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On Isothermal Squeeze Films

It is shown that squeeze film damping cutoff frequencies can be computed directly from the lowest eigenvalue of the Helmholtz equation. A kinematic mode is proposed and analyzed for the computation of those frequencies and it is demonstrated that Griffin's calculations may underestimate considerably those frequencies. New results are given for the squeeze film behavior between rectangular plates, annuli which are not necessarily thin and plate sectors.

Introduction

Some modern seismic accelerometers employ a damping device consisting of two plates which squeeze a thin film of gas from between them. This device is particularly useful when the differential displacement of the elements to be damped is small.

Griffin et al. [1] calculated damping coefficients and cutoff frequencies in compressible squeeze films by determining first the film response to a sudden step change in the film thickness and then using a well known technique which consists of applying the principle of superposition combined with the convolution integral formulation to determine the film response to any displacement function [2]. Finally they truncated the infinite series solution which they obtained to one term only, i.e., retaining only the first harmonic approximation of their solution, from which they extracted the damping coefficient and the cutoff frequency of the system. With this procedure, parallel motion of infinite strips, thin annuli and circular plates were analyzed. The tilting motion of thin annuli was also dealt with approximately, by neglecting circumferential flow of the gas, since they maintained that it is very difficult to analyze without this assumption.

A close inspection of the Griffin procedure leads to the immediate conclusion that the cutoff frequency can be obtained directly by calculating the lowest eigenvalue of the Helmholtz equation with trivial boundary conditions. Moreover, the damping coefficient can be derived from an incompressible formulation since the condition which is imposed by Griffin and his collaborators that the excitation frequency be much smaller than the cutoff frequency is equivalent to requiring that the squeeze number be small. This stipulation implies nearly incompressible conditions.

In this paper, the physical implication of Griffin's approximation is demonstrated. It is then shown that the cutoff frequencies are derived from the lowest eigenvalue of the Helmholtz equation. Cutoff frequencies are given for finite rectangular plates (of which infinite strips are a special case), circular plates, and annuli which are not necessarily thin and for plate sectors. The tilting mode of annuli is also analyzed without requiring that they be thin, and without imposing Griffin's assumption that the flow be radial only.

Some damping coefficients for the incompressible formulation are compared to those obtained by Griffin.

To complete the analysis, a formulation is given in which no series truncations are needed. Results of this analysis show that, in some configurations, there exists a considerable error in the cutoff frequencies as calculated by Griffin.

Analytical Formulation

Consider a compressible fluid which is squeezed between two plates moving one with respect to the other. The surfaces of the plates are substantially parallel (small deviations from parallelity are permissible). The contours of the plates are arbitrary. It is assumed that most of the motion is perpendicular to the surfaces of the plates, and that the fluid undergoes an isothermal process during the entire time of the motion. The equation governing the fluid pressure is the well known compressible gas-film Reynolds equation [3].

Assume that the motion of the plate is restricted to be small and therefore the resulting pressure variation from ambient is also small. Introduce pressure and film thickness perturbation parameters. The compressible Reynolds gas-film equation can then be linearized to the form of equation (5) of [1]. This equation, when written in nondimensional form yields:

$$\nabla^2\psi - \sigma\frac{\partial\psi}{\partial\tau} = \sigma\frac{\partial\eta}{\partial\tau} \tag{1}$$

Griffin et al. determined first the film response to a sudden step change in the film, i.e., they solved the homogeneous part of equation (1), which is the Helmholtz equation, resulting in an infinite series solution in the system eigenvalues. Then, in order to compute the response due to any displacement function, they used a procedure which is described in [2], utilizing superpositions and the convolution integral theorem. They finally truncated their series to retain only the first harmonic. Instead, it is proposed here to compute the film response, right from the outset, to a simple harmonic motion excitation.

Assume, therefore, that the variation of the plate spacing is given by:

$$e = e_0 \cos\tau \tag{2}$$

where e_0 may vary over the plate surface, but is not a function of time. Equation (2) is, in nondimensional form:

$$\eta = \epsilon\cos\tau;\quad \epsilon = \frac{e_0}{h_m} \tag{3}$$

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Equation (1) then becomes:

$$\nabla^2\psi-\sigma\frac{\partial\psi}{\partial\tau}=-\sigma\epsilon\sin\tau \tag{3}$$

Assume the solution has the form:

$$\psi=\psi_1\cos\tau+\psi_0\sin\tau \tag{5}$$

i.e., the pressure distribution has a component $\psi_1\cos\tau$ which is in phase with the film thickness disturbance $e_0\cos\tau$, and a second component which is "out of phase" with the film thickness disturbance, but in phase with the squeeze velocity $e_0\sin\tau$. Another interpretation of equation (5) is that the fluid film acts as a spring (ψ_1) and as a viscous damper (ψ_0). Insert equation (5) into equation (4) and equate coefficients of $\cos\tau$ and $\sin\tau$ separately:

$$\nabla^2\psi_1-\sigma\psi_0=0; \tag{6}$$

$$\nabla^2\psi_0+\sigma\psi_1=-\sigma; \tag{7}$$

with trivial boundary conditions on both ψ_0 and ψ_1.

The cutoff frequency is that frequency where the spring and damping force amplitudes are equal in magnitude, or, in terms of our notation;

$$f_0=f_1 \tag{8}$$

where f_0 and f_1 are the nondimensional damping and spring pressure force amplitudes:[1]

$$f_0=\frac{1}{A}\int_A\psi_0dA;\quad f_1=-\frac{1}{A}\int_A\psi_1dA \tag{8a}$$

Denote:

$$\psi_1+\psi_0=\delta \tag{9}$$

Eliminate ψ_1 from equations (6), (7) by means of equation (9):

$$\nabla^2\delta+\sigma\delta-2\sigma\psi_0=-\sigma\epsilon; \tag{10}$$

$$\nabla^2\psi_0+\sigma\delta-\sigma\psi_0=-\sigma\epsilon \tag{11}$$

with trivial boundary conditions on δ.

[1] The forces themselves are obtained by multiplying f_0 and f_1 by p_aA.

Now, suppose that at cutoff frequency excitation the spring and damping pressure distributions are nearly equal. This would be one way (but, of course, not the only way) to satisfy equation (8). Then it can be assumed that:

$$\delta<<\psi_0 \tag{12}$$

Neglecting δ with respect to ψ_0 in equation (11) leads to:

$$\nabla^2\psi_0-\sigma\psi_0=-\sigma\epsilon \tag{13}$$

The solution for δ is decomposed into the particular and homogeneous solutions:

$$\delta=\delta_p+\delta_h \tag{14}$$

It is easily verified by equations (10), (13) that a particular solution is:

$$\delta_p=\psi_0 \tag{15}$$

The homogeneous part of the solution is governed by equation (10) which becomes, after using equations (13), (14), (15):

$$\nabla^2\delta_h+\sigma\delta_h=0 \tag{16}$$

which is the Helmholtz equation. The boundary conditions on δ_h are trivial, in view of equations (14), (15).

Now, δ_h must be nontrivial (in fact, it is of the order of ψ_0, in order to satisfy equation (12)). Therefore, σ must be an eigenvalue of equation (16). The decision as to which eigenvalue it is rests upon the spatial nature of the excitation amplitude, ϵ, as compared to the topographical nature of the various eigenfunctions. Parallel motion excitation, for example, implies physically, that the solution be of the nature of the first eigenfunction. Thus σ is the lowest eigenvalue of equation (16).

Rectangular Plates in Parallel Motion. In this case the domain is rectangular, of dimensions $a\times b$. The squeeze number is defined as

$$\sigma=\frac{12\mu a^2}{p_a h_m{}^2}\omega \tag{17}$$

Nomenclature

A = plate area
a,b = rectangular plate dimensions
D_{r0} = rectangular plate incompressible damping coefficient
D_{c0} = circular plate incompressible damping coefficient
D_{rs} = rectangular plate compressible damping coefficient
D_{cs} = circular plate compressible damping coefficient
e = film thickness perturbation
e_0 = film thickness perturbation amplitude
f_1,f_0 = spring and damping nondimensional pressure forces, respectively
f_{r0},f_{r1} = values of f_0 and f_1 for rectangular plates
f_{c0},f_{c1} = values of f_0 and f_1 for circular plates
h_m = mean film thickness
δp = film pressure perturbation
p_a = ambient pressure
r = radial coordinate
R = circular plate outer radius
R_{in},R_{out} = annular plate inner and outer radii, respectively
t = time
$\beta=\frac{b}{a}$ = plate aspect ratio
$\epsilon=\frac{e_0}{h_m}$ = nondimensional film thickness perturbation amplitude
$\zeta=\frac{r}{R}$ = nondimensional radial coordinate
$\eta=\frac{e}{h_m}$ = nondimensional plate displacement
$x_r=\frac{D_{rs}}{D_{r0}}$ = rectangular plate damping coefficients ratio
$x_c=\frac{D_{cs}}{D_{c0}}$ = circular plate damping coefficients ratio
μ = gas absolute viscosity
$\xi=\frac{R_{in}}{R_{out}}$ = annular plate inner to outer radii ratio
σ = squeeze number
σ_c = cutoff squeeze number
$\tau=\omega t$ = nondimensional time
$\psi=\frac{\delta p}{p_a}$ = nondimensional pressure perturbation
ω = excitation frequency
ω_c = cutoff frequency

The eigenvalues of equation (16) are:

$$\pi^2\left(m^2+\frac{n^2}{\beta^2}\right); \quad m,n=1,2,3,\ldots$$

The lowest eigenvalue is obtained by setting m, n = 1. Thus, the squeeze number at cutoff frequency excitation is:

$$\sigma_c=\pi^2\left(1+\frac{1}{\beta^2}\right) \tag{17a}$$

Solving for ω_c, the cutoff frequency is obtained:

$$\omega_c=\frac{\pi^2 p_a h_m{}^2}{12\mu}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \tag{18}$$

If $b>>a$ (infinitely wide plate), Griffin's result (equation (21) of [1]) is obtained.

Langlois [4], gives the exact solution for the in- and out-of-phase forces in infinitely wide plates. Equating those two forces, results in the transcendental equation:

$$k=\frac{2\sinh k}{\cosh k+\cos k}; \quad k=\sqrt{\sigma/2} \tag{19}$$

the solution of which is k = 2.2510, from which σ_c = 10.1342, as compared with the value of π^2 = 9.8696 which is predicted by Griffin's analysis.

Circular Plates in Parallel Motion. Here the domain is circular, of radius R. The squeeze number is defined by:

$$\sigma=\frac{12\mu R^2}{p_a h_m{}^2}\omega \tag{20}$$

The characteristic equation of equation (16) is $J_0(\sqrt{\sigma})$ = 0, which involves the Bessel function of zero order. The lowest root is σ_c = 5.784 which, when entered into equation (26) gives a cutoff frequency which is identical to equation (30) of [1].

Annular Plates in Parallel Motion. Griffin et al. used the results for infinitely wide rectangular plates to analyze thin annular plates. In the present analysis, no restriction on the inner to outer plate radius ratio is imposed.

The characteristic equation for an annular domain of inner radius R_{in} and outer radius R_{out} is:

$$J_0(\sqrt{\sigma})Y_0(\xi\sqrt{\sigma})-Y_0(\sqrt{\sigma})J_0(\xi\sqrt{\sigma})=0; \; \xi=\frac{R_{in}}{R_{out}} \tag{21}$$

This equation involves both kinds of Bessel functions of zero order. The squeeze number is defined as in equation (26), where R is replaced by R_{out}. The lowest roots of this equation are tabulated below versus the radii ratio:

ξ	0.02	0.1	0.2	0.4	0.6	0.8
σ_c	8.32	10.96	14.55	26.86	61.28	246.45

Assuming that ξ differs only slightly from 1 results, after asymptotic expansions of equation (21), in:

$$\sigma_c=\frac{\pi^2}{(1-\xi)^2} \tag{22}$$

which is identical to equation (26) of (1).

In Fig. 1, σ_c, as predicted both by equations (21) and (22) is plotted versus ξ. Judgment can be passed on the extent of the validity of equaton (22).

Annular Plates in a Tilting Mode. Griffin and his collaborators continuing the assumption that the annuli are thin, calculated the cutoff frequency under the further assumption that there is no circumferential flow between the plates. Here, again, no restriction is imposed on the radii ratio.

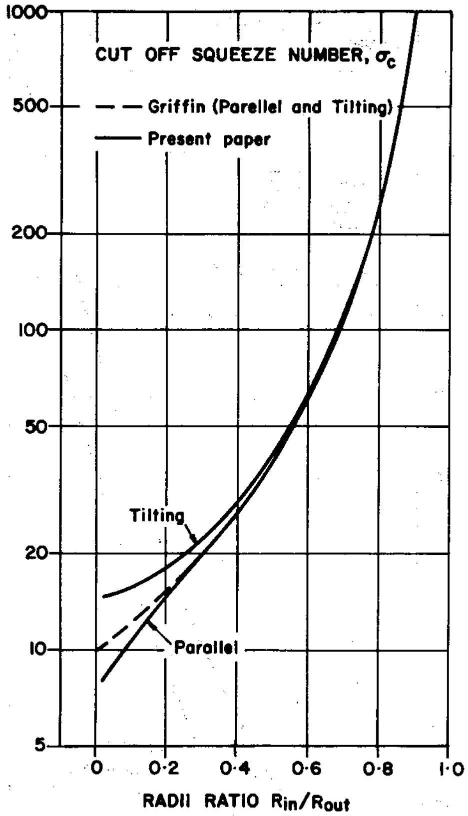

Fig. 1 Cutoff squeeze number for annuli in parallel and tilting motion

The pressure distribution in the tilting mode would correspond to the eigenfunctions combination which are comprised of the Bessel functions of the first kind. The characteristic equation is therefore:

$$J_1(\sqrt{\sigma})Y_1(\xi\sqrt{\sigma})-Y_1(\sqrt{\sigma})J_1(\xi\sqrt{\sigma})=0 \tag{23}$$

giving the following roots:

ξ	0.02	0.1	0.2	0.4	0.6	0.8
σ_c	14.71	15.52	17.94	29.05	62.88	247.67

For thin annuli, equation (23) reduces to an equation which is identical with equation (22), and which is also identical to Griffin's results (equation (35)) of [1].

In Fig. 1, the tilting mode values of σ_c, as predicted by equation (23) are plotted versus ξ, and can be compared with Griffin's prediction.

Circular Plate Sectors in Parallel Motion. For plates having a sector angle α and radius R, the characteristic equation is:

$$J_{\pi/\alpha}(\sqrt{\sigma})=0 \tag{24}$$

The squeeze number, σ, is defined by equation (20). Lowest roots of this equation for three sector angles are given below:

π/α	1	2	3
σ_c	14.68	26.38	40.70

Inserting those values of σ_c into equation (20) gives the cutoff frequencies for the various angles.

The Complete Solution

The kinematic mode for squeeze films which is dealt with here renders itself to a complete solution since the solution of the governing equations (6), (7) is straightforward.

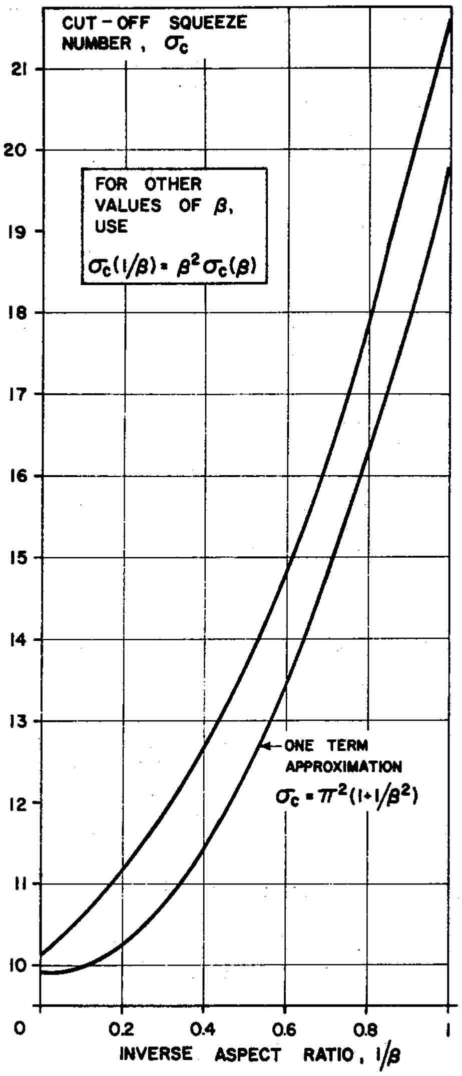

Fig. 2 Cutoff squeeze number for translatory motion of rectangular plates

Parallel Motion of Rectangular Plates. The solutions of equations (6), (7) are given in the Appendix.[2] The nondimensional damping and spring force amplitudes are obtained by equation (8*a*):

$$f_{r0} = \frac{64\sigma\epsilon}{\pi^6} \sum_{\substack{m,n \\ \text{odd}}} \frac{m^2 + (n/\beta)^2}{(mn)^2\{[m^2 + (n/\beta)^2]^2 + \sigma^2/\pi^4\}}; \tag{25}$$

$$f_{r1} = \frac{64\sigma^2\epsilon}{\pi^8} \sum_{\substack{m,n \\ \text{odd}}} \frac{1}{(mn)^2\{[m^2 + (n/\beta)^2]^2 + \sigma^2/\pi^4\}} \tag{26}$$

Equating the force amplitudes gives the following algebraic equation for the cutoff frequency

$$\frac{\sigma}{\pi^2} = \frac{\displaystyle\sum_{\substack{m,n \\ \text{odd}}} \frac{m^2 + (n/\beta)^2}{(mn)^2\{[m^2 + (n/\beta)^2]^2 + \sigma^2/\pi^4\}}}{\displaystyle\sum_{\substack{m,n \\ \text{odd}}} \frac{1}{(mn)^2\{[m^2 + (n/\beta)^2]^2 + \sigma^2/\pi^4\}}} \tag{27}$$

[2] See also [7].

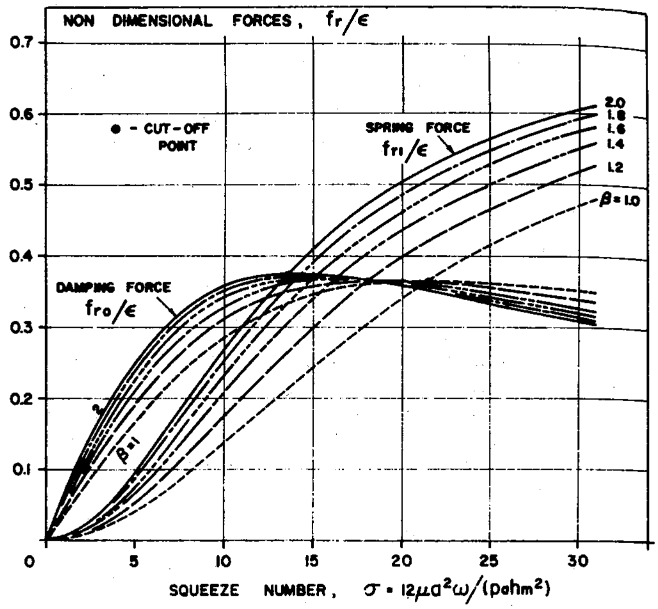

Fig. 3 Forces on rectangular plate

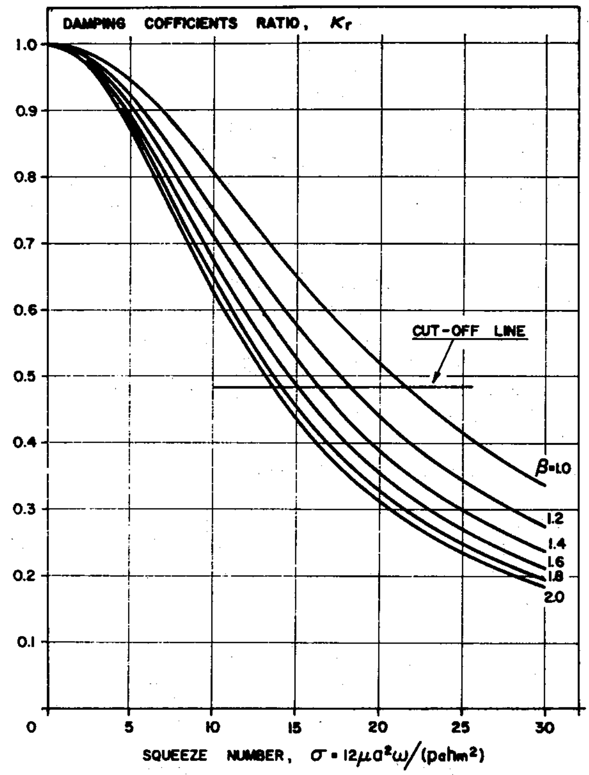

Fig. 4 Damping coefficients ratio for translatory motion of rectangular plates

Equation (27) is solved in an iterative form to yield the cutoff squeeze number, σ_c, from which the cutoff frequency is calculated. In Fig. 2, σ_c is plotted versus β, and compared with the approximate analysis results.

If only one term is maintained in the summations of equation (27), the cutoff squeeze number yields:

$$\sigma_c = \pi^2(1 + 1/\beta^2) \tag{28}$$

which is identical with the prediction of the approximate analysis.

The nondimensional spring and damping forces are plotted in Fig. 3 versus the squeeze number with the plate aspect ratio

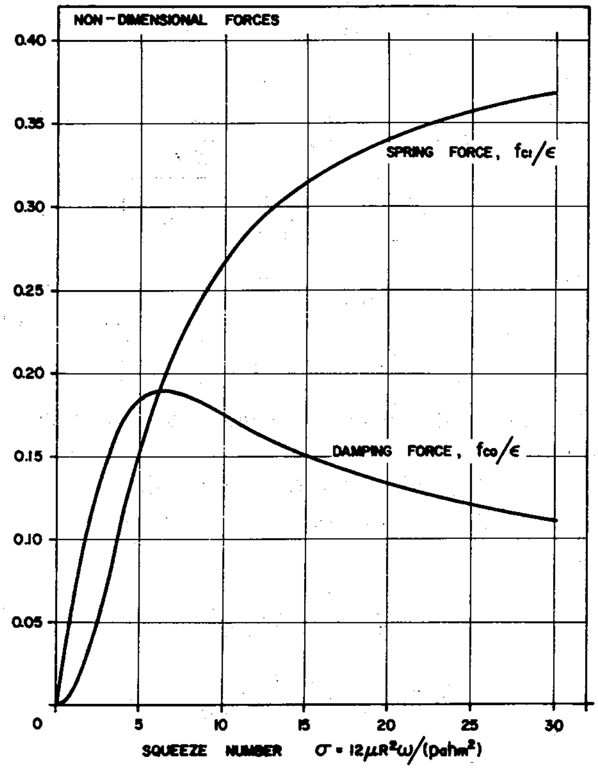

Fig. 5 Squeeze film forces in circular plate

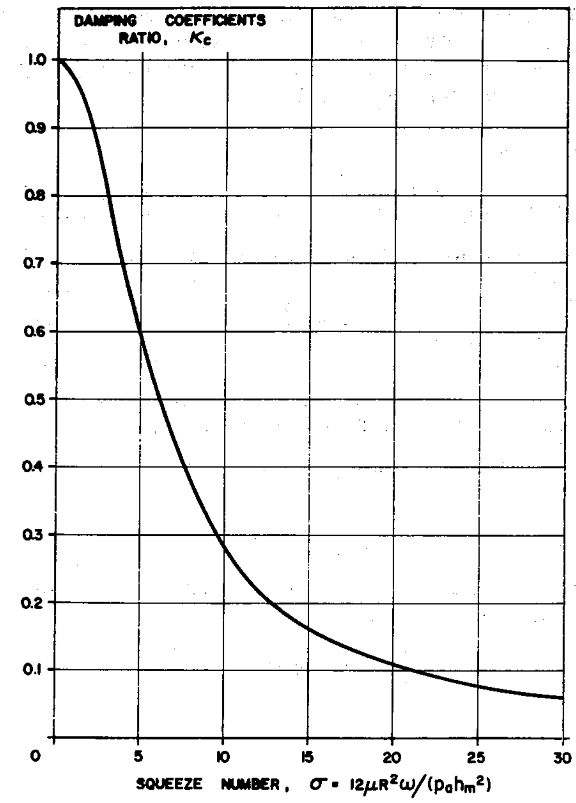

Fig. 6 Compressible to incompressible damping coefficients ratio

as a parameter. The ratio of the compressible to incompressible damping coefficients are plotted in Fig. 4.[3]

Circular Plates in Parallel Motion. The solutions of equations (6) and (7) for the circular domain were derived by Crandall [6].[4] They involve Kelvin functions and are given in the Appendix. The nondimensional spring and damping force amplitudes, are in this case:

$$f_{c0} = -\sqrt{\frac{2}{\sigma}}\,[A_c(\mathrm{ber}_1\sqrt{\sigma} - \mathrm{bei}_1\sqrt{\sigma}) + B_c(\mathrm{ber}_1\sqrt{\sigma} + \mathrm{bei}_1\sqrt{\sigma})]\epsilon \qquad (29)$$

$$f_{c1} = \left\{1 + \sqrt{\frac{2}{\sigma}}\,[A_c(\mathrm{ber}_1\sqrt{\sigma} + \mathrm{bei}_1\sqrt{\sigma}) + B_c(\mathrm{ber}_1\sqrt{\sigma} - \mathrm{bei}_1\sqrt{\sigma})]\epsilon\right\} \qquad (30)$$

where the constants A_c, B_c are given in the Appendix. Equating the force amplitudes yields the following transcendental equation in the squeeze number:

$$\sigma = 8\left(\frac{\mathrm{ber}\sqrt{\sigma}\,\mathrm{bei}_1\sqrt{\sigma} - \mathrm{bei}\sqrt{\sigma}\,\mathrm{ber}_1\sqrt{\sigma}}{\mathrm{ber}^2\sqrt{\sigma} + \mathrm{bei}^2\sqrt{\sigma}}\right)^2 \qquad (31)$$

which has the solution $\sigma_c = 6.2333$. Introducing this solution into equation (20) yields, for the cutoff frequency:

$$\omega_c = \frac{p_a h_m{}^2}{1.93\mu R^2} \qquad (32)$$

which is by 7 percent higher than Griffin's estimate.[5]

In Fig. 5, the spring and damping forces are plotted versus the squeeze number. Figure 6 illustrates the compressible to incompressible damping coefficient ratio variation with the squeeze number.[6]

The analysis of annuli in parallel motion is performed in a similar fashion. It will involve two additional kinds of Kelvin functions. This analysis is disposed with here for the sake of brevity.

Discussion

The one term approximation to the cutoff frequency is found to be, in parallel motion, an underestimate of this frequency. This can be seen in Fig. 2 for rectangular plates, where the error in the estimate ranges from 2.6 percent (infinite strips) to 9 percent (square plates). For circular plates the one term approximation underestimates this frequency by 7 percent.

The damping coefficient decreases with increasing squeeze number (or compressibility) as is demonstrated by Figs. 4, 6. At the cutoff frequency, both circular and rectangular plates, exhibit a reduction of over 50 percent in their value in a purely incompressible state.

The damping force reaches its maximum value approximately (but not exactly) at the cutoff frequency. Excitation above or below this frequency results in a decreasing damping force, independently of the plate shape. This result is demonstrated in Figs. 3, 5.

An inspection of Fig. 1 reveals that Griffin's assumption that circumferential gas flow can be neglected in annular plate motion gives very good results for a wide range of inner to outer radius ratio. The annuli need not necessarily be thin.

[3] For the incompressible damping coefficient see, for example, [5].

[4] Interestingly enough, Langlois attributes the solution to G. I. Taylor. Neither Langlois nor Taylor gave note to Crandall's solution which appeared in 1917.

[5] Griffin gives the number 2.07 in the denominator.

[6] For the incompressible damping coefficient see, for example, [5].

This range is smaller for the tilting mode than for the parallel motion mode, and can be expected to be so, since the circumferential variation of squeeze velocity is expected to increase the asymmetry in the flow.

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APPENDIX

The pressure distribution components for rectangular plates in parallel motion are:

$$\psi_0 = \epsilon \sum_{\substack{m,n \\ \text{odd}}} a_{mn} \cos m\pi X \cos \frac{n\pi Y}{\beta}; \tag{A1}$$

$$\psi_1 = \epsilon \sum_{m,n} b_{mn} \cos m\pi X \cos \frac{n\pi Y}{\beta}; \tag{A2}$$

where

$$a_{mn} = \frac{16\sigma(-1)^{\frac{m+n}{2}} \left[(m\pi)^2 + \left(\frac{n\pi}{\beta}\right)^2 \right]}{\pi^2 mn \left\{ \left[(m\pi)^2 + \left(\frac{n\pi}{\beta}\right)^2 \right]^2 + \sigma^2 \right\}}; \tag{A3}$$

$$b_{mn} = \frac{16\sigma^2(-1)^{\frac{m+n}{2}}}{\pi^2 mn \left\{ \left[(m\pi)^2 + \left(\frac{n\pi}{\beta}\right)^2 \right]^2 + \sigma^2 \right\}}; \tag{A4}$$

The pressure distribution components for circular plates in parallel motion are:

$$\psi_0 = [A_c \mathrm{ber}(\sqrt{\sigma}\zeta) + B_c \mathrm{bei}(\sqrt{\sigma}\zeta)]\epsilon; \tag{A5}$$

$$\psi_1 = [-1 + A_c \mathrm{bei}(\sqrt{\sigma}\zeta) - B_c \mathrm{ber}(\sqrt{\sigma}\zeta)]\epsilon, \tag{A6}$$

where

$$A_c = \frac{\mathrm{bei}\sqrt{\sigma}}{(\mathrm{ber}^2\sqrt{\sigma} + \mathrm{bei}^2\sqrt{\sigma})}; \tag{A7}$$

$$B_c = -\frac{\mathrm{ber}\sqrt{\sigma}}{(\mathrm{ber}^2\sqrt{\sigma} + \mathrm{bei}^2\sqrt{\sigma})}. \tag{A8}$$