Journal
of
The Franklin Institute

Devoted to Science and the Mechanic Arts

Vol. 235 MARCH, 1943 No. 3

THEORY OF SUSPENSION BRIDGES

BY

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PART I.

INTRODUCTION.

In this paper various methods of analysis of suspension bridges are discussed together with their application to several particular bridges. In the first two articles, are discussed the cases of a perfectly flexible cable and of an unstiffened suspension bridge and equations for calculation of deflections and changes in cable tension produced by live load are developed. It is shown also that in the case of heavy long span suspension bridges, deflections produced by live load are very small and a stiffening truss is not required. In the third article, the fundamental equations for stiffened suspension bridges are derived and the errors introduced in these equations by various assumptions, usually made in the process of derivation, are discussed in detail. In the fourth article an analysis of a single span stiffened suspension bridge is given and it is shown that the derivation of the necessary equations is simplified by using the method of superposition. For the determination of the most unfavorable distribution of live load, the use of influence lines is recommended. In articles 5 and 6 the application of trigonometric series in calculating deflections is discussed and it is shown that by using these series not only the calculation of deflections and bending moments can be considerably simplified but also the calculation of cable tension produced by live load can be made with better accuracy. In article 7 the theory of a suspension bridge with a continuous stiffening truss is discussed and there are derived equations for cable tension produced by live load and for bending moments at the towers. The use of the method of superposition considerably simplifies the derivation of these equations. In the last article, the case of a stiffening truss of variable cross section is discussed and it is shown that by using trigonometric series this problem can be treated without much difficulty.

1. PARABOLIC FUNICULAR CURVE.

Assume that a uniform and perfectly flexible cable fixed at points A and B, Fig. 1, is submitted to the action of dis-

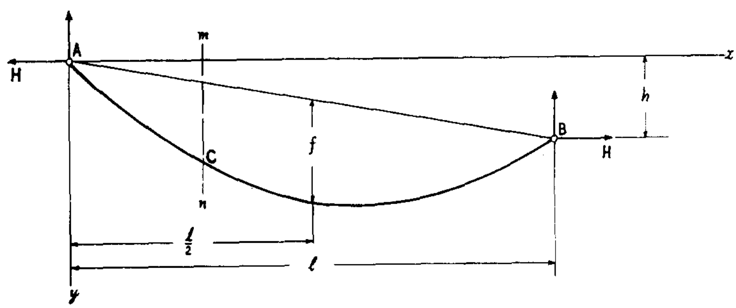

FIG. 1.

tributed vertical load. Then the ordinate y of any point C of the cable is obtained from the equation of moments with respect to C of forces to the left of this point which gives

$$\mathfrak{M}_x + H\frac{h}{l}x - Hy = 0. \qquad (a)$$

In this equation $\mathfrak{M}_x$ denotes the bending moment at the cross-section mn of a simply supported beam of span l and carrying the load acting on the cable. H is the horizontal component of the tensile force in the cable, and h is the difference in elevation of the ends of the cable. In the particular case when the load of intensity w is uniformly distributed along the horizontal projection of the cable we have

$$\mathfrak{M}_x = \frac{wx}{2}(l - x)$$

and equation (a) gives

$$y = \frac{wx}{2H}(l - x) + \frac{h}{l}x \tag{1}$$

which shows that the funicular curve in this case is a parabola with vertical axis. If the ends of the cable are on the same level, we obtain

$$y = \frac{wx}{2H}(l - x). \tag{2}$$

Applying this equation to the mid-point of the cable, where the ordinate of the funicular curve represents the sag f, we obtain

$$f = \frac{wl^2}{8H} \quad \text{or} \quad H = \frac{wl^2}{8f} \cdot \tag{3}$$

These equations hold also in the more general case shown in Fig. 1 if f is measured from the middle of the line AB joining the ends of the cable. In our further discussion the length s of the funicular curve will be required. It is obtained from the equation

$$s = \int_0^l (1 + y'^2)^{1/2} dx$$

Developing the expression under the integral sign into a series and substituting expression (2) for y, we obtain

$$s = l\left(1 + \frac{8}{3}\frac{f^2}{l^2} - \frac{32}{5}\frac{f^4}{l^4} + \frac{256}{7}\frac{f^6}{l^6} - \cdots\right) \cdot$$

In the case of flat parabolic curves, say $f/l \leqq 1/10$, we can take only the two first terms of the series and use the approximate formula:

$$s = l\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right) \cdot \tag{4}$$

To establish the relation between the change in length of the curve and the change in its sag, we differentiate equation

(4), which gives

$$\Delta s = \frac{16}{3} \cdot \frac{f}{l} \Delta f \tag{5}$$

and

$$\Delta f = \frac{3}{16} \cdot \frac{l}{f} \Delta s. \tag{6}$$

To find the change Δf due to a rise in tempersture of t degrees, we substitute $\Delta s = \epsilon t s$ into equation (6) and obtain

$$\Delta f = \frac{3}{16} \frac{l^2}{f} \epsilon t \left(1 + \frac{8}{3} \frac{f^2}{l^2} \right). \tag{7}$$

The elastic elongation of the cable is obtained from the equation

$$\Delta s = \int_0^s \frac{H}{A_c E_c} \frac{ds}{dx} ds = \int_0^l \frac{H}{A_c E_c} (1 + y'^2) dx$$

in which A_c is the cross-sectional area of the cable and E_c, its modulus of elasticity. Substituting equation (2) for y and integrating, we obtain

$$\Delta s = \frac{Hl}{A_c E_c} \left(1 + \frac{16}{3} \frac{f^2}{l^2} \right). \tag{8}$$

The corresponding change in sag, from equation (6), is *

$$\Delta f = \frac{3}{16} \frac{Hl^2}{A_c E_c f} \left(1 + \frac{16}{3} \frac{f^2}{l^2} \right). \tag{9}$$

2. DEFLECTIONS OF UNSTIFFENED SUSPENSION BRIDGES.

In the case of a suspension bridge of large span the dead load uniformly distributed on a horizontal plane is usually many times larger than that uniformly distributed along the cables, and we can assume with sufficient accuracy that the curve of the cable under the action of dead load is a parabola. Let us consider now deflections in the cable produced by live load. As a first example we consider the symmetrical case in which the load of intensity p is uniformly distributed along the distance 2ξ of the span, Fig. 2. The full line indicates

* In the derivation of equations (7) and (9) it is assumed that the change Δf in sag is small and its effect on the horizontal tension H is neglected.

the shape of the cable under the action of dead load w only. Let f_w and H_w denote the corresponding values of the sag of the cable and of the horizontal component of the tensile force in the cable. The length of the cable then is

$$s = l\left(1 + \frac{8}{3}\frac{f_w^2}{l^2}\right) = l\left(1 + \frac{1}{24}\frac{w^2 l^2}{H_w^2}\right). \qquad (b)$$

After application of live load p the shape of the cable will be as shown by the dotted line ACD. It consists of two parabolic curves AC and CD having a common tangent at C.

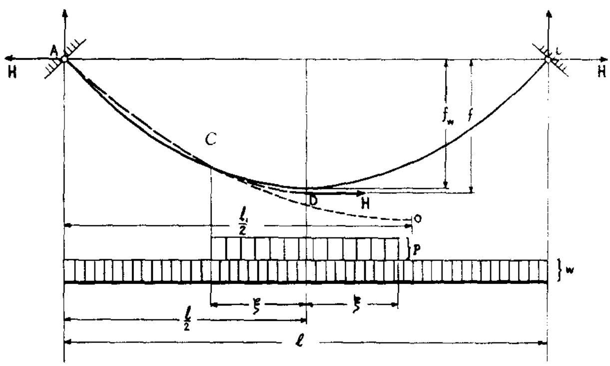

FIG. 2.

The curve CD carries the load of intensity $w + p$ and the curve AC carries the load of intensity w. The distance $l_1/2$ of the vertex o of this latter curve from the vertical through A will be found from the condition that the total load along the portion CO of the curve ACO is the same as the total load along the curve CD. Hence

$$\frac{l_1}{2} = \frac{l}{2} + \xi\frac{p}{w}. \qquad (c)$$

We denote by f and H the sag of the cable and the tensile force in the cable at D after application of the live load. One relation between these two quantities is obtained by making the equation of moments with respect to point D which gives

$$f = \frac{1}{H}\left[\frac{wl^2}{8} + \frac{p\xi}{2}(l - \xi)\right]. \qquad (d)$$

The second equation is obtained from the condition that the length of the cable remains unchanged * during application of the live load. Using for the length of the curves AO, CO and CD the approximate equation (b), the condition of inextensibility of the cable is

$$\frac{l}{2}\left(1 + \frac{8}{3}\frac{f_w^2}{l^2}\right) = \frac{l_1}{2}\left(1 + \frac{1}{24}\frac{w^2 l_1^2}{H^2}\right)$$
$$- \frac{\xi(w+p)}{w}\left[1 + \frac{\xi^2(w+p)^2}{6H^2}\right] + \xi\left[1 + \frac{1}{6}\frac{\xi^2(w+p)^2}{H^2}\right]. \qquad (e)$$

Introducing notations

$$\frac{p}{w} = n, \qquad \frac{2\xi}{l} = z, \qquad (f)$$

we obtain from equation (e)

$$H = H_w\sqrt{1 + 3nz + 3n^2z^2 - (2n^2 + n)z^3}. \qquad (10)$$

Substituting this value of H into equation (d), we obtain

$$f = f_w \frac{1 + n(2z - z^2)}{\sqrt{1 + 3nz + 3n^2z^2 - (2n^2 + n)z^3}}. \qquad (11)$$

To find what portion of the span must be loaded to produce the maximum deflection at the middle of the bridge, we put equal to zero the derivative of expression (11) with respect to z which gives the equation

$$(2n^2 + n)z^4 - 2n(n - 1)z^3 - 3(n - 1)z^2 - 4z + 1 = 0.$$

Solving this equation for several numerical values of n, we obtain for z the values given in Table I. Substituting these values of z into equation (11), we find the values of the sag f of the cable after application of the live load. The calculated ratios of the change in sag to the initial sag are given in the third line of Table I. In the case of long span bridges, the ratio p/w is usually small,† say smaller than 1/4, and it

* The small influence on the deflection of an elastic elongation of the cable will be discussed later.

† In the case of the Washington bridge over Hudson River this ratio is about 1/6.

may be seen from Table I that the deflection at the middle is of the order of one hundredth of the sag f_w or of one thousandth of the span l of the bridge. Such deflections can be considered as sufficiently small to make the use of any stiffening truss unnecessary.

TABLE I.

$n =$	0	0.10	0.25	0.50	1.00
$z =$	0.333	0.322	0.306	0.289	0.253
$\frac{f - f_w}{f_w} =$	0	0.0069	0.0151	0.0281	0.0456
$\frac{H}{H_w} =$	1	1.047	1.112	1.213	1.379

To calculate the deflection at the middle due to elastic deformation of the cable produced by live load, we use the approximate formula (9) in which $H - H_w$, instead of H, must be substituted. Then the deflection due to the elongation of the cable is

$$\Delta f = \frac{3}{16} \cdot \frac{H_w l}{A_c E_c} \cdot \frac{l}{f} \left(\frac{H}{H_w} - 1 \right) \left(1 + \frac{16}{3} \frac{f^2}{l^2} \right) \cdot \qquad (g)$$

The values of the ratio H/H_w for various values of n calculated from equation (10) are given in the last line of Table I. Taking for a numerical example $n = 1/4$, $H_w/A_cE_c = 0.002$,* $f/l = 0.1$, we find $H/H_w = 1.112$ and equation (g) gives $\Delta f = 0.00044l$. This deflection must be added to the deflection $0.00151l$, calculated from equation (11), to obtain the total deflection produced by live load.

Let us consider now the deflection of the cable produced by a concentrated force applied at the middle of the span. This force can be considered as a load distributed along a very short distance, and equation (11) can be used also in this case. From notation (f) it follows that

$$nz = \frac{2p\xi}{wl} = \frac{P}{Q} = \psi \qquad (h)$$

* This value depends evidently on the allowable stresses produced by dead load.

where ψ denotes the ratio of the live load P to the dead load Q of the bridge. Substituting ψ for nz and zero for z into equation (11), we obtain

$$f = f_w \frac{1 + 2\psi}{\sqrt{1 + 3\psi + 3\psi^2}} \cdot \qquad (12)$$

In the case of long span bridges, the concentrated load P is small in comparison with the dead load Q of the bridge and ψ is a small quantity. Developing then the radical in the denominator of expression (12) into a series and taking only the first three terms of that series, we obtain

$$\sqrt{1 + 3\psi + 3\psi^2} \approx 1 + \frac{3}{2}\psi + \frac{3}{8}\psi^2$$

and equation (12) gives

$$f = f_w \left(1 + \frac{1}{2}\psi - \frac{9}{8}\psi^2 \right) \cdot$$

Hence the deflection produced in the cable by a concentrated force applied at the middle is

$$\Delta f = f - f_w = f_w \frac{\psi}{2}\left(1 - \frac{9}{4}\psi \right) \cdot \qquad (13)$$

Assuming, for example, $\psi = 0.01$ * and $f_w/l = 0.1$, we obtain

$$\Delta f = 0.000489l$$

which is a very small deflection. To find the deflection due to elastic elongation of the cable produced by a concentrated force, we calculate first the change $H - H_w$ in the horizontal tensile force. Equation (10) in this case gives

$$H = H_w\sqrt{1 + 3\psi + 3\psi^2} \approx H_w \left(1 + \frac{3}{2}\psi + \frac{3}{8}\psi^2 \right)$$

and we obtain

$$H - H_w = \frac{3}{2} H_w\psi \left(1 + \frac{1}{4}\psi \right) \cdot$$

* In the case of the Washington Bridge, this value of ψ corresponds to a concentrated load of 570 tons.

Substituting this for H in equation (9), we find

$$\Delta f = \frac{9}{32} \cdot \frac{H_w}{A_c E_c} \frac{l}{f} \psi \left(1 + \frac{1}{4} \psi \right) \left(1 + \frac{16}{3} \frac{f_w^2}{l^2} \right) l.$$

For small ψ which we have in the case of large spans, this deflection is a very small one.

We discussed up to now the symmetrical case of loading, Fig. 2. The case of a non-symmetrical distribution of uniform live load can be treated in a similar manner. Let us consider now a general case of vertical live load acting on a cable with both ends on the same level. The initial ordinates of the funicular curve are obtained from equation of moments which gives

$$y = \frac{\mathfrak{M}_w}{H_w} \cdot \qquad (i)$$

In this equation $\mathfrak{M}_w$ is bending moment due to dead load calculated as for a simply supported beam and H_w the horizontal component of the tensile force produced in the cable by dead load. If live load is now applied, the bending moment calculated as for a simple beam becomes $\mathfrak{M}_w + \mathfrak{M}_p$ and the horizontal component of cable tension becomes $H_w + H_p$. Denoting by η the vertical deflections of the cable, we obtain, from equation of moments,

$$y + \eta = \frac{\mathfrak{M}_w + \mathfrak{M}_p}{H_w + H_p} \qquad (j)$$

Subtracting equation (i) from this equation, we obtain

$$\eta = \frac{\mathfrak{M}_p - H_p y}{H_w + H_p} \cdot \qquad (14)$$

It is seen that the vertical deflections η can be readily calculated provided we know the horizontal component H_p of cable tension produced by live load. This later quantity can be found from geometrical considerations.

Let us consider an infinitely small element ab of the cable, Fig. 3. Under the action of live load this element elongates somewhat and takes a new position a_1b_1. We denote by ξ and η the horizontal and the vertical components of the small displacement of point a. The initial length of the element is obtained from the equation

$$ds^2 = dx^2 + dy^2. \tag{k}$$

The length of the same element after application of live load is found from the equation

$$(ds + \Delta ds)^2 = (dx + d\xi)^2 + (dy + d\eta)^2, \tag{l}$$

in which Δds is the elongation of the element produced by

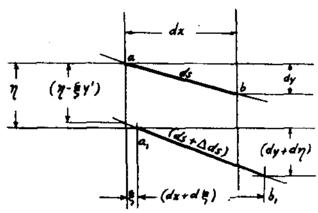

FIG. 3.

live load. Neglecting the small change in slope of the cable produced by live load,* we put

$$\Delta ds = \frac{dsH_p}{A_cE_c}\frac{ds}{dx}. \tag{m}$$

Since $H_p ds/dx$ is that part of the tensile force in the cable which is produced by live load and which is usually much smaller than the part produced by dead load, the unit elongation $\Delta ds/ds$ is usually much smaller than one thousandth. In such a case $(\Delta ds)^2$ in equation (l) can be neglected. From the same reason, and from the observation that the curve of the cable is a flat curve, we neglect also $d\xi^2$. Then we obtain, from equations (k) and (l),

$$ds\Delta ds = dxd\xi + dyd\eta + \tfrac{1}{2}d\eta^2,$$

which gives

$$d\xi = \frac{ds}{dx}\Delta ds - \frac{dy}{dx}d\eta - \frac{1}{2}\frac{d\eta}{dx}d\eta.$$

Substituting expression (m) for Δds in this equation and

* The error introduced by this omission will be discussed later, see p. 227.

integrating, we obtain

$$\xi = \frac{H_p}{A_c E_c}\int_0^x \left(\frac{ds}{dx}\right)^3 dx - \int_0^x y'\eta' dx - \frac{1}{2}\int_0^x \eta'^2 dx. \quad (15)$$

With the values of y' and η' which are encountered in long span bridges, the value of ξ usually does not surpass one thousandth of x.* At the ends of the cable ξ vanishes and we obtain from equation (15)

$$\frac{H_p}{A_c E_c}\int_0^l \left(\frac{ds}{dx}\right)^3 dx = \int_0^l y'\eta' dx + \frac{1}{2}\int_0^l \eta'^2 dx. \quad (n)$$

The integral on the left side of this equation for the assumed parabolic shape of the cable can be readily evaluated and we obtain

$$\int_0^l \left(\frac{ds}{dx}\right)^3 dx = \int_0^l (1 + y'^2)^{3/2} dx = l\left\{\frac{1}{4}\left(\frac{5}{2} + \frac{16f^2}{l^2}\right)\left(1 + \frac{16f^2}{l^2}\right)^{1/2} + \frac{3l}{32f}\log_e\left[\frac{4f}{l} + \left(1 + \frac{16f^2}{l^2}\right)^{1/2}\right]\right\}. \quad (o)$$

On the right side of equation (n) we make integration by parts. Observing that η vanishes at the ends of the cable, and using equation (2) we obtain

$$\left.\begin{aligned} \int_0^l y'\eta' dx &= \Big|y'\eta\Big|_0^l - \int_0^l y''\eta dx = \frac{w}{H_w}\int_0^l \eta dx. \\ \frac{1}{2}\int_0^l \eta'^2 dx &= \frac{1}{2}\Big|\eta'\eta\Big|_0^l - \frac{1}{2}\int_0^l \eta''\eta dx = -\frac{1}{2}\int_0^l \eta''\eta dx. \end{aligned}\right\} \quad (p)$$

Substituting expressions (o) and (p) into equation (n) and denoting the integral (o) by L, we obtain

$$\frac{H_p}{A_c E_c} L = \frac{w}{H_w}\int_0^l \eta dx - \frac{1}{2}\int_0^l \eta''\eta dx. \quad (16)$$

This equation, together with equation (14), give the system of equations sufficient for calculation of vertical deflections of the cable.

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* The maximum value of ξ/x occurs near the supports where η' and y' usually have their largest numerical values.

Let us apply these equations to the above discussed case shown in Fig. 2. In this case

$$\mathfrak{M}_p = p\xi x, \quad \text{for} \quad x < \frac{l}{2} - \xi;$$

$$\mathfrak{M}_p = p\xi x - \frac{1}{2}p\left(x - \frac{l}{2} + \xi\right)^2, \quad \text{for} \quad \frac{l}{2} + \xi > x > \frac{l}{2} - \xi;$$

$$\eta = \frac{1}{H_w + H_p}\left[p\xi x - \frac{w}{2H_w}x(l - x)H_p\right], \qquad \text{for} \quad x < \frac{l}{2} - \xi;$$

$$\eta = \frac{1}{H_w + H_p}\left[p\xi x - \frac{1}{2}p\left(x - \frac{l}{2} + \xi\right)^2 - \frac{w}{2H_w}x(l - x)H_p\right], \quad \text{for} \quad \frac{l}{2} + \xi > x > \frac{l}{2} - \xi;$$

$$\eta'' = \frac{wH_p}{H_w(H_w + H_p)}, \quad \text{for} \quad x < \frac{l}{2} - \xi;$$

$$\eta'' = \frac{-pH_w + wH_p}{H_w(H_w + H_p)}, \quad \text{for} \quad \frac{l}{2} + \xi > x > \frac{l}{2} - \xi.$$

Substituting these expressions into equation (16), assuming that the cable is inextensible, and introducing our previous notations (*f*), we obtain, for calculation of H_p, the following quadratic equation

$$\left(\frac{H_p}{H_w}\right)^2 + 2\left(\frac{H_p}{H_w}\right) - 3nz - 3n^2z^2 + nz^3 + 2n^2z^3 = 0,$$

which gives for $H_p + H_w$ the same value as obtained from our previous equation (10).

Sometimes equation (16) is simplified by omitting the second term on the right side and taking

$$\frac{H_p}{A_cE_c}L = \frac{w}{H_w}\int_0^l \eta dx. \tag{17}$$

Considering an inextensible cable and substituting for η its expression (14) we obtain, from equation (17),

$$H_p = \frac{\int_0^l \mathfrak{M}_p dx}{\int_0^l y dx} = \frac{3}{2fl}\int_0^l \mathfrak{M}_p dx. \tag{18}$$

In the case shown in Fig. 2 equation (18) gives

$$H_p = \frac{nH_w}{2}(3z - z^3).$$

Applying this approximate formula to the numerical examples given in Table I, p. 219, we find that the results obtained are in good agreement with those given in the last line of Table I.

3. FUNDAMENTAL EQUATIONS FOR STIFFENED SUSPENSION BRIDGES.

It is seen from the preceding discussion that the deflection of the cable produced by live load is small only in the case of heavy long span bridges. Otherwise the deflections may be

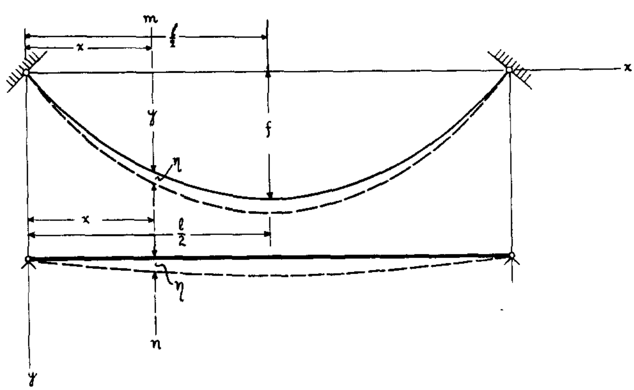

FIG. 4.

considerable. In order to reduce them, stiffening trusses are usually introduced. A simplest structure of this kind, shown in Fig. 4, consists of a single-span cable stiffened by a simply supported truss of constant cross section. It is assumed that by a proper assembly the dead load of the structure, uniformly distributed along the span, is entirely transmitted to the cable which takes the parabolic form shown in the figure by full line. A live load produces deflection of the cable and of the truss as indicated in the figure by dotted lines. We assume that both these deflections are equal.* The spacing of hangers is assumed small as compared with the length of span so that the action of the hangers on the cable and on the truss can be considered as continuously distributed along the span.

Let us consider first the case when the structure is carrying only dead load. The truss does not suffer bending in this case, and the equation of moments for the forces to the left of a cross section *mn*, Fig. 4, then gives

$$\mathfrak{M}_w - H_w y = 0. \qquad (a)$$

When live load is applied and deflections η are produced, there will be bending moment M acting in a cross section *mn* of the truss and the equation of moments for the forces to the left of this cross section is

$$\mathfrak{M}_w + \mathfrak{M}_p - (H_w + H_p)(y + \eta) - M = 0. \qquad (b)$$

Subtracting equation (a) from equation (b), we obtain

$$M = \mathfrak{M}_p - (H_w + H_p)\eta - H_p y. \qquad (19)$$

From this equation, the bending moment at any cross section of the truss can be calculated provided the horizontal component of the tensile force in the cable and deflection η are known.

In the case of very rigid stiffening trusses, the deflections η can be neglected and we obtain

$$M = \mathfrak{M}_p - H_p y. \qquad (20)$$

This bending moment is independent of deflections and can be evaluated by the methods used in the analysis of rigid statically indeterminate structures. Investigations show that the stiffening trusses in large-span bridges are usually very flexible, and in calculation of bending moments recourse must be made to the more complete equation (19) which

* That is, the elongation of the hangers and their small inclination during deformation are neglected in this discussion.

requires the calculation of deflections η * of the truss. Using for the truss the differential equation of the deflection curve of a beam

$$EI\frac{d^2\eta}{dx^2} = -M,$$

we obtain, by using expression (19), the following equation for calculation η

$$EI\frac{d^2\eta}{dx^2} - (H_w + H_p)\eta = H_p y - \mathfrak{M}_p. \qquad (21)$$

The quantity $\mathfrak{M}_p$ in this equation can be readily calculated for any distribution of live load over the span. The quantities y and H_w are given by equations (2) and (3), and only the quantity H_p is unknown. It depends on the deflections η, and for its determination equation (16) of the preceding article is used. Equation (21) together with equation (16) completely defined the deflections of the stiffening truss. In solution of these equations the trial and error method is used. We assume a certain value for H_p, for instance the value obtained for the unstiffened cable, and with this value solve equation (21). The obtained expression for η we substitute in the integrals on the right side of equation (16). Since H_p was taken arbitrarily, the result of this substitution usually will not equal the left side of equation (16), and it will be necessary to repeat the calculation with a new assumed value of H_p. These trial calculations are continued so far as to obtain H_p with a sufficient accuracy. The procedure of this calculation with all details will be discussed in the next two articles.

Now we will discuss how accurate equations (16) and (21) are and what is the magnitude of errors introduced in these equations by neglecting various small quantities during their derivation. We begin with the discussion of elongation of the cable. In the derivation of equation (m) of the preceding article, we neglected the change in the deflection of the cable

* J. Melan was the first who indicated the importance of considering deflections in analysis of suspension bridges, see his book, "Theorie der eisernen Bogenbrücken und der Hängebrücken," Leipzig 1906. English translation by D. B. Steinman, Chicago, 1913. Melan's theory has been widely used in analysis of large span suspension bridges in this country.

produced by live load. Taking into account this additional deflection we obtain

$$\Delta ds = \frac{dsH_p}{A_cE_c}[1 + (y' + \eta')^2]^{\frac{1}{2}} \approx \frac{dsH_p}{A_cE_c}\left(\frac{ds}{dx} + y'\eta' + \frac{1}{2}\eta'^2\right).$$

Using this more accurate expression for Δds we obtain, instead of equation (n) of the preceding article, the following equation

$$\frac{H_p}{A_cE_c}\int_0^l \left(\frac{ds}{dx}\right)^3 dx = \int_0^l y'\eta' dx$$
$$+ \frac{1}{2}\int_0^l \eta'^2 dx - \frac{H_p}{A_cE_c}\int_0^l \left(\frac{ds}{dx}\right)^2 \left(y'\eta' + \frac{1}{2}\eta'^2\right) dx.$$

The last term on the right side of this equation represents the required correction. Since H_p/A_cE_c is usually smaller than one thousandth we conclude that the relative error in the magnitude of the right side of equation (n) due to the use of the approximate expression (m) is of the order of one thousandth, which can be disregarded in a practical analysis.

Let us consider now the effect on the bending moment in the truss of horizontal displacements ξ in the cable which were entirely disregarded in our previous derivation. To take these displacements into account, we observe that the vertical distances between the full line and dotted line curves, marked by η in Fig. 4, are more accurately equal to $\eta - \xi dy/dx$, as shown in Fig. 3. We note also that each element of the load transmitted to the cable, and approximately equal to $-dx(H_w + H_p)d^2y/dx^2$, has a horizontal displacement ξ, which produces the change in moment of this element equal to

$$-\xi dx(H_w + H_p)\frac{d^2y}{dx^2}.$$

With these two considerations we obtain, instead of expression (19), the following more accurate value of the bending moment:

$$M = \mathfrak{M}_p - (H_w + H_p)\eta - H_py + (H_w + H_p)\xi\frac{dy}{dx}$$
$$- (H_w + H_p)\int_0^x \frac{d^2y}{dx^2}\xi dx. \quad (22)$$

The required correction in the bending moment is represented by the last two terms in this expression. To get a clearer idea regarding the magnitude of this correction, let us calculate the intensity of the load acting on the truss. This intensity is obtained as the second derivative with respect to x of the bending moment (22), taken with opposite sign, which gives

$$-\frac{d^2M}{dx^2} = p + H_p\frac{d^2y}{dx^2} + (H_w + H_p)\frac{d^2\eta}{dx^2} - (H_w + H_p)\frac{d}{dx}\left(\frac{d\xi}{dx}\frac{dy}{dx}\right). \quad (23)$$

The last term in this expression represents the correction due to horizontal displacements of the cable.

The same expression for the intensity of the load on the truss is obtained also in another manner by subtracting the intensity of the upward pull of the hangers on the truss from the combined intensity $w + p$ of the downward loading. The intensity of the vertical loading on the cable at a distance x from the left support is

$$q = -(H_w + H_p)\frac{d^2}{dx^2}\left(y + \eta - \xi\frac{dy}{dx}\right). \quad (c)$$

The upward pull transmitted to a length dx of the truss at x is the downward pull on a horizontal length $dx[1 + (d\xi/dx)]$ at $x + \xi$ on the deflected cable. Hence the required intensity of the load on the truss is

$$w + p - \left(q + \xi\frac{dq}{dx}\right)\left(1 + \frac{d\xi}{dx}\right) = w + p - q - \frac{d}{dx}(q\xi). \quad (d)$$

Substituting for q its expression (c) and neglecting small terms of higher order, we obtain the previous expression (23).*

* The correction due to horizontal displacements ξ of the cable has been discussed in a recent paper by R. J. Atkinson and R. V. Southwell, Proc. of the Institution of Civil Engineers, session 1938–1939, p. 289. These authors overlooked the term $-\xi(dy/dx)$ in the above expression (c) and did not obtain a satisfactory expression for the correction.

Substituting for ξ the first two terms in expression (15) into the last term of expression (23), we find that the correction in the intensity of the load on the truss, due to horizontal displacements ξ, is

$$-(H_w + H_p)\frac{d}{dx}(\xi' y')$$
$$= -(H_w + H_p)\left\{ -\frac{w}{H_w}\left[\frac{H_p}{A_cE_c}s'^3 - y'\eta'\right] + y'\left[\frac{3H_p}{A_cE_c}s'^2s'' + \frac{w}{H_w}\eta' - y'\eta''\right]\right\},$$

where primes denote the first derivative with respect to x.

For flat curves we can take

$$s' \approx 1 + \frac{1}{2}y'^2, \qquad s'' \approx -\frac{w}{H_w}y',$$

which gives

$$-(H_w + H_p)\frac{d}{dx}(\xi' y')$$
$$= \frac{(H_w + H_p)w}{H_w}\left[\frac{H_p}{A_cE_c}(s'^3 + 3s'^2y'^2) - 2y'\eta'\right] + (H_w + H_p)\eta''y'^2. \quad (e)$$

From our previous discussion we conclude that the first term on the right side of this equation is of the order of one thousandth of w and can be neglected in practical calculation. The second term, $(H_w + H_p)\eta''y'^2$, can also be considered as small in comparison with the term $(H_w + H_p)\eta''$ representing the effect on the intensity of the load of vertical deflections of the cable. Hence the total effect of horizontal displacements of the cable can be considered as small and usually can be neglected in practical calculation. If we would like to take that effect into consideration, we can begin with a solution of equations (16) and (21). Knowing η we calculate ξ from equation (15) and the more accurate value for bending moment from equation (22).

In a similar manner the effect of extension of hangers on the magnitude of bending moment can be evaluated. The calculation shows that this effect is also very small and can be neglected.*

Let us consider now the effect of shearing force on deflection of a stiffening truss. For this purpose we take the differential equation of the deflection curve in the following form:

$$EI\frac{d^2\eta}{dx^2} = -M + mEI\frac{d^2M}{dx^2}, \qquad (f)$$

in which the second term on the right side represents the effect of shearing force on deflection. The magnitude of factor m in this term depends on the kind of structure used for the stiffening truss. In the case of an I-beam, we take

$$m = \frac{1}{A_wG} \qquad (g)$$

where A_w is the cross-sectional area of the web of the beam and G is the modulus of elasticity of the material in shear.

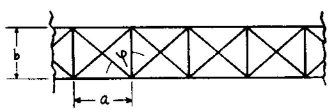

FIG. 5.

In the case of a truss, as shown in Fig. 5, we take †

$$m = \frac{1}{A_dE\sin\varphi\cos^2\varphi} \qquad (h)$$

where A_d is the sum of cross-sectional area of the two diagonals in a panel.

Substituting for M in equation (f) its expression (19), we obtain

$$EI[1 + m(H_w + H_p)]\frac{d^2\eta}{dx^2} - (H_w + H_p)\eta$$
$$= -\mathfrak{M}_p + H_py - mEI\left(p - \frac{H_p}{H_w}w\right). \qquad (24)$$

* Such calculations were made by F. E. Turneaure, see the book "Modern Framed Structures" by Johnson, Bryan and Turneaure, Vol. 2, 9th ed., p. 299, 1917.

† See writer's book, "Theory of Elastic Stability," 1936, p. 143.

This equation is of the same form as equation (21) and we see that the effect of shearing force can be readily taken into account provided the factor m is known.

In derivation of equation (16) the elastic elongation of the cable alone was considered. The equation can be readily generalized and extended to those cases in which an elongation of the cable depends also on a change in its temperature. Instead of equation (m) of the preceding article, we use in such a case the equation

$$\Delta ds = \frac{dsH_s}{A_cE_c}\frac{ds}{dx} + ds\epsilon t \qquad (i)$$

where ϵ is the coefficient of thermal expansion, and H_s is the horizontal component of the tensile force produced in the cable by the combined action of live load and temperature change. Using equation (i), instead of equation (m), and introducing the notation

$$\int_0^l \left(\frac{ds}{dx}\right)^2 dx = L_1$$

we obtain

$$\frac{H_s}{A_cE_c}L + \epsilon tL_1 = \frac{w}{H_w}\int_0^l \eta dx - \frac{1}{2}\int_0^l \eta''\eta dx. \qquad (25)$$

This equation, instead of equation (16), must be used if we are considering a simultaneous action of live load and temperature change.

4. ANALYSIS OF STIFFENING TRUSSES.

Let us begin with the case in which a single concentrated load P is acting on the truss. Making the second derivative of equation (21), we find that deflections of the truss in this case are the same as those occurring in a simply supported beam subjected to the combined action of an axial tensile force $H_w + H_p$, of a uniformly distributed upward lateral load of intensity H_pw/H_w, and a concentrated load P as shown in Fig. 6. In such a case, with notation

$$\frac{H_w + H_p}{EI} = k^2 \qquad (26)$$

the deflections produced by the load P in the part of the beam to the left of this load, $(x < l - c)$, are *

$$\eta_1 = -\frac{P}{H_w + H_p} \cdot \frac{\sinh kc}{k \sinh kl} \sinh kx + \frac{Pcx}{(H_w + H_p)l} \cdot \qquad (27)$$

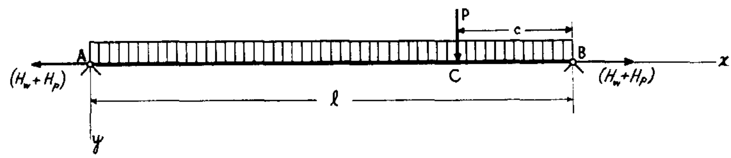

FIG. 6.

For the portion of the beam to the right of the load, $(x > l - c)$ the deflections are

$$\eta_1 = -\frac{P}{(H_w + H_p)} \frac{\sinh k(l - c)}{k \sinh kl} \sinh k(l - x) + \frac{P(l - c)(l - x)}{(H_w + H_p)l} \cdot \qquad (28)$$

The deflections produced by the upward pull are †

$$\eta_2 = -\frac{H_p}{H_w} \cdot \frac{wl^2}{H_w + H_p} \left\{ \frac{\cosh\left(\frac{kl}{2} - kx\right)}{k^2l^2 \cosh \frac{kl}{2}} - \frac{1}{k^2l^2} + \frac{x(l - x)}{2l^2} \right\} \cdot \qquad (29)$$

The total deflections η of the truss are obtained by superposing deflections η_1 on deflections η_2.

To determine the magnitude of tension H_p, entering into equations (27), (28), and (29), we use equation (17), which is obtained from equation (16) by omitting the second term on the right side.‡ Substituting expressions (27) to (29) into

* See "Strength of Materials," Vol. 2, p. 39.

† See "Strength of Materials," Vol. 2, p. 40.

‡ This term usually has only a small effect on the magnitude of H_p. This effect will be discussed in the next article.

equation (17) and performing the integration, we obtain, for calculating H_p, the equation

$$H_p\left[\frac{H_w + H_p}{A_cE_c}\cdot\frac{L}{l} + \frac{1}{12}\left(\frac{8f}{l}\right)^2 \times\left(1 - \frac{12}{k^2l^2} + \frac{24}{k^3l^3}\tanh\frac{kl}{2}\right)\right] = P\frac{8f}{l}\left\{\frac{1}{2}\frac{c}{l}\left(1 - \frac{c}{l}\right) - \frac{1}{k^2l^2\sinh kl} \times[\sinh kl - \sinh kc - \sinh k(l - c)]\right\}. \quad (30)$$

In the case of long span bridges the quantity kl is usually a number of considerable magnitude * and all terms in equation (30), containing k are small and in a first approximation can be neglected. The term $(H_w + H_p)/A_cE_c$ can also be neglected as very small, and we obtain

$$H_p = \frac{3}{4}P\frac{l}{f}\cdot\frac{c}{l}\left(1 - \frac{c}{l}\right).$$

For $c = l/2$ this gives

$$H_p = \frac{3}{2}P\frac{l}{8f}.$$

The same result we obtain, for small ψ, from equation (10), which indicates that by omitting all terms containing k we obtain, from equation (30), for H_p the same value as in the case of an unstiffened suspension bridge.

Equation (30) can be used for calculating the influence line for H_p. In such a case we assume that P is a small load moving along the truss. Then H_p can be neglected in comparison with H_w, $kl \approx l\sqrt{H_w/EI}$, and we obtain

$$H_p = P\frac{8f}{l}\,\frac{\frac{1}{2}\frac{c}{l}\left(1 - \frac{c}{l}\right) - \frac{1}{k^2l^2\sinh kl}\times[\sinh kl - \sinh kc - \sinh k(l - c)]}{\frac{H_w}{A_cE_c}\frac{L}{l} + \frac{1}{12}\left(\frac{8f}{l}\right)^2\left(1 - \frac{12}{k^2l^2} + \frac{24}{k^3l^3}\tanh\frac{kl}{2}\right)}.$$

* In the case of the Ambassadore Bridge (Detroit) $kl = 9.52$. In the case of the Washington Bridge, after placing the planned stiffening truss, $kl = 35$.

The magnitude of H_p depends not only on position of the load P but also on the quantities kl, H_w/A_cE_c, and f/l. In Fig. 7*a* is shown the influence line for H_p calculated on the assumption that $kl = 10$, $H_w/A_cE_c = 0.002$ and $f/l = 0.1$. For

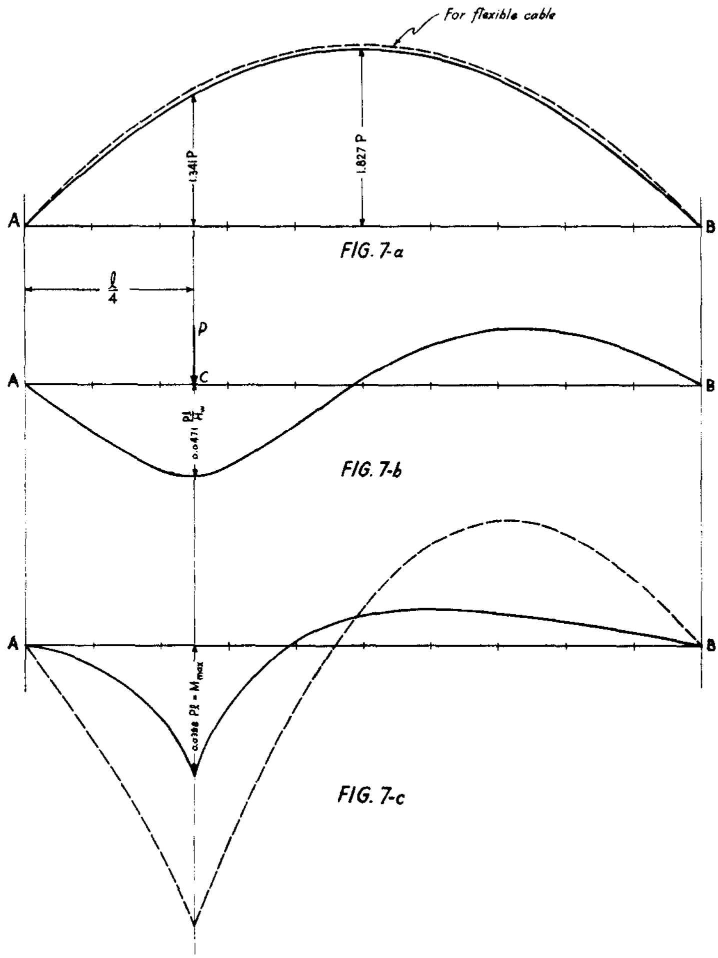

FIG. 7.

comparison there are shown in the same figure by dotted line the values of H_p for a non-stiffened cable. It is seen that for the assumed value of kl the stiffened truss has only a small effect on the magnitude of H_p.

Having H_p and using equations (27) to (29), we can calculate the deflections of the truss. In Fig. 7*b* the deflection curve is constructed for the case where $c = 0.75l$. Since H_p is neglected in comparison with H_w, the deflections become proportional to P and the principle of superposition holds. The reciprocity theorem also holds, and the deflection curve in Fig. 7*b* is the influence line for the deflection at the quarter point C. Using this line we can readily construct the influence line for bending moment at C. Neglecting H_p in comparison with H_w in expression (19), we obtain

$$M = \mathfrak{M}_p - H_p y - H_w \eta.$$

The first two terms on the right side of this equation give the bending moment if the influence of deflections of the truss is disregarded. The corresponding influence line is given by the dotted line in Fig. 7*c*. The last term on the right side gives the effect of deflection of the truss on the bending moment. Taking this into account, we obtain the full line in Fig. 7*c*. It is seen that in this case the deflections have a very large effect on the bending moment and cannot be disregarded.

In using the derived influence line for calculation of bending moment it must be noted that in our derivation the increase of tension in the cable produced by live load was neglected. Hence the influence line will give a satisfactory result only if the live load is very small in comparison with the dead load. If it is not small, the influence line will not give an accurate value of the moment, and can be used with sufficient accuracy only for a determination of the limits within which the live load must be distributed to produce the maximum value of the moment.* Calculations of deflections and moments must then be accomplished by using equations (27) to (30) within which H_p has been retained.

Assume, for example, that live load is distributed as shown in Fig. 8. Then the equation for calculating H_p is obtained from equation (30) by substituting *pdc*, instead of P, and integrating the right side of the equation from $c = a$

* The use of influence lines in analysis of suspension bridges was first proposed by T. Godard, "Annales des Ponts et Chaussées" 7e série, Vol. 8, 1894, p. 105.

to $c = b$, which gives

$$H_p\left[\frac{H_w + H_p}{A_cE_c}\cdot\frac{L}{l} + \frac{1}{12}\left(\frac{8f}{l}\right)^2\left(1 - \frac{12}{k^2l^2} + \frac{24}{k^3l^3}\tanh\frac{kl}{2}\right)\right]$$

$$= 8fp\left\{\frac{3l(b^2 - a^2) - 2(b^3 - a^3)}{12l^3} - \frac{1}{k^2l^2}\right.$$

$$\left.\times\left[\frac{b-a}{l} - \frac{\cosh kb - \cosh ka + \cosh k(l-a) - \cosh k(l-b)}{kl\sinh kl}\right]\right\}. \qquad (31)$$

The value of H_p can now be calculated from this equation by successive approximations. We start by omitting all terms

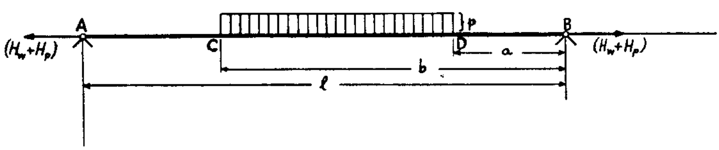

FIG. 8.

containing k and neglect also the term with $(H_w + H_p)/A_cE_c$. In this way we obtain the first approximation for H_p which will be close to the true value, if kl is of considerable magnitude,* say $kl = 10$. To get a better accuracy, we use the approximate value of H_p to calculate k from equation (26) and then substitute this value of k into equation (31), which gives then the second approximation for H_p which is usually accurate enough for practical application. If necessary, further approximations can be calculated in the same manner. When H_p has been calculated, the deflection curve will be found by using equations (27) to (29) together with the method of superposition. To find the deflections for the portion AC of the truss, Fig. 8, we substitute pdc, for P, into equation (27) and integrate from $c = a$ to $c = b$. This gives

* An example of such calculation is given in a paper by A. A. Jakkula, Publications of the International Association for Bridge and Structural Engineering, Vol. 4, p. 333, 1936.

(for $x < l - b$)

$$\eta_1 = \frac{pl}{H_w+H_p}\int_a^b \left[-\frac{\sinh kc}{kl \sinh kl} \sinh kx + \frac{cx}{l^2} \right] dc$$

$$= \frac{pl^2}{H_w+H_p}\left[\frac{\cosh ka - \cosh kb}{k^2l^2 \sinh kl} \sinh kx + \frac{(b^2-a^2)x}{2l^3} \right].$$

To obtain the complete deflection, we superpose on this deflection the deflection η_2 produced by the upward pull (eq. 29) which gives, for $x < l - b$,

$$\eta = \eta_1 + \eta_2 = \frac{pl^2}{H_w+H_p}$$

$$\times \left[\frac{\cosh ka - \cosh kb}{k^2l^2 \sinh kl} \sinh kx + \frac{(b^2-a^2)x}{2l^3} \right]$$

$$- \frac{H_p w l^2}{H_w(H_w+H_p)} \left[\frac{\cosh\left(\frac{kl}{2} - kx\right)}{k^2l^2 \cosh \frac{kl}{2}} - \frac{1}{k^2l^2} + \frac{x(l-x)}{2l^2} \right].$$

In a similar manner the deflections in the portions *CD* and *DB* of the truss can be obtained. A simpler method of calculation deflections is shown in the next article.

If a combined action of live load and temperature change is considered, the equation for calculation of the additional horizontal tension H_s is obtained by substitution into equation (31) H_s for H_p and $H_sL/A_cE_c + \epsilon t L_1$ for H_pL/A_cE_c (see p. 232).