Ray tracing analysis of uniaxial birefringent optical components

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ABSTRACT

A polarization ray tracing algorithm which calculates intensities of rays propagating in uniaxial birefringent media is presented. Calculations in this algorithm are performed on vectors in the global coordinate system, obviating the need for frequent conversions between global and local coordinate systems. For the first time, to the best of author's knowledge, a full ray tracing analysis of a Wollaston prism is presented as a calculation example.

Keywords: polarization ray tracing, amplitude calculation, birefringent medium

1. INTRODUCTION

Ray tracing analysis of optical setups containing uniaxial brefringent components is an established optical design technique. In most cases only paths of rays propagating in the analyzed optical setup are calculated, while directions and amplitudes of electric and magnetic field vectors remain unknown. Therefore, Poynting vectors cannot be calculated and intensity of propagating rays cannot be determined, thus depriving the designer of information important in certain applications, such as interferometry and polarimetry.

A ray tracing algorithm providing amplitudes of field vectors should be compact and should perform calculations in the global coordinate system, contrary to well-established approaches[1]. Its complexity can be further reduced, without substantial loss of functionality, by limiting its analysis capabilities to isotropic and uniaxial materials, as the great majority of optical components are made from these two materials classes.

The primary objective of this paper is to devise a compact ray tracing algorithm for isotropic and uniaxial media which allows for calculation of Poynting vectors of propagating rays. Section 2 gives an outline of theory needed to introduce, in Section 3, expressions for wave and ray vectors' directions. In Section 4 expressions for field vectors are derived and the algorithm is presented. Finally, in Section 5 the algorithm is applied to analysis of a Wollaston prism.

2. THEORY

Light propagation in a homogeneous nonconducting medium is described by the Maxwell equations and two material equations:

$$\begin{aligned} &\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\mathbf{D} = 0 \qquad \mathbf{B} = \mu\mu_0\mathbf{H} \\ &\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} \qquad \nabla\mathbf{B} = 0 \qquad \mathbf{D} = \varepsilon\varepsilon_0\mathbf{E} \end{aligned} \tag{1}$$

where $\mathbf{E}$ – electric field vector, $\mathbf{H}$ – magnetic field vector, $\mathbf{D}$ – electric displacement vector, $\mathbf{B}$ – magnetic induction vector, ε_0, ε – absolute and relative electric permittivites, respectively, μ_0, μ – absolute and relative magnetic permittivities, respectively.

In an isotropic medium both permittivities are scalars and relative electric permittivity ε is equal to the square of the refractive index n of the medium:

$$\varepsilon = n^2 \tag{2}$$

In a nonabsorbing and non-optically active uniaxial medium ε is a symmetric tensor whose elements are real numbers. For such a tensor a coordinate system exists in which only the diagonal elements of the tensor are non-zero, i.e.

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$$\boldsymbol{\varepsilon} = \begin{bmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix} \tag{3}$$

where n_o – ordinary refractive index of the medium and n_e – extraordinary refractive index. In this coordinate system, referred to as the principal coordinate system, the z axis is parallel to the optical axis of the medium. In order to facilitate subsequent discussion, a unit vector **A** parallel to the z axis of the principal coordinate system is introduced.

Additionally, at surfaces where optical properties (i.e. refractive indices or direction **A** of the optical axis) change, field vectors must satisfy certain boundary conditions. Since in optics we can safely assume that neither surface charge (ρ=0) n_{or} surface current (j=0) are present on the boundary, the conditions can be written as ([2], pp. 1÷7):

1°, the normal component of the magnetic induction vector is continuous across the boundary, i.e.:

$$\boldsymbol{\eta} \cdot (\mathbf{B}^{(2)} - \mathbf{B}^{(1)}) = 0 \tag{4}$$

where $\boldsymbol{\eta}$ – unit vector normal to the boundary between the two media, $\mathbf{B}^{(1)}$ – magnetic induction vector on the boundary in the first medium, $\mathbf{B}^{(2)}$ – magnetic induction vector on the boundary in the second medium.

2°, the normal component of the electric displacement vector is continuous across the boundary, i.e.:

$$\boldsymbol{\eta} \cdot (\mathbf{D}^{(2)} - \mathbf{D}^{(1)}) = 0 \tag{5}$$

where $\mathbf{D}^{(1)}$, $\mathbf{D}^{(2)}$ – electric displacement vector on the boundary, in the first and in the second medium, respectively.

3°, the tangential component of the electric field vector is continuous across the boundary, i.e.:

$$\boldsymbol{\eta} \times (\mathbf{E}^{(2)} - \mathbf{E}^{(1)}) = 0 \tag{6}$$

where $\mathbf{E}^{(1)}$, $\mathbf{E}^{(2)}$ – electric field vector on the boundary, in the first and in the second medium, respectively,

4°, the tangential component of the magnetic field vector is continuous across the boundary, i.e.:

$$\boldsymbol{\eta} \times (\mathbf{H}^{(2)} - \mathbf{H}^{(1)}) = 0 \tag{7}$$

where $\mathbf{H}^{(1)}$, $\mathbf{H}^{(2)}$ – magnetic field vector on the boundary, in the first and in the second medium, respectively.

In ray tracing analysis a plane harmonic wave is considered, whose electric field vector **E** is ([2], p. 667):

$$\mathbf{E}(t, \mathbf{r}) = \mathbf{E}_0 \exp\left[j\omega \left(\frac{\mathbf{r} \cdot \mathbf{s}}{v_p} - t \right) \right] \tag{8}$$

where $\mathbf{E}_0$ – direction of electric field vector, ω – angular frequency, **s** – unit wave vector (i.e. unit wave-normal), **r** – position vector, v_p – phase velocity, t – time. Similar expressions can be written for vectors **B**, **H** and **D**. The unit wave vector **s** is normal to the surfaces of constant phase (called also wave surfaces) of the wave.

Direction of energy transport, which is the ray direction in ray tracing analysis, is given by Poynting vector **S**:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{9}$$

instead of which a unit ray vector **t**, defined as

$$\mathbf{t} = \frac{\mathbf{S}}{|\mathbf{S}|} \tag{10}$$

is often used. In isotropic media the ray vector equals the wave vector (i.e. **t**=**s**), while in anisotropic media, directions of these vectors differ (**t**≠**s**).

Phase velocity v_p of the plane wave given by (8) propagating in an isotropic medium does not depend on direction of **s**. It can be expressed as:

$$v_p = \frac{c}{n} \tag{11}$$

where c – velocity of light in vacuum, n – refractive index of the medium. In an anisotropic medium two types of waves, an *ordinary wave* and an *extraordinary wave*, can exist for a given unit wave vector **s**. Their phase velocities, v_o and v_e respectively, are given by:

$$v_o = \frac{c}{n_o} \qquad v_e^2 = \left(\frac{c}{n}\right)^2 = \left(\frac{c}{n_o}\right)^2 \cos^2\psi + \left(\frac{c}{n_e}\right)^2 \sin^2\psi \tag{12}$$

where n_o – ordinary refractive index, n_e – extraordinary refractive index, ψ – angle between the unit vector **A** and the unit wave vector **s**. From Eq. (12) follows that while the velocity of the ordinary wave is independent of **s** (like that of a wave in an isotropic medium), the velocity of the extraordinary wave v_e is a function of the angle between **A** and **s**. Multiplying the expression for v_e by $\left(n_e n_o n/c\right)^2$ and using identity $\sin^2\psi = 1 - \cos^2\psi$ we arrive at:

$$n_e^2 n_o^2 = n^2 n_e^2 \cos^2\psi + n^2 n_o^2 - n^2 n_o^2 \cos^2\psi \,. \tag{13}$$

Dividing the resulting equation (Eq. (13)) by n_o^2 we obtain a relation

$$n^2 = n_e^2 - \frac{n_e^2 - n_o^2}{n_o^2} n^2 \cos^2\psi = n_e^2 - \frac{n_e^2 - n_o^2}{n_o^2} n^2 \left(\mathbf{s}\cdot\mathbf{A}\right)^2 \tag{14}$$

which will be later used in calculation of directions of wave and ray vectors.

Vector **D** of a wave propagating in an isotropic medium is perpendicular to the wave vector **s**, but its direction in that plane can be arbitrary. As a result, it is impossible to determine the direction of **D** of such a wave based on **s**. In contrast, directions of vectors **D** of the ordinary and extraordinary wave propagating in an anisotropic medium can be conveniently expressed relative to the principal plane, i.e. the plane containing wave vector s and the optical axis vector **A** (cf.[2], p. 680). The direction of the vector **D** of the ordinary wave ($\mathbf{D}_1$ in Eq. (15)) is perpendicular to the principal plane, while the direction of the vector **D** of the extraordinary wave ($\mathbf{D}_2$ in Eq. (15)) lies in that plane and is perpendicular to **s**, viz.:

$$\begin{aligned} \mathbf{D}_1 &= \left|\mathbf{D}_1\right| \frac{\mathbf{s}\times\mathbf{A}}{\left|\mathbf{s}\times\mathbf{A}\right|} & \mathbf{D}_1 &= \left|\mathbf{D}_1\right| \mathbf{i}_1 \\ \mathbf{D}_2 &= \left|\mathbf{D}_2\right| \frac{\mathbf{s}\times\mathbf{A}}{\left|\mathbf{s}\times\mathbf{A}\right|} \times \mathbf{s} & \mathbf{D}_2 &= \left|\mathbf{D}_2\right| \mathbf{i}_2 \end{aligned} \tag{15}$$

where $|\mathbf{D}_1|$ and $|\mathbf{D}_2|$ are amplitudes of corresponding vectors and $\mathbf{i}_1$ and $\mathbf{i}_2$ unit vectors in their directions. It should be noted here that amplitudes $|\mathbf{D}_1|$ and $|\mathbf{D}_2|$ can assume also negative values, since $\mathbf{D}_1$ and $\mathbf{D}_2$ can have directions opposite to those defined by Eq. (15).

Relations (15) reduce to 0/0 when **s** is parallel to **A**. This case corresponds to propagation along the optical axis, where the direction of **D** cannot be obtained from (15) and should be calculated using boundary conditions (4)÷(7).

3. CALCULATION OF WAVE AND RAY VECTORS

Formulae for wave and ray vectors directions of rays refracted on and reflected from the boundary surface of two media of different optical properties are derived, based on the elegant approach presented in[3]. Not requiring any local coordinate system, the algorithm presented therein can be used for tracing both ordinary and extraordinary rays at boundaries between two media, both of which can be isotropic or uniaxial anisotropic. We will first consider refraction and reflection of the extraordinary ray on a boundary between two uniaxial media, shown in Fig. 1, and subsequently explain how resulting formulae can be applied to other cases.

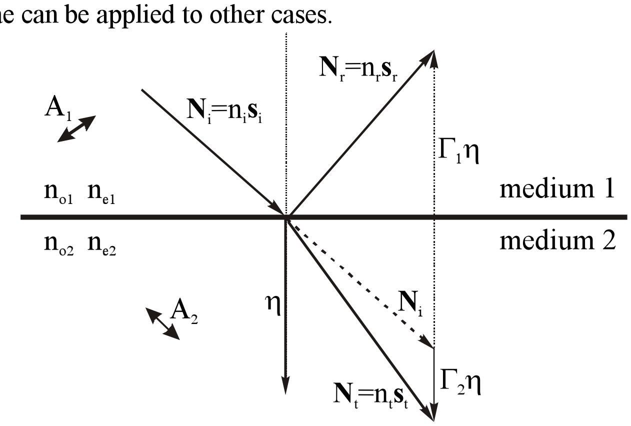

Fig. 1. Reflection and transmission of incident wave on the boundary between two uniaxial anisotropic media (explanations in the text).

Let us start from the condition stating that phases of the fields given by Eq. (8) on both sides of the boundary must be equal in any point r on the boundary and for any time t, viz.:

$$\frac{\mathbf{r}\cdot\mathbf{s}_i}{v_i}-t=\frac{\mathbf{r}\cdot\mathbf{s}_r}{v_r}-t=\frac{\mathbf{r}\cdot\mathbf{s}_t}{v_t}-t \tag{16}$$

where the indices i, r and t correspond to incident, reflected and transmitted wave, respectively. Eq. (16) can be expressed as

$$\mathbf{r}\cdot\left(\frac{\mathbf{s}_i}{v_i}-\frac{\mathbf{s}_t}{v_t}\right)=0 \quad \mathbf{r}\cdot\left(\frac{\mathbf{s}_i}{v_i}-\frac{\mathbf{s}_r}{v_r}\right)=0\,. \tag{17}$$

Using Eq. (17) it can be shown that $\mathbf{s}_r$ and $\mathbf{s}_t$ lie in the plane of incidence, i.e. the plane containing $\mathbf{s}_i$ and a unit vector $\boldsymbol{\eta}$ perpendicular to the boundary and that vectors $(n_i\mathbf{s}_i-n_t\mathbf{s}_t)$ and $(n_i\mathbf{s}_i-n_r\mathbf{s}_r)$ are perpendicular to the boundary, i.e.:

$$n_r\,\mathbf{s}_r-n_i\,\mathbf{s}_i=\Gamma_1\boldsymbol{\eta} \qquad n_t\,\mathbf{s}_t-n_i\,\mathbf{s}_i=\Gamma_2\boldsymbol{\eta} \tag{18}$$

where n_i, n_r, n_t – refractive indices of incident, reflected and transmitted wave respectively,
Γ_1 and Γ_2 are scaling constants which, depending on direction of $\boldsymbol{\eta}$ and values of refractive indices, can assume positive or negative values. Introducing normalized wave vector $\mathbf{N}$, given by:

$$\mathbf{N}=n\mathbf{s}=|\mathbf{N}|\mathbf{s} \tag{19}$$

and rewriting Eq. (18) in terms of it yields

$$\mathbf{N}_r=\mathbf{N}_i+\Gamma_1\boldsymbol{\eta} \qquad \mathbf{N}_t=\mathbf{N}_i+\Gamma_2\boldsymbol{\eta}\,. \tag{20}$$

In order to calculate normalized wave vectors $\mathbf{N}_t$ and $\mathbf{N}_r$, expressions for Γ_1 and Γ_2 in terms of a priori known quantities must be found. These quantities are: n_{o1} and n_{e1} – ordinary and extraordinary refractive index of the first medium (cf. Fig. 1), n_{o2} and n_{e2} – ordinary and extraordinary refractive index of the second medium, $\mathbf{A}_1$ and $\mathbf{A}_2$ – directions of the optical axis in the first and second medium, and normalized wave vector $\mathbf{N}_i$ of the incident ray.
Let us start by taking the dot product of each of Eq. (20) with itself, viz.:

$$\begin{aligned}\mathbf{N}_r\cdot\mathbf{N}_r&=\mathbf{N}_i\cdot\mathbf{N}_i+2\Gamma_1\boldsymbol{\eta}\cdot\mathbf{N}_i+\Gamma_1^2(\boldsymbol{\eta}\cdot\boldsymbol{\eta})\\ \mathbf{N}_t\cdot\mathbf{N}_t&=\mathbf{N}_i\cdot\mathbf{N}_i+2\Gamma_2\boldsymbol{\eta}\cdot\mathbf{N}_i+\Gamma_2^2(\boldsymbol{\eta}\cdot\boldsymbol{\eta})\end{aligned}\,. \tag{21}$$

Using definition (19) of normalized wave vectors $\mathbf{N}_r$ and $\mathbf{N}_t$ Equation (14) can be rewritten as:

$$\begin{aligned}|\mathbf{N}_r|^2&=n_{e1}^2-\frac{n_{e1}^2-n_{o1}^2}{n_{o1}^2}(\mathbf{N}_r\cdot\mathbf{A}_1)^2\\ |\mathbf{N}_t|^2&=n_{e2}^2-\frac{n_{e2}^2-n_{o2}^2}{n_{o2}^2}(\mathbf{N}_t\cdot\mathbf{A}_2)^2\end{aligned}\,. \tag{22}$$

By substituting (20) into the right hand side of Eq. (22) we obtain

$$\begin{aligned}|\mathbf{N}_r|^2&=n_{e1}^2-q_1(\mathbf{N}_i\cdot\mathbf{A}_1+\Gamma_1\,\boldsymbol{\eta}\cdot\mathbf{A}_1)^2\\ |\mathbf{N}_t|^2&=n_{e2}^2-q_2(\mathbf{N}_i\cdot\mathbf{A}_2+\Gamma_2\,\boldsymbol{\eta}\cdot\mathbf{A}_2)^2\end{aligned} \tag{23}$$

where

$$q_1=\frac{n_{e1}^2-n_{o1}^2}{n_{o1}^2} \qquad q_2=\frac{n_{e2}^2-n_{o2}^2}{n_{o2}^2} \tag{24}$$

Since $\mathbf{N}\cdot\mathbf{N}=|\mathbf{N}|^2$, we can substitute right hand sides of Eq. (23) into Eq. (21), which, after collecting terms of Γ, becomes

$$\begin{aligned}&\Gamma_1^2\left[1+q_1(\boldsymbol{\eta}\cdot\mathbf{A}_1)^2\right]+2\Gamma_1\left[\mathbf{N}_i\cdot\boldsymbol{\eta}+q_1(\mathbf{N}_i\cdot\mathbf{A}_1)(\boldsymbol{\eta}\cdot\mathbf{A}_1)\right]+\left[|\mathbf{N}_i|^2-n_{e1}^2+q_1(\mathbf{N}_i\cdot\mathbf{A}_1)^2\right]=0\\ &\Gamma_2^2\left[1+q_2(\boldsymbol{\eta}\cdot\mathbf{A}_2)^2\right]+2\Gamma_2\left[\mathbf{N}_i\cdot\boldsymbol{\eta}+q_2(\mathbf{N}_i\cdot\mathbf{A}_2)(\boldsymbol{\eta}\cdot\mathbf{A}_2)\right]+\left[|\mathbf{N}_i|^2-n_{e2}^2+q_2(\mathbf{N}_i\cdot\mathbf{A}_2)^2\right]=0\end{aligned}\,. \tag{25}$$

All vectors and scalars in Eq. (25), apart from scaling constants Γ_1 and Γ_2, are known, therefore Eq. (25) is a set of two quadratic equations, from which Γ_1 and Γ_2 can be calculated.
If direction of $\boldsymbol{\eta}$ is from medium 1 to medium 2 (as shown in Fig. 1), i.e.

$$\mathbf{N}\cdot\boldsymbol{\eta}\geq 0 \tag{26}$$

then scaling constant Γ_1 for reflected ray is obtained with the negative square root in quadratic formula, while scaling constant Γ_2 for transmitted ray is obtained with the positive square root, viz.:

$$\Gamma_1 = \frac{-b_1 - \sqrt{b_1^2 - 4a_1c_1}}{2a_1} \qquad \Gamma_2 = \frac{-b_2 + \sqrt{b_2^2 - 4a_2c_2}}{2a_2} \tag{27}$$

where a_1, b_1, c_1, a_2, b_2 and c_2 are respective coefficients in Eq. (25). Finally, knowing Γ_1 and Γ_2, vectors $\mathbf{N}_t$ and $\mathbf{N}_r$ are calculated from Eq. (20).

Formulae (20)÷(27) can be used also to calculate directions of ordinary rays or directions of rays propagating in isotropic media. When an ordinary reflected ray is to be calculated, substitution $n_{e1} \leftarrow n_{o1}$ is performed, q_1 becomes zero and |Nr| becomes n_{o1} (cf. Eq. (23)). Nr is then computed using Eqs. (25), (27) and (20). Similarly, for an ordinary transmitted ray, substitution $n_{e2} \leftarrow n_{o2}$ is done, after which q_2 becomes zero, $|\mathbf{N}_t|$ becomes n_{o2} and Eqs. (25), (27) and (20) are employed to obtain $\mathbf{N}_r$. In order to find directions of rays propagating in isotropic media, substitutions $n_{e1} \leftarrow n_{o1}$ or $n_{e2} \leftarrow n_{o2}$ are used for reflected and transmitted ray respectively, after which q_1 or q_2 become zero. Inspecting Eq. (25) we note that when q_i=0 (i=1,2), all terms containing $\mathbf{A}_i$ vanish, since they are multiplied q_i. Therefore, any $\mathbf{A}_i$ can be used in calculations, which was to be expected since isotropic media do not have optical axis. Because numerical calculations are conducted with finite precision, it is advisable to use a zero vector $\mathbf{A}_i$, in order to minimize errors arising when computer representation of q_i differs from its correct value.

Directions $\boldsymbol{\sigma}_r$, $\boldsymbol{\sigma}_t$ of reflected and transmitted rays are calculated from

$$\begin{aligned}\boldsymbol{\sigma}_r &= n_{e1}^2\left(\frac{\mathbf{N}_r \cdot \mathbf{A}_1}{|\mathbf{N}_r|}\right)\mathbf{A}_1 + n_{o1}^2\left[\frac{\mathbf{N}_r - (\mathbf{N}_r \cdot \mathbf{A}_1)\mathbf{A}_1}{|\mathbf{N}_r|}\right]\\ \boldsymbol{\sigma}_t &= n_{e2}^2\left(\frac{\mathbf{N}_t \cdot \mathbf{A}_2}{|\mathbf{N}_t|}\right)\mathbf{A}_2 + n_{o2}^2\left[\frac{\mathbf{N}_t - (\mathbf{N}_t \cdot \mathbf{A}_2)\mathbf{A}_2}{|\mathbf{N}_t|}\right]\end{aligned} \tag{28}$$

which expressed in terms of wave vectors $\mathbf{s}$ becomes

$$\begin{aligned}\boldsymbol{\sigma}_r &= n_{e1}^2(\mathbf{s}_r \cdot \mathbf{A}_1)\mathbf{A}_1 + n_{o1}^2[\mathbf{s}_r - (\mathbf{s}_r \cdot \mathbf{A}_1)\mathbf{A}_1]\\ \boldsymbol{\sigma}_t &= n_{e2}^2(\mathbf{s}_t \cdot \mathbf{A}_2)\mathbf{A}_2 + n_{o2}^2[\mathbf{s}_t - (\mathbf{s}_t \cdot \mathbf{A}_2)\mathbf{A}_2].\end{aligned} \tag{29}$$

Directions of reflected or transmitted ordinary rays, or rays propagating in isotropic media, can be calculated by applying substitutions $n_{e1} \leftarrow n_{o1}$ or $n_{e2} \leftarrow n_{o2}$. Using definition (19) in Eq. (29) we can show that in these cases $\boldsymbol{\sigma}_r$, $\boldsymbol{\sigma}_t$ are equal to $n_{o1}\cdot\mathbf{N}_r$ and $n_{o2}\cdot\mathbf{N}_t$ respectively. Therefore, it is possible to calculate $\boldsymbol{\sigma}_r$, $\boldsymbol{\sigma}_t$ using $\mathbf{N}_r$ and $\mathbf{N}_t$ along with Eq. (19), viz.:

$$\boldsymbol{\sigma}_r = |\mathbf{N}_r|\mathbf{N}_r \qquad \boldsymbol{\sigma}_t = |\mathbf{N}_t|\mathbf{N}_t. \tag{30}$$

To conclude, directions of wave vectors are calculated from Eqs. (20)÷(27), applying substitutions $n_{e1} \leftarrow n_{o1}$ for reflected ordinary rays or reflected rays in isotropic media, or $n_{e2} \leftarrow n_{o2}$ for transmitted ordinary rays or transmitted rays in isotropic media. Following, ray vectors are computed from formulae (28) or (29) for extraordinary rays, and from Eq. (30) for ordinary rays or rays propagating in isotropic media.

4. CALCULATION OF FIELD VECTORS

The method presented here uses tree steps to find vectors **E** and **D** of reflected and transmitted rays. First, directions of wave (**s**) and ray (**σ**) vectors are calculated using the method discussed in the previous Section. Second, directions of **D** and **E** vectors in anisotropic media are computed. Third, amplitudes of **D** and **E** vectors in anisotropic media and directions and amplitudes of **D** and **E** vectors in isotropic media are calculated. Based on **D** and **E** it is possible to calculate the Poynting vector **S** as well as **H** and **B** vectors.

Following discussion of the two latter steps will cover all types of boundaries in which at least one medium is anisotropic, viz.: 1°, boundary between isotropic and anisotropic medium, 2°, boundary between two anisotropic media, and 3°, boundary between anisotropic and isotropic medium (The boundary between two isotropic media is not discussed here, as it has been extensively treated elsewhere, e.g.[2]). We also assume that directions of wave (**s**) and ray (**σ**) vectors of incident ray are known and those of reflected and transmitted rays have been calculated. Since the equations for each type of boundary differ, the discussion will be divided into three subsections.

4.1. Boundary between isotropic and anisotropic medium

Let us consider a ray propagating in an isotropic medium in the direction of its wave vector **s**. Incident on a plane boundary Σ with an anisotropic medium, as shown in Fig. 2, this ray gives rise to a reflected ray whose wave vector is $\mathbf{s}^{(I)}$, propagating in the isotropic medium, and two transmitted rays in the anisotropic medium. Wave vectors of the transmitted rays are $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ for ordinary and extraordinary ray respectively.

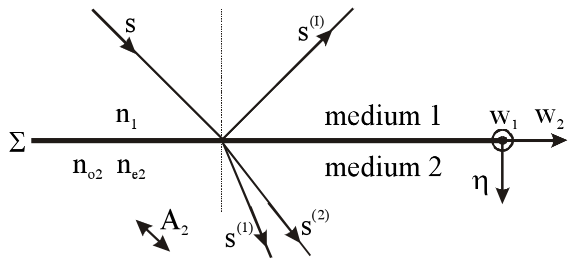

Fig. 2. Reflection and transmission of incident wave on the boundary Σ between an isotropic medium 1 and a uniaxial anisotropic medium 2 (explanations in the text).

Unit direction vectors $\mathbf{i}_{d1}$ and $\mathbf{i}_{d2}$ of electric displacement vectors $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ of the transmitted ordinary and extraordinary rays can be obtained from (15)

$$\mathbf{i}_{d1} = \frac{\mathbf{s}^{(1)} \times \mathbf{A}_2}{\left|\mathbf{s}^{(1)} \times \mathbf{A}_2\right|} \qquad \mathbf{i}_{d2} = \frac{\mathbf{s}^{(2)} \times \mathbf{A}_2}{\left|\mathbf{s}^{(2)} \times \mathbf{A}_2\right|} \times \mathbf{s}^{(2)}. \tag{31}$$

Direction $\mathbf{i}_{dI}$ of electric displacement vector $\mathbf{D}^{(I)}$ of the reflected ray in the isotropic medium cannot be determined in this way. It will be calculated using boundary conditions, together with magnitudes (amplitudes) of the $\mathbf{D}^{(1)}$, $\mathbf{D}^{(2)}$ and $\mathbf{D}^{(I)}$ vectors.

In order to facilitate writing of boundary conditions, let us introduce a coordinate system $\mathbf{w}_2\mathbf{w}_1\boldsymbol{\eta}$ defined as follows: $\boldsymbol{\eta}$ - unit vector perpendicular to the boundary Σ pointing from medium 1 to medium 2, as shown in Fig. 2, $\mathbf{w}_1$ – unit vector given by

$$\mathbf{w}_1 = \frac{\boldsymbol{\eta} \times \mathbf{s}}{\left|\boldsymbol{\eta} \times \mathbf{s}\right|} \tag{32}$$

and $\mathbf{w}_2$ – unit vector perpendicular to $\boldsymbol{\eta}$ and $\mathbf{w}_1$

$$\mathbf{w}_2 = \mathbf{w}_1 \times \boldsymbol{\eta}. \tag{33}$$

For the case of normal incidence (i.e. $\boldsymbol{\eta} \| \mathbf{s}$), Eq. (32) cannot be used due to its singularity in such a case. Instead, the unit direction vector id of the incident ray can be used as $\mathbf{w}_1$:

$$\mathbf{w}_1 = \mathbf{i}_d \tag{34}$$

as it is perpendicular to **s** (and therefore to $\boldsymbol{\eta}$). Subsequently, $\mathbf{w}_2$ is calculated from (33).

Electric displacement vector $\mathbf{D}^{(I)}$ of the reflected ray is perpendicular to its wave vector $\mathbf{s}^{(I)}$. It can therefore be expressed as a linear combination of two vectors perpendicular to $\mathbf{s}^{(I)}$ and perpendicular to each other, e.g.:

$$\mathbf{D}^{(I)} = \alpha\,\mathbf{w}_1 + \beta\,(\mathbf{w}_1 \times \mathbf{s}^{(I)}) \tag{35}$$

where α, β –coefficients of the linear combination.

Now boundary condition (6) can be expressed in the $\mathbf{w}_2\mathbf{w}_1\boldsymbol{\eta}$ coordinate system as:

$$\begin{aligned} &\mathbf{w}_1\mathbf{E}_1 - \mathbf{w}_1\mathbf{E}_2 = 0 \quad \mathbf{w}_1\mathbf{E} + \mathbf{w}_1\mathbf{E}^{(I)} - \mathbf{w}_1\mathbf{E}^{(1)} - \mathbf{w}_1\mathbf{E}^{(2)} = 0 \\ &\mathbf{w}_2\mathbf{E}_1 - \mathbf{w}_2\mathbf{E}_2 = 0 \quad \mathbf{w}_2\mathbf{E} + \mathbf{w}_2\mathbf{E}^{(I)} - \mathbf{w}_2\mathbf{E}^{(1)} - \mathbf{w}_2\mathbf{E}^{(2)} = 0 \end{aligned} \tag{36}$$

where $\mathbf{E}_1$ – electric field vector in the first medium, $\mathbf{E}_2$ – electric field vector in the second medium, $\mathbf{E}^{(I)}$ – electric field vector of the reflected ray, $\mathbf{E}^{(1)}$, $\mathbf{E}^{(2)}$ – electric field vectors of the ordinary and extraordinary ray, respectively.

Similarly, boundary condition (7) can be expressed in the $\mathbf{w}_2\mathbf{w}_1\boldsymbol{\eta}$ coordinate system as:

$$\begin{aligned} &\mathbf{w}_1\mathbf{H}_1 - \mathbf{w}_1\mathbf{H}_2 = 0 \quad \mathbf{w}_1\mathbf{H} + \mathbf{w}_1\mathbf{H}^{(I)} - \mathbf{w}_1\mathbf{H}^{(1)} - \mathbf{w}_1\mathbf{H}^{(2)} = 0 \\ &w_2\mathbf{H}_1 - w_2\mathbf{H}_2 = 0 \quad \mathbf{w}_2\mathbf{H} + \mathbf{w}_2\mathbf{H}^{(I)} - \mathbf{w}_2\mathbf{H}^{(1)} - \mathbf{w}_2\mathbf{H}^{(2)} = 0 \end{aligned} \tag{37}$$

where $\mathbf{H}_1$ – magnetic field vector in the first medium, $\mathbf{H}_2$ – magnetic field vector in the second medium, $\mathbf{H}^{(I)}$ – magnetic field vector of the reflected ray, $\mathbf{H}^{(1)}$, $\mathbf{H}^{(2)}$ – magnetic field vectors of the ordinary and extraordinary ray, respectively.

Equations (36) and (37) will be presently used to calculate amplitudes and directions of unknown **E**, **D** and **H** vectors. We begin with expressing **E** and **H** vectors appearing in these equations in terms of **D** vectors.
Since the first medium is isotropic, electric field vector $\mathbf{E}^{(\mathrm{I})}$ can be expressed as:

$$\mathbf{E}^{(\mathrm{I})} = \frac{1}{n_1^2}\mathbf{D}^{(\mathrm{I})}, \tag{38}$$

where n_1 – refractive index of the first medium. Magnetic field vector $\mathbf{H}^{(\mathrm{I})}$ then becomes

$$\mathbf{H}^{(\mathrm{I})} = n_1\left(\mathbf{s}^{(\mathrm{I})} \times \mathbf{E}^{(\mathrm{I})}\right) = \frac{\alpha}{n_1}\left(\mathbf{s}^{(\mathrm{I})} \times \mathbf{w}_1\right) + \frac{\beta}{n_1}\left(\mathbf{w}_1 - \mathbf{s}^{(\mathrm{I})}\left(\mathbf{s}^{(\mathrm{I})} \cdot \mathbf{w}_1\right)\right). \tag{39}$$

Similar set of equations can be written for the ordinary ray in the second medium, viz.:

$$\mathbf{E}^{(1)} = \frac{1}{n_{o2}^2}\mathbf{D}^{(1)} = \frac{\left|\mathbf{D}^{(1)}\right|}{n_{o2}^2}\mathbf{i}_{d1} \tag{40}$$

where n_{o2} – ordinary refractive index of the second medium,

$$\mathbf{H}^{(1)} = n_{o2}\left(\mathbf{s}^{(1)} \times \mathbf{E}^{(1)}\right) = \frac{\left|\mathbf{D}^{(1)}\right|}{n_{o2}}\left(\mathbf{s}^{(1)} \times \mathbf{i}_{d1}\right) \tag{41}$$

For the extraordinary ray we use the approach presented in ([2], p. 671), i.e. first we will find the component $\mathbf{D}_\perp$ of vector $\mathbf{D}^{(2)}$ which is perpendicular to the vector $\mathbf{E}^{(2)}$:

$$\mathbf{D}_\perp^{(2)} = \mathbf{D}^{(2)} - \mathbf{t}^{(2)}\left(\mathbf{D}^{(2)} \cdot \mathbf{t}^{(2)}\right) = \left|\mathbf{D}^{(2)}\right|\left[\mathbf{i}_{d2} - \mathbf{t}^{(2)}\left(\mathbf{i}_{d2} \cdot \mathbf{t}^{(2)}\right)\right] = \left|\mathbf{D}^{(2)}\right|\mathbf{e}^{(2)} \tag{42}$$

where $\mathbf{e}^{(2)}$ – direction of $\mathbf{D}_\perp$, defined as:

$$\mathbf{e}^{(2)} = \mathbf{i}_{d2} - \mathbf{t}^{(2)}\left(\mathbf{i}_{d2} \cdot \mathbf{t}^{(2)}\right). \tag{43}$$

We should note that $\mathbf{e}^{(2)}$ is not in general a unit vector, therefore it is not referred to as $\mathbf{i}_{e2}$. Following, we can express $\mathbf{E}^{(2)}$ in terms of $\mathbf{D}_\perp$.:

$$\mathbf{E}^{(2)} = \frac{\left|\mathbf{D}^{(2)}\right|}{n_{r2}^2}\mathbf{e}^{(2)} \tag{44}$$

where n_{r2} – ray refractive index of the extraordinary ray, given by:

$$n_{r2} = n_{n2}\cos(\varphi_2) = n_{n2}\left(\mathbf{s}^{(2)} \cdot \mathbf{t}^{(2)}\right) \tag{45}$$

where n_{n2} – refractive index of the extraordinary ray. Knowing $\mathbf{E}^{(2)}$, we can express magnetic field vector $\mathbf{H}^{(2)}$ as:

$$\mathbf{H}^{(2)} = n_{n2}\left(\mathbf{s}^{(2)} \times \mathbf{E}^{(2)}\right) = \frac{\left|\mathbf{D}^{(2)}\right| n_{n2}}{n_{r2}^2}\left(\mathbf{s}^{(2)} \times \mathbf{e}^{(2)}\right). \tag{46}$$

Using electric and magnetic field vectors given by Eqs. (38)÷(46) in boundary conditions (36) and (37), we arrive, after lengthly but relatively straightforward calculations, at the system of equations:

$$\begin{bmatrix} -\dfrac{\mathbf{w}_1\mathbf{i}_{d1}}{n_{o2}^2} & \dfrac{\mathbf{w}_1\mathbf{e}^{(\mathrm{II})}}{n_{r2}^2} & \dfrac{1}{n_1^2} & 0 \\ -\dfrac{\mathbf{w}_2\mathbf{i}_{d1}}{n_{o2}^2} & -\dfrac{\mathbf{w}_2\mathbf{e}^{(\mathbf{II})}}{n_{r2}^2} & 0 & \dfrac{1}{n_1^2}\left(\mathbf{s}^{(\mathrm{I})} \cdot \boldsymbol{\eta}\right) \\ -\dfrac{\mathbf{w}_1\left(\mathbf{s}^{(1)} \times \mathbf{i}_{d1}\right)}{n_{o2}} & -\dfrac{n_{n2}}{n_{r2}^2}\mathbf{w}_1\left(\mathbf{s}^{(2)} \times \mathbf{e}^{(2)}\right) & 0 & \dfrac{1-\left(\mathbf{s}^{(\mathrm{I})}\mathbf{w}_1\right)^2}{n_1} \\ -\dfrac{\mathbf{w}_2\left(\mathbf{s}^{(1)} \times \mathbf{i}_{d1}\right)}{n_{o2}} & -\dfrac{n_{n2}}{n_{r2}^2}\mathbf{w}_2\left(\mathbf{s}^{(2)} \times \mathbf{e}^{(2)}\right) & -\dfrac{\left(\mathbf{s}^{(\mathrm{I})} \cdot \boldsymbol{\eta}\right)}{n_1} & 0 \end{bmatrix} \cdot \begin{bmatrix} \left|\mathbf{D}^{(1)}\right| \\ \left|\mathbf{D}^{(2)}\right| \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\mathbf{w}_1\mathbf{E} \\ -\mathbf{w}_2\mathbf{E} \\ -n_1\mathbf{w}_1\left(\mathbf{s} \times \mathbf{E}\right) \\ -n_1\mathbf{w}_2\left(\mathbf{s} \times \mathbf{E}\right) \end{bmatrix}. \tag{47}$$

Solving this system we obtain amplitudes of electric displacement vectors $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ as well as coefficients α and β using which vector $\mathbf{D}^{(\mathrm{I})}$ is calculated. Finally, **E** and **H** vectors are calculated using Eqs. (38)÷(46).

4.2. Boundary between two anisotropic media

Let us consider a ray propagating in an anisotropic medium in the direction of its wave vector **s**. Incident on a plane boundary Σ with another anisotropic medium, as shown in Fig. 3, this ray gives rise to four rays: two reflected rays whose wave vectors are $\mathbf{s}^{(\mathrm{I})}$ and $\mathbf{s}^{(\mathrm{II})}$, for ordinary and extraordianry ray respectively, and two transmitted rays. Wave vectors of the transmitted rays are $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ for the ordinary and extraordinary ray respectively.

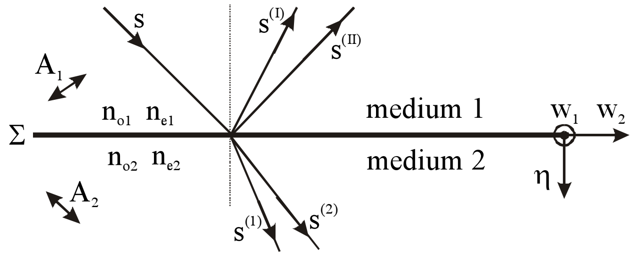

Fig. 3. Reflection and transmission of incident wave on the boundary Σ between a uniaxial anisotropic medium 1 and a uniaxial anisotropic medium 2 (explanations in the text).

Unit direction vectors $\mathbf{i}_{d\mathrm{I}}$ and $\mathbf{i}_{d\mathrm{II}}$ of electric displacement vectors $\mathbf{D}^{(\mathrm{I})}$ and $\mathbf{D}^{(\mathrm{II})}$ of the transmitted ordinary and extraordinary rays can be obtained from (15):

$$\mathbf{i}_{dI} = \frac{\mathbf{s}^{(\mathrm{I})} \times \mathbf{A}_1}{\left|\mathbf{s}^{(\mathrm{I})} \times \mathbf{A}_1\right|} \qquad \mathbf{i}_{dII} = \frac{\mathbf{s}^{(\mathrm{II})} \times \mathbf{A}_1}{\left|\mathbf{s}^{(\mathrm{II})} \times \mathbf{A}_1\right|} \times \mathbf{s}^{(\mathrm{II})}. \tag{48}$$

Similarly, unit direction vectors $\mathbf{i}_{d1}$ and $\mathbf{i}_{d2}$ of electric displacement vectors $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ of the reflected ordinary and extraordinary rays can be written as:

$$\mathbf{i}_{d1} = \frac{\mathbf{s}^{(1)} \times \mathbf{A}_2}{\left|\mathbf{s}^{(1)} \times \mathbf{A}_2\right|} \qquad \mathbf{i}_{d2} = \frac{\mathbf{s}^{(2)} \times \mathbf{A}_2}{\left|\mathbf{s}^{(2)} \times \mathbf{A}_2\right|} \times \mathbf{s}^{(2)}. \tag{49}$$

In order to facilitate writing of boundary conditions, let us introduce a coordinate system $\mathbf{w}_2\mathbf{w}_1\boldsymbol{\eta}$ defined, as in the previous sub-section, using Eqs. (32)÷(34).
Now boundary condition (6) can be expressed in the $\mathbf{w}_2\mathbf{w}_1\boldsymbol{\eta}$ coordinate system as

$$\begin{aligned} &\mathbf{w}_1\mathbf{E}_1 - \mathbf{w}_1\mathbf{E}_2 = 0 \quad \mathbf{w}_1\mathbf{E} + \mathbf{w}_1\mathbf{E}^{(\mathrm{I})} + \mathbf{w}_1\mathbf{E}^{(\mathrm{II})} - \mathbf{w}_1\mathbf{E}^{(1)} - \mathbf{w}_1\mathbf{E}^{(2)} = 0 \\ &\mathbf{w}_2\mathbf{E}_1 - \mathbf{w}_2\mathbf{E}_2 = 0 \quad \mathbf{w}_2\mathbf{E} + \mathbf{w}_2\mathbf{E}^{(\mathrm{I})} + \mathbf{w}_2\mathbf{E}^{(\mathrm{II})} - \mathbf{w}_2\mathbf{E}^{(1)} - \mathbf{w}_2\mathbf{E}^{(2)} = 0 \end{aligned}. \tag{50}$$

Similarly, boundary condition (7) can be expressed as:

$$\begin{aligned} &\mathbf{w}_1\mathbf{H}_1 - \mathbf{w}_1\mathbf{H}_2 = 0 \quad \mathbf{w}_1\mathbf{H} + \mathbf{w}_1\mathbf{H}^{(\mathrm{I})} + \mathbf{w}_1\mathbf{H}^{(\mathrm{II})} - \mathbf{w}_1\mathbf{H}^{(1)} - \mathbf{w}_1\mathbf{H}^{(2)} = 0 \\ &\mathbf{w}_2\mathbf{H}_1 - \mathbf{w}_2\mathbf{H}_2 = 0 \quad \mathbf{w}_2\mathbf{H} + \mathbf{w}_2\mathbf{H}^{(\mathrm{I})} + \mathbf{w}_2\mathbf{H}^{(\mathrm{II})} - \mathbf{w}_2\mathbf{H}^{(1)} - \mathbf{w}_2\mathbf{H}^{(2)} = 0 \end{aligned}. \tag{51}$$

Using equations (50) and (51) amplitudes and directions of unknown **E**, **D** and **H** vectors will presently be calculated. We begin with expressing **E** and **H** vectors appearing in these equations in terms of **D** vectors.
We can express electric field vector $\mathbf{E}^{(\mathrm{I})}$ of the ordinary reflected ray as:

$$\mathbf{E}^{(\mathrm{I})} = \frac{1}{n_{o1}^2}\mathbf{D}^{(\mathrm{I})} = \frac{\left|\mathbf{D}^{(\mathrm{I})}\right|}{n_{o1}^2}\mathbf{i}_{d\mathrm{I}} \tag{52}$$

and its magnetic field vector $\mathbf{H}^{(\mathrm{I})}$ as:

$$\mathbf{H}^{(\mathrm{I})} = n_{o1}\left(\mathbf{s}^{(\mathrm{I})} \times \mathbf{E}^{(\mathrm{I})}\right) = \frac{\left|\mathbf{D}^{(\mathrm{I})}\right|}{n_{o1}}\left(\mathbf{s}^{(\mathrm{I})} \times \mathbf{i}_{d\mathrm{I}}\right). \tag{53}$$

For the extraordinary reflected ray we will employ again the method presented in ([2], p. 671), obtaining:

$$\mathbf{D}_{\perp}^{(\mathrm{II})} = \mathbf{D}^{(\mathrm{II})} - \mathbf{t}^{(\mathrm{II})}\left(\mathbf{D}^{(\mathrm{II})} \cdot \mathbf{t}^{(\mathrm{II})}\right) = \left|\mathbf{D}^{(\mathrm{II})}\right|\left[\mathbf{i}_{d\mathrm{II}} - \mathbf{t}^{(\mathrm{II})}\left(\mathbf{i}_{dII} \cdot \mathbf{t}^{(\mathrm{II})}\right)\right] = \left|\mathbf{D}^{(\mathrm{II})}\right|\mathbf{e}^{(\mathrm{II})}. \tag{54}$$

Following, we can express $\mathbf{E}^{(\mathrm{II})}$ in terms of $\mathbf{D}_{\perp}$:

$$\mathbf{E}^{(\mathrm{II})} = \frac{\left|\mathbf{D}^{(\mathrm{II})}\right|}{n_{r1}^2}\mathbf{e}^{(\mathrm{II})} \tag{55}$$

where n_{r1} – ray refractive index of the reflected extraordinary ray given by:

$$n_{r1} = n_{n1}\cos(\varphi_1) = n_{n1}\left(\mathbf{s}^{(\mathrm{II})} \cdot \mathbf{t}^{(\mathrm{II})}\right) \tag{56}$$

where n_{n1} – refractive index of the reflected extraordinary ray. Knowing $\mathbf{E}^{(\mathrm{II})}$, we can express $\mathbf{H}^{(\mathrm{II})}$ as:

$$\mathbf{H}^{(\mathrm{II})} = n_{n1}\left(\mathbf{s}^{(\mathrm{II})} \times \mathbf{E}^{(\mathrm{II})}\right) = \frac{\left|\mathbf{D}^{(\mathrm{II})}\right| n_{n1}}{n_{r1}^2}\left(\mathbf{s}^{(\mathrm{II})} \times \mathbf{e}^{(\mathrm{II})}\right). \tag{57}$$

For the ordinary ray in the second medium, we have:

$$\mathbf{E}^{(1)} = \frac{1}{n_{o2}^2}\mathbf{D}^{(1)} = \frac{\left|\mathbf{D}^{(1)}\right|}{n_{o2}^2}\mathbf{i}_{d1} \tag{58}$$

where n_{o2} – ordinary refractive index of the second medium,

$$\mathbf{H}^{(1)} = n_{o2}\left(\mathbf{s}^{(1)} \times \mathbf{E}^{(1)}\right) = \frac{\left|\mathbf{D}^{(1)}\right|}{n_{o2}}\left(\mathbf{s}^{(1)} \times \mathbf{i}_{d1}\right). \tag{59}$$

For the extraordinary ray in the second medium, we can write:

$$\mathbf{D}_{\perp}^{(2)} = \mathbf{D}^{(2)} - \mathbf{t}^{(2)}\left(\mathbf{D}^{(2)} \cdot \mathbf{t}^{(2)}\right) = \left|\mathbf{D}^{(2)}\right|\left[\mathbf{i}_{d2} - \mathbf{t}^{(2)}\left(\mathbf{i}_{d2} \cdot \mathbf{t}^{(2)}\right)\right] = \left|\mathbf{D}^{(2)}\right|\mathbf{e}^{(2)} \tag{60}$$

where $\mathbf{e}^{(2)}$ – direction of $\mathbf{D}_{\perp}$, defined as:

$$\mathbf{e}^{(2)} = \mathbf{i}_{d2} - \mathbf{t}^{(2)}\left(\mathbf{i}_{d2} \cdot \mathbf{t}^{(2)}\right). \tag{61}$$

We should note that $\mathbf{e}^{(2)}$ is not in general a unit vector, therefore it is not referred to as $\mathbf{i}_{e2}$. Following, we can express $\mathbf{E}^{(2)}$ in terms of $\mathbf{D}_{\perp}$.:

$$\mathbf{E}^{(2)} = \frac{\left|\mathbf{D}^{(2)}\right|}{n_{r2}^2}\mathbf{e}^{(2)} \qquad 62)$$

where n_{r2} – ray refractive index of the extraordinary ray, given by:

$$n_{r2} = n_{n2}\cos(\varphi_2) = n_{n2}\left(\mathbf{s}^{(2)} \cdot \mathbf{t}^{(2)}\right) \tag{63}$$

where n_{n2} – refractive index of the extraordinary ray. Knowing $\mathbf{E}^{(2)}$, we can express magnetic field vector $\mathbf{H}^{(2)}$ as:

$$\mathbf{H}^{(2)} = n_{n2}\left(\mathbf{s}^{(2)} \times \mathbf{E}^{(2)}\right) = \frac{\left|\mathbf{D}^{(2)}\right| n_{n2}}{n_{r2}^2}\left(\mathbf{s}^{(2)} \times \mathbf{e}^{(2)}\right) \tag{64}$$

Using electric and magnetic field vectors given by Eqs. (52)÷(64) in boundary conditions (50) and (51), we arrive, after lengthly but relatively straighforward calculations, at the system of four linear equations:

$$\begin{bmatrix} \frac{\mathbf{w}_1\mathbf{i}_{d\mathrm{I}}}{n_{o1}^2} & \frac{\mathbf{w}_1\mathbf{e}^{(\mathrm{II})}}{n_{r1}^2} & -\frac{\mathbf{w}_1\mathbf{i}_{d1}}{n_{o2}^2} & -\frac{\mathbf{w}_1\mathbf{e}^{(2)}}{n_{r2}^2} \\ \frac{\mathbf{w}_2\mathbf{i}_{d\mathrm{I}}}{n_{o1}^2} & \frac{\mathbf{w}_2\mathbf{e}^{(\mathrm{II})}}{n_{r1}^2} & -\frac{\mathbf{w}_2\mathbf{i}_{d1}}{n_{o2}^2} & -\frac{\mathbf{w}_2\mathbf{e}^{(2)}}{n_{r2}^2} \\ \frac{\mathbf{w}_1\left(\mathbf{s}^{(\mathrm{I})} \times \mathbf{i}_{d\mathrm{I}}\right)}{n_{o1}} & \frac{n_{n1}}{n_{r1}^2}\mathbf{w}_1\left(\mathbf{s}^{(\mathrm{II})} \times \mathbf{e}^{(\mathrm{II})}\right) & -\frac{\mathbf{w}_1\left(\mathbf{s}^{(1)} \times \mathbf{i}_{d1}\right)}{n_{o2}} & -\frac{n_{n2}}{n_{r2}^2}\mathbf{w}_1\left(\mathbf{s}^{(2)} \times \mathbf{e}^{(2)}\right) \\ \frac{\mathbf{w}_2\left(\mathbf{s}^{(\mathrm{I})} \times \mathbf{i}_{d\mathrm{I}}\right)}{n_{o1}} & \frac{n_{n1}}{n_{r1}^2}\mathbf{w}_2\left(\mathbf{s}^{(\mathrm{II})} \times \mathbf{e}^{(\mathrm{II})}\right) & -\frac{\mathbf{w}_2\left(\mathbf{s}^{(1)} \times \mathbf{i}_{d1}\right)}{n_{o2}} & -\frac{n_{n2}}{n_{r2}^2}\mathbf{w}_2\left(\mathbf{s}^{(2)} \times \mathbf{e}^{(2)}\right) \end{bmatrix} \cdot \begin{bmatrix} \left|\mathbf{D}^{(I)}\right| \\ \left|\mathbf{D}^{(II)}\right| \\ \left|\mathbf{D}^{(1)}\right| \\ \left|\mathbf{D}^{(2)}\right| \end{bmatrix} = \begin{bmatrix} -\mathbf{w}_1\mathbf{E} \\ -\mathbf{w}_2\mathbf{E} \\ -n\mathbf{w}_1\left(\mathbf{s} \times \mathbf{E}\right) \\ -n\mathbf{w}_2\left(\mathbf{s} \times \mathbf{E}\right) \end{bmatrix}. \tag{65}$$

By solving the system (65), amplitudes of all four electric displacement vectors $\mathbf{D}^{(\mathrm{I})}$, $\mathbf{D}^{(\mathrm{II})}$, $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ are obtained, allowing us to calculate $\mathbf{E}$ and $\mathbf{H}$ vectors using Eqs. (52)÷(64).

4.3. Boundary between anisotropic and isotropic medium

Finally, let us consider a ray propagating in an anisotropic medium in the direction of its wave vector $\mathbf{s}$. Incident on a plane boundary Σ with an isotropic medium, as shown in Fig. 4, this ray gives rise to two reflected rays whose wave vectors are $\mathbf{s}^{(\mathrm{I})}$ and $\mathbf{s}^{(\mathrm{II})}$, for ordinary and extraordianry ray respectively, and a transmitted ray having wave vector $\mathbf{s}^{(1)}$.
Unit direction vectors $\mathbf{i}_{d\mathrm{I}}$ and $\mathrm{i}_{d\mathrm{II}}$ of electric displacement vectors $\mathbf{D}^{(\mathrm{I})}$ and $\mathbf{D}^{(\mathrm{II})}$ of the transmitted ordinary and extraordinary rays can be obtained from (15):

$$\mathbf{i}_{dI} = \frac{\mathbf{s}^{(I)} \times \mathbf{A}_1}{\left|\mathbf{s}^{(I)} \times \mathbf{A}_1\right|} \qquad \mathbf{i}_{dII} = \frac{\mathbf{s}^{(II)} \times \mathbf{A}_1}{\left|\mathbf{s}^{(II)} \times \mathbf{A}_1\right|} \times \mathbf{s}^{(II)}. \tag{66}$$

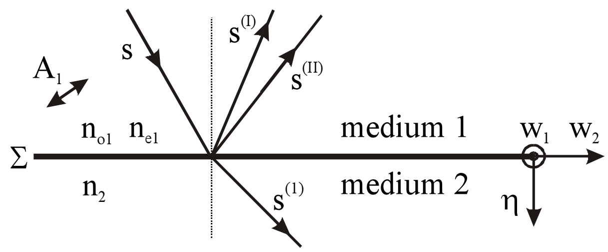

Fig. 4. Reflection and transmission of incident wave on the boundary Σ between a uniaxial anisotropic medium 1 and an isotropic medium 2 (explanations in the text).

Direction $\mathbf{i}_{d1}$ of electric displacement vector $\mathbf{D}^{(1)}$ of the transmitted ray in the isotropic medium cannot be determined in this way. It will be calculated using boundary conditions, together with magnitudes of the $\mathbf{D}^{(I)}$, $\mathbf{D}^{(II)}$ and $\mathbf{D}^{(1)}$ vectors. Introducing again a coordinate system $\mathbf{w}_2\mathbf{w}_1\boldsymbol{\eta}$ defined using Eqs. (32)÷(34), we can express boundary condition (6) as:

$$\begin{array}{ll} \mathbf{w}_1\mathbf{E}_1 - \mathbf{w}_1\mathbf{E}_2 = 0 & \mathbf{w}_1\mathbf{E} + \mathbf{w}_1\mathbf{E}^{(I)} + \mathbf{w}_1\mathbf{E}^{(II)} - \mathbf{w}_1\mathbf{E}^{(1)} = 0 \\ \mathbf{w}_2\mathbf{E}_1 - \mathbf{w}_2\mathbf{E}_2 = 0 & \mathbf{w}_2\mathbf{E} + \mathbf{w}_2\mathbf{E}^{(I)} + \mathbf{w}_2\mathbf{E}^{(II)} - \mathbf{w}_2\mathbf{E}^{(1)} = 0 \end{array}. \tag{67}$$

Similarly, boundary condition (7) can be expressed as:

$$\begin{array}{ll} \mathbf{w}_1\mathbf{H}_1 - \mathbf{w}_1\mathbf{H}_2 = 0 & \mathbf{w}_1\mathbf{H} + \mathbf{w}_1\mathbf{H}^{(I)} + \mathbf{w}_1\mathbf{H}^{(II)} - \mathbf{w}_1\mathbf{H}^{(1)} = 0 \\ \mathbf{w}_2\mathbf{H}_1 - \mathbf{w}_2\mathbf{H}_2 = 0 & \mathbf{w}_2\mathbf{H} + \mathbf{w}_2\mathbf{H}^{(I)} + \mathbf{w}_2\mathbf{H}^{(II)} - \mathbf{w}_2\mathbf{H}^{(1)} = 0 \end{array}. \tag{68}$$

Using equations (67) and (68) amplitudes and directions of unknown **E**, **D** and **H** vectors will presently be calculated. We begin by expressing **E** and **H** vectors appearing in these equations in terms of **D** vectors.
We can express electric field vector $\mathbf{E}^{(I)}$ of the ordinary reflected ray as:

$$\mathbf{E}^{(I)} = \frac{1}{n_{o1}^2}\mathbf{D}^{(I)} = \frac{\left|\mathbf{D}^{(I)}\right|}{n_{o1}^2}\mathbf{i}_{dI} \tag{69}$$

and its magnetic field vector $\mathbf{H}^{(I)}$ as:

$$\mathbf{H}^{(I)} = n_{o1}\left(\mathbf{s}^{(I)} \times \mathbf{E}^{(I)}\right) = \frac{\left|\mathbf{D}^{(I)}\right|}{n_{o1}}\left(\mathbf{s}^{(I)} \times \mathbf{i}_{dI}\right). \tag{70}$$

For the extraordinary reflected ray we will employ again the method presented in ([2], p. 671), obtaining:

$$\mathbf{D}_{\perp}^{(II)} = \mathbf{D}^{(II)} - \mathbf{t}^{(II)}\left(\mathbf{D}^{(II)} \cdot \mathbf{t}^{(II)}\right) = \left|\mathbf{D}^{(II)}\right|\left[\mathbf{i}_{dII} - \mathbf{t}^{(II)}\left(\mathbf{i}_{dII} \cdot \mathbf{t}^{(II)}\right)\right] = \left|\mathbf{D}^{(II)}\right|\mathbf{e}^{(II)}. \tag{71}$$

Following, we can express $\mathbf{E}^{(II)}$ in terms of $\mathbf{D}_{\perp}$:

$$\mathbf{E}^{(II)} = \frac{\left|\mathbf{D}^{(II)}\right|}{n_{r1}^2}\mathbf{e}^{(II)} \tag{72}$$

where n_{r1} – ray refractive index of the reflected extraordinary ray given by:

$$n_{r1} = n_{n1}\cos(\varphi_1) = n_{n1}\left(\mathbf{s}^{(II)} \cdot \mathbf{t}^{(II)}\right) \tag{73}$$

where n_{n1} – refractive index of the reflected extraordinary ray. Knowing $\mathbf{E}^{(II)}$, we can express magnetic field vector $\mathbf{H}^{(II)}$ as:

$$\mathbf{H}^{(II)} = n_{n1}\left(\mathbf{s}^{(II)} \times \mathbf{E}^{(II)}\right) = \frac{\left|\mathbf{D}^{(II)}\right| n_{n1}}{n_{r1}^2}\left(\mathbf{s}^{(II)} \times \mathbf{e}^{(II)}\right). \tag{74}$$

Furthermore, electric displacement vector $\mathbf{D}^{(1)}$ of the transmitted ray in the isotropic medium can be expressed as a linear combination of two vectors perpendicular to $\mathbf{s}^{(1)}$ and perpendicular to each other, e.g.:

$$\mathbf{D}^{(1)} = \alpha\,\mathbf{w}_1 + \beta(\mathbf{w}_1 \times \mathbf{s}^{(1)}) \tag{75}$$

Using Eq. (75) we can express electric field vector $\mathbf{E}^{(1)}$ terms of $\mathbf{D}^{(1)}$:

$$\mathbf{E}^{(1)} = \frac{1}{n_2^2}\mathbf{D}^{(1)} \tag{76}$$

and, subsequently, write magnetic field vector $\mathbf{H}^{(1)}$ as:

$$\mathbf{H}^{(1)} = n_2\left(\mathbf{s}^{(1)} \times \mathbf{E}^{(1)}\right) = \frac{\alpha}{n_2}\left(\mathbf{s}^{(1)} \times \mathbf{w}_1\right) + \frac{\beta}{n_2}\left(\mathbf{w}_1 - \mathbf{s}^{(1)}\left(\mathbf{s}^{(1)} \cdot \mathbf{w}_1\right)\right). \tag{77}$$

Using electric and magnetic field vectors given by Eqs. (69)÷(77) in boundary conditions (67) and (68), we arrive, after lengthly but relatively straighforward calculations, at the system of four linear equations:

$$\begin{bmatrix} \frac{\mathbf{w}_1 \mathbf{i}_{dI}}{n_{o1}^2} & \frac{\mathbf{w}_1 \mathbf{e}^{(II)}}{n_{r1}^2} & -\frac{1}{n_2^2} & 0 \\ \frac{\mathbf{w}_2 \mathbf{i}_{dI}}{n_{o1}^2} & \frac{\mathbf{w}_2 \mathbf{e}^{(II)}}{n_{r1}^2} & 0 & -\frac{\left(\mathbf{s}^{(1)} \cdot \boldsymbol{\eta}\right)}{n_2^2} \\ \frac{\mathbf{w}_1\left(\mathbf{s}^{(I)} \times \mathbf{i}_{dI}\right)}{n_{o1}} & \frac{n_{n1}}{n_{r1}^2}\mathbf{w}_1\left(\mathbf{s}^{(II)} \times \mathbf{e}^{(II)}\right) & 0 & \frac{\left(\mathbf{s}^{(1)} \cdot \mathbf{w}_1\right)^2 - 1}{n_2} \\ \frac{\mathbf{w}_2\left(\mathbf{s}^{(I)} \times \mathbf{i}_{dI}\right)}{n_{o1}} & \frac{n_{n1}}{n_{r1}^2}\mathbf{w}_2\left(\mathbf{s}^{(II)} \times \mathbf{e}^{(II)}\right) & \frac{\mathbf{s}^{(1)} \cdot \boldsymbol{\eta}}{n_2} & 0 \end{bmatrix} \cdot \begin{bmatrix} \left|\mathbf{D}^{(I)}\right| \\ \left|\mathbf{D}^{(II)}\right| \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\mathbf{w}_1\mathbf{E} \\ -\mathbf{w}_2\mathbf{E} \\ -n\mathbf{w}_1\left(\mathbf{s} \times \mathbf{E}\right) \\ -n\mathbf{w}_2\left(\mathbf{s} \times \mathbf{E}\right) \end{bmatrix}. \tag{78}$$

By solving the system (78), amplitudes of electric displacement vectors **D**(I), **D**(II), are obtained. Additionally, two coefficients, α and β of linear combination forming $\mathbf{D}^{(1)}$ are also obtained, allowing $\mathbf{D}^{(1)}$ to be calculated from (75). Finally, **E** and **H** vectors are calculated using Eqs. (66), (69)÷(77).

5. COMPUTATIONAL EXAMPLE

First, a series of tests, aimed at detection of errors, was conducted for various combinations of isotropic and anisotropic materials, and for different orientations of boundary surface. Following, a ray tracing in a Wollaston prism, shown in Fig. 5, was performed. A highly birefringent material YVO_4 (n_e=2.2154 and n_o=1.9929 for λ=633 nm) has been chosen in order to determine the impact of high birefringence of the prism's material on the intensities of two beams emerging from the prism. Thickness of the prism was assumed to be 4 mm and the apex angle α=20° was chosen. Calculations were performed for the case of normal incidence. The opical ray incident on the prism was linearly polarized in the plane inclined at 45° to the plane of the Fig. 5. and its intensity was 1.0.

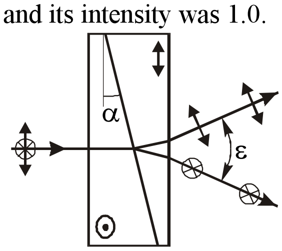

Fig. 5. Wollaston prism used in the ray tracing. α – apex angle of the prism, ε – angle between the beams emerging from the prism.

Conducted calculations yielded intensities of the two rays leaving the prism to be I_1=0.3766 and I_2=0.3887. Their intensities differ by about 3.2%, which may adversely affect performance of optical setups using such prisms as beamsplitters and relying on the equal power or amplitude division, as is often the case in e.g. Optical Coherence Tomography balanced detection setups.

Let us now consider the effect of this intensity difference on the performance of optical setups in which interference of these beams is used, such as the detection setup for polarization interferometric sensors[5] presented in Fig. 6. Intensity I of interfering beams can be expressed as ([2], p. 259):

$$I(\varphi) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\varphi \tag{79}$$

where I_1 and I_2 – intensities of interfering beams, φ – phase difference between them. Visibility V then becomes:

$$V = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} \tag{80}$$

and in our case is equal 0.999876. The visibility is very close to unity, being much higher than expected. Therefore, the influence of intensity difference on the performance of this class of optical setups is in most cases negligible.

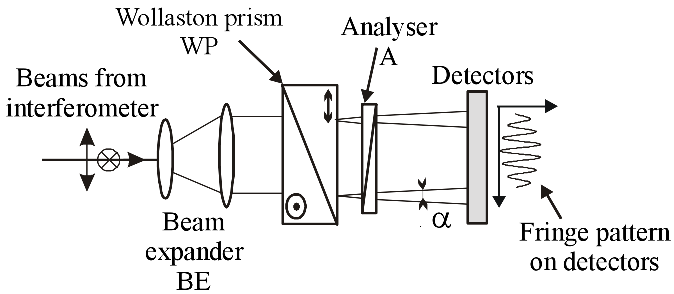

Fig. 6. Polarization interferometer detection setup using Wollaston prism.

Finally, the angle ε between the beams emerging from the prism was calculated using scalar product of their s vectors and was found to be equal to 9.28527°. This value was subsequently compared with angle ε calculated using well-known formula[4]:

$$\varepsilon = 2\left|n_e - n_o\right|\tan(\alpha) \tag{81}$$

which for the prism yielded result 9.28001°. Since both results agree only to within 6 parts in 10000, the source of this difference is worth investigating. On consulting[6], one can find that in the derivation of (81) the use is made of approximations $\cos(x) \approx 1$ and $\sin(x) \approx x$ which are valid only for small angles. This explains the obtained difference.

6. CONCLUSIONS

Presented method allows for the calculation of amplitudes of rays transmitted and reflected from a boundary between two media of which at least one is uniaxial, birefringent and non-absorbing. Being based on a vectorial formulation, and extending approach developped by Trollinger *et. al.*[3], the method provides a compact framework which can be used to gain new insights from ray tracing analysis of existing and new optical setups.

Calculation results obtained for a Wollaston prism made from a highly birefringent material indicate that the intensity difference of the two beams emerging from such prism may adversely affect operation of optical setups relying on the equal power or amplitude division performed by such prisms. Observed intensity difference has little impact on the performance of optical setups in which interference of these beams is used, such as detection setups for polarization interferometric sensors or polarization optical fiber sensors.

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