_n) \to EX_1$, as $n \to \infty$.

From the definition of Y_n's we have

$$\sum_{n=1}^{\infty} P[X_n \neq Y_n] = \sum_{n=1}^{\infty} P[X_n > n] = \sum_{n=1}^{\infty} P[X_1 > n] < \infty.$$

Thus $P[X_n \neq Y_n \text{ i.o.}] = 0$, so we conclude that $k_n^{-1} S_{k_n} \to EX_1$, a.s. as $n \to \infty$.

Let us observe that for any $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that $[\alpha^{k(n)-1}] < n \leqslant [\alpha^{k(n)}]$. Thus, taking into account monotonicity of S_n with respect to n, we get

$$\alpha^{-1} k_{k(n)-1}^{-1} S_{k_{k(n)-1}} \leqslant n^{-1} S_n \leqslant k_{k(n)-1}^{-1} S_{k_{k(n)}} \leqslant \alpha k_{k(n)}^{-1} S_{k_{k(n)}}.$$

From these inequalities follows that

$$\alpha^{-1} EX_1 \leqslant \liminf_{n \to \infty} n^{-1} S_n \leqslant \limsup_{n \to \infty} n^{-1} S_n \leqslant \alpha EX_1 \quad \text{a.s.}$$

Since α may be arbitrarily close to 1, we conclude that $n^{-1} S_n \to EX_1$, almost surely as $n \to \infty$. □

Now, let us consider random variables with multidimentional indices. Let $\mathbb{N}^d$, where $d \geqslant 2$ is an integer, denote the positive integer d-dimentional lattice points. The notation $\boldsymbol{m} \leqslant \boldsymbol{n}$, where $\boldsymbol{m} = (m_1, \ldots, m_d)$ and $\boldsymbol{n} = (n_1, \ldots, n_d)$ are elements of $\mathbb{N}^d$, means that $m_i \leqslant n_i$, $i = 1, \ldots, d$. $|\boldsymbol{n}|$ is used for $\prod_{i=1}^n n_i$ and $\boldsymbol{1} = (1, \ldots, 1)$. In what follows $\ln_+ x = \max(1, \ln x)$.

Theorem 2. Let $(X_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d}$ *be a sequence of pairwise NQD random variables with the same distribution. If* $E | X_{\boldsymbol{1}} | \ln_+^{d-1} | X_{\boldsymbol{1}} | < \infty$, *then*

$$\lim_{|\boldsymbol{n}| \to \infty} |\boldsymbol{n}|^{-1} \sum_{\boldsymbol{k} \leqslant \boldsymbol{n}} X_{\boldsymbol{k}} = EX_{\boldsymbol{1}} \quad a.s.$$

Proof. Without loss of generality we may assume $X_{\boldsymbol{k}} \geqslant 0$, setting $Y_{\boldsymbol{k}} = X_{\boldsymbol{k}} I[X_{\boldsymbol{k}} \leqslant |\boldsymbol{k}|] + |\boldsymbol{k}| I[X_{\boldsymbol{k}} > |\boldsymbol{k}|]$, we see that $Y_{\boldsymbol{k}}$'s are also pairwise NQD. Thus following the lines of the proof of Theorem 1 and proof of Theorem 2 of Etemadi (1981) we get the desired conclusion. □

3. Convergence of series of negatively associated random variables

Theorem 3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of negatively associated random variables with finite second moments. If $\sum_{n=1}^{\infty}\operatorname{Var}(X_n)<\infty$, then $\sum_{n=1}^{\infty}(X_n-EX_n)$ converges a.s.*

The above theorem and the Kronecker lemma imply the following corollary.

Corollary. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of negatively associated random variables with finite second moments and $\sum_{n=1}^{\infty}\operatorname{Var}(X_n)/n^2<\infty$, then $(S_n-ES_n)/n\to 0$ a.s. as $n\to\infty$.* □

We need the following Kolmogorov-type inequality.

Lemma 4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of negatively associated random variables with finite second moments and zero mean. Then, for every $\varepsilon>0$,*

$$P\left[\max(|S_1|,\ldots,|S_n|)>\varepsilon\right]\leqslant 8\varepsilon^{-2}\sum_{k=1}^{n}\operatorname{Var}(X_k).$$

Proof. We have

$$\max(|S_1|,\ldots,|S_n|)\leqslant\max(0,S_1,\ldots,S_n)+\max(0,\ -S_1,\ldots,\ -S_n),$$

thus

$$\begin{aligned}
&P\left[\max(|S_1|,\ldots,|S_n|)>\varepsilon\right]\\
&\quad\leqslant P\left[\max(0,\ S_1,\ldots,S_n)>\tfrac{1}{2}\varepsilon\right]+P\left[\max(0,\ -S_1,\ldots,\ -S_n)>\tfrac{1}{2}\varepsilon\right]\\
&\quad\leqslant 4\varepsilon^{-2}E\left(\max(0,\ S_1,\ldots,S_n)\right)^2+4\varepsilon^{-2}E\left(\max(0,\ -S_1,\ldots,\ -S_n)\right)^2\\
&\quad\leqslant 4\varepsilon^{-2}E\left(\max(S_1,\ldots,S_n)\right)^2+4\varepsilon^{-2}E\left(\max(-S_1,\ldots,\ -S_n)\right)^2. \qquad (*)
\end{aligned}$$

We see that $M_n:=\max(S_1,\ldots,S_n)=X_1+\max(0,\ X_2,\ X_2+X_3,\ldots,X_2+\cdots+X_n)$, and X_1 and $\max(0,\ X_2,\ X_2+X_3,\ldots,X_2+\cdots+X_n)$ are negatively associated as nondecreasing functions of disjoint subsets of X_n's, so negatively correlated. Thus we get

$$\begin{aligned}
EM_n^2&=EX_1^2+2EX_1\max(0,\ X_2,\ X_2+X_3,\ldots,X_2+\cdots+X_n)\\
&\qquad+E\left(\max(0,\ X_2,\ X_2+X_3,\ldots,X_2+\cdots+X_n)\right)^2\\
&\leqslant EX_1^2+E\left(\max(X_2,\ X_2+X_3,\ldots,X_2+\cdots+X_n)\right)^2.
\end{aligned}$$

Now, by induction we have

$$EM_n^2\leqslant\sum_{k=1}^{n}\operatorname{Var}(X_k).$$

Replacing X_n's by $-X_n$'s we get similar inequality for $E(\max(-S_1,\ldots,\ -S_n))^2$, thus taking into account inequality $(*)$ we get the required conjecture. □

Proof of Theorem 3. Without loss of generality we assume $EX_k=0$, $k\in\mathbb{N}$. Let ε be a positive number, then

$$P\left[\sup_{k,m\geqslant n}|S_k-S_m|>\varepsilon\right]\leqslant P\left[\sup_{k\geqslant n}|S_k-S_n|>\tfrac{1}{2}\varepsilon\right]+P\left[\sup_{m\geqslant n}|S_m-S_n|>\tfrac{1}{2}\varepsilon\right]$$

$$\leqslant 2 \lim_{N\to\infty} P\left[\max_{n\leqslant k\leqslant N} |S_k - S_n| > \tfrac{1}{2}\varepsilon \right].$$

Now using Lemma 4 we get

$$P\left[\sup_{k,m\geqslant n} \leqslant |S_k - S_m| > \varepsilon \right] \leqslant 64\varepsilon^{-2} \sum_{i=n}^{\infty} \mathrm{Var}(X_i) \to 0, \quad \text{as } n\to\infty,$$

thus we may conclude that the sequence $(S_n)_{n\in\mathbb{N}}$ is a.s. Cauchy and therefore a.s. convergent. □

Let us put

$$X^c = \begin{cases} X, & \text{if } |X| \leqslant c, \\ -c, & \text{if } X < -c, \\ c, & \text{if } X > c, \end{cases} \quad \text{and} \quad S_n^c = \sum_{k=1}^{n} X_k^c \quad \text{for some } c > 0.$$

Theorem 4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of negatively associated random variables. If for some $c>0$ series $\sum_{n=1}^{\infty} EX_n^c$, $\sum_{n=1}^{\infty}\mathrm{Var}(X_n^c)$, $\sum_{n=1}^{\infty} P[|X_n| \geqslant c]$ are convergent, then $\sum_{n=1}^{\infty} X_n$ is convergent a.s.*

Proof. The proof is standard, so we omit details. □

Remark. Since from the convergence of $\sum_{n=1}^{\infty} X_n$ follows $X_n \to 0$ almost surely, thus $X_n^+ \to 0$, $X_n^- \to 0$ almost surely, so taking into account Lemma 1 we see, that condition $\sum_{n=1}^{\infty} P[|X_n| \geqslant c] < \infty$, for some $c > 0$, is necessary for the convergence of $\sum_{n=1}^{\infty} X_n$.

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