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General Transport Theory for Weak Inhomogeneities and Quantum Solids in High External Fields

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The exact transport equation obtained in a recent work on new renormalization methods for single-time Green's functions is applied to weakly inhomogeneous quantum systems. The initial density matrix and the external fields are assumed to vary slowly over microscopic length scales like the de Broglie wavelength λ_F of the particles, the range λ_V of the interaction and the lattice constant a. In addition, the Hamiltonian without the external fields has to fulfil (quasi-)momentum conservation. The final result is given by a set of local and nonlinear integral equations for the Weyl transforms of all one-particle distribution functions occurring in the system. As an application a quantum solid of Bloch electrons, phonons, and impurities in arbitrarily time-dependent and weakly inhomogeneous electric and magnetic fields is studied. The fields can be moderately high in the sense that $r_L \gg \max\{\lambda_F, \lambda_V, a\}$ and $eE \max\{\lambda_F, \lambda_V, a\} \ll \varepsilon_F$. Here r_L denotes the radius of the Landau orbits, E is the electric field, and ε_F is the fermi energy of the electrons. In Born approximation, a set of generalized Boltzmann equations is obtained which include interband transition terms from scattering and Zeener tunneling as well as collisional broadening and intracollisional field effects.

1. INTRODUCTION

In a previous paper [1] (hereafter to be referred to as I) we have derived an exact transport equation for all single-time distribution functions

$$F^{\pm}_{ll'}(t; bb') = \mp \langle b'_{l'} b_l \rangle_{\rho(t)}, \tag{1.1}$$

occurring in an arbitrary multi-component quantum many-body system (b and b' can be creation or annihilation operators of any component). The purpose of the present work is to specialize this result to the case of weak inhomogeneities and to apply it to quantum solids in high electromagnetic fields.

In the conventional Keldysh formalism for double-time Green's functions, a standard method to obtain kinetic equations for small variations in time and space is to use a so-called gradient expansion [2, 3]. However, since (1.1) is already a

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single-time quantity, this procedure is only needed here for the spatial variables. Thus, in our formalism there is no restriction with respect to the time dependence of the external fields. Moreover, we will show that the precise conditions for the application of a gradient expansion in **x** are:

1. The Hamiltonian without the external fields has to fulfil (quasi-)momentum conservation.

2. The wavelength λ_0 of the inhomogeneities in the initial density matrix $\rho(t_0)$ and in the external fields has to be large according to

$$\lambda_0 \gg \lambda_F, \lambda_V, a, \tag{1.2}$$

where λ_F denotes the de Broglie wavelength of the particles, λ_V is the range of the interaction potential V, and a is the lattice constant.

The final transport equation for the Weyl transform $F^{+\mathrm{ph}}_{\alpha\alpha'}(\mathbf{x}; \mathbf{l}; t; bb')$ (we use the notation $l = \mathbf{l}, \alpha$, where $\mathbf{l}$ is the quasimomentum and α denotes the set of the remaining quantum numbers) will have a local structure in the spatial coordinates. This means that all quantities in the generalized collision integral depend on the same parameter **x** and the structure in the remaining variables $\mathbf{l}, \alpha$ is just like in the homogeneous case. Consequently, the result can be written independent of the representation by transforming it to an operator form where the spatial and temporal coordinates appear simply as parameters. The homogeneity of all these operators, together with the possibility of using other sets of one-particle wave functions, will be especially important for the treatment of Bloch electrons in external fields.

The transport equation is presented as a systematic perturbation expansion in the coupling parameter, each order being defined by the general Feynman rules developed in I. Furthermore, the derivation is independent of the special form of the interaction.

The second part of this paper is devoted to the case of quantum solids in time-dependent and weakly inhomogeneous electric or magnetic fields. We will consider a system of Bloch electrons, phonons, and impurities which can interact in all possible ways: electron–electron (e-e), electron–phonon (e-p), electron–impurity (e-i), phonon–phonon (p-p), and phonon–impurity (p-i) interaction.

In order to apply our general transport equation to this system we will take care of the fact that weak inhomogeneities in the external fields and in the electromagnetic potentials are not necessarily equivalent. We will show that this problem can be removed by imposing the conditions

$$\frac{\lambda_F e B}{\hbar c} \ll \min\left\{\frac{1}{\lambda_F}, \frac{1}{\lambda_V}, \frac{1}{a}\right\} \tag{1.3}$$

and

$$eE \max\{\lambda_F, \lambda_V, a\} \ll \varepsilon_F \tag{1.4}$$

on the magnitude of the magnetic field B and the electric field E; ε_F, e, and m denote the fermi energy, charge, and mass of the electrons. Equations (1.3) and (1.4) mean that the radius $r_L = \hbar c/\lambda_F eB$ of the Landau orbits is large compared to $\max\{\lambda_F, \lambda_V, a\}$ and that the energy transfer to an electron over a distance λ_F, λ_V, or a by the electric field is small compared to ε_F.

A second problem arises from the question of which representation of one-particle wave functions is the most suitable for the Bloch electrons. For homogeneous fields one often uses the lattice Weyl transform [3–5] with respect to magnetic Wannier functions [6–8], also called Roth functions in **k**-space. Although it is possible to generalize these functions to the case of weak inhomogeneities, we will use a more obvious choice in the present work. It consists in a complete and orthonormal set of instantaneous and local eigenfunctions of the unperturbed Hamiltonian including the external fields. As a result we will obtain a driving term consistent with the well-known relation [9, 10]

$$\hbar\dot{\mathbf{k}} = e\mathbf{E}(\mathbf{x}, t) + \frac{e}{\hbar c}\frac{\partial\varepsilon_\alpha^{(0)}}{\partial\mathbf{k}}(\mathbf{k}) \wedge (\mathbf{x}, t) \tag{1.5}$$

for the time evolution of the wave vector **k** of a Bloch electron ($\varepsilon_\alpha(k)$ denotes the dispersion law for the band α).

For small tunneling rates, being equivalent to the condition (1.4), we will also consider the electronic Wigner function $F_{\alpha\alpha'}^{+\mathrm{ph}}$ for $\alpha \neq \alpha'$. This leads to a correction of the driving term by interband transition effects due to Zeener tunneling. For the case of e-p and e-i interactions in homogeneous electric fields, the same result has been obtained in [11].

The collision integral is calculated in Born approximation. However, since the external fields can be very high, the transition probabilities are not given by the usual golden rule. Actually the whole equation has a non-Markovian form and the fields enter the collision term, a property which is known as the intracollisional field effect [11–17]. Finally, as has been shown in detail in I, one can also include collisional broadening effects by simply replacing the unperturbed evolution operators by the retarded or advanced Green's functions.

Transport theory of quantum solids in electric or magnetic fields has been the subject of many previous works [2–5, 11–21]. However, all these treatments are resticted to homogeneous fields and there are further assumptions depending on the formalism used (see especially Section 5 for a detailed discussion).

The work is organized as follows. In Section 2 we will give a short review over the general theory developed in I. All basic notations in the exact transport equation (2.24) are explained in detail, but for the graphical techniques and the general Feynman rules, the reader is referred to I. The properties of weakly inhomogeneous quantum systems and a generalized Weyl transformation are presented in Section 3. The Weyl transform of the exact formula (2.24) will lead to the quantum transport equation (3.31) for weak inhomogeneities. Section 4 is devoted to the case of quantum solids as described above. Finally, Section 5 contains some conclusions and a comparison with other methods in the literature.

2. The Exact Transport Equation

In this section we will give a short summary of the results obtained in I. Thereby we will try to use a notation which is as compact as possible. The final equations are exact and can be used for weak as well as strong inhomogeneities.

We consider a general quantum many-body system of several components characterized by discrete one-particle quantum numbers l. An arbitrary creation or annihilation operator of any sort of particle is denoted by b_l. The Hamiltonian is written as

$$H(t) = H_0(t) + V, \tag{2.1}$$

where $H_0(t)$ is assumed to have the bilinear form

$$H_0(t) = \tfrac{1}{2} \sum_{ll'} \sum_{bb'} \varepsilon_{l'l}(t; b'^{\dagger} b^{\dagger})\, N(b'_{l'} b_l) \tag{2.2}$$

and V describes the general interaction

$$V = \sum_r \frac{1}{r!} \sum_{l_1 \cdots l_r} \sum_{b^1 \cdots b^r} \bar{v}(l_1 \cdots l_r; b^{1\dagger} \cdots b^{r\dagger})\, N(b^1_{l_1} \cdots b^r_{l_r}) \tag{2.3}$$

with $N(\cdots)$ being the normal ordering.

The full double-time Green's functions are defined by

$$\begin{aligned} G^+_{ll'}(t, t'; bb') &= \mp \langle b'_{l'}(t')_{\mathrm{H}}\, b_l(t)_{\mathrm{H}} \rangle_{\rho(t_0)} \\ G^-_{ll'}(t, t'; bb') &= \langle b_l(t)_{\mathrm{H}}\, b'_{l'}(t')_{\mathrm{H}} \rangle_{\rho(t_0)}, \end{aligned} \tag{2.4}$$

where the upper (lower) sign always corresponds to fermions (bosons) and $\langle \cdots \rangle_{\rho(t_0)}$ denotes the expectation value with respect to the initial density matrix $\rho(t_0)$. The Heisenberg and interaction operators of b_l are given by $b_l(t)_{\mathrm{H}}$ and $b_l(t)$, respectively. For the unperturbed Green's functions we write

$$\begin{aligned} G^{(0)+}_{ll'}(t, t'; bb') &= \mp \langle b'_{l'}(t')\, b_l(t) \rangle_{\rho(t_0)} \\ G^{(0)-}_{ll'}(t, t'; bb') &= \langle b_l(t)\, b'_{l'}(t') \rangle_{\rho(t_0)} \end{aligned} \tag{2.5}$$

and the symbol F is used for the corresponding single-time quantities

$$F^{\pm}_{ll'}(t; bb') = G^{\pm}_{ll'}(t, t; bb'), \qquad F^{(0)\pm}_{ll'}(t; bb') = G^{(0)\pm}_{ll'}(t, t; bb'). \tag{2.6}$$

The connection between $G^{(0)}$ and $F^{(0)}$ can easily be obtained by using the transformation law I (2.29)

$$b_l(t)=\sum_{l'}\sum_{b'} W^{(0)}_{ll'}(t,t';bb'^{\dagger})\,b'_{l'}(t') \tag{2.7}$$

with

$$W^{(0)}(t,t';bb')=\begin{cases}G^{(0)\,B}(t,t';bb') & \text{if} \quad b'\sim c^{\dagger}\\ G^{(0)\,B}(t,t';bb')^{\mathrm{T}} & \text{if} \quad b'\sim c\end{cases} \tag{2.8}$$

and

$$G^{(0)\,B}_{ll'}(t,t';bb')=\langle [b_l(t),b'_{l'}(t')]_{\pm}\rangle_{\rho(t_0)}, \tag{2.9}$$

where, in (2.8), we have used the general definition of a transposed operator

$$A(t,t';bb')^{\mathrm{T}}_{ll'}=A(t',t;b'b)_{l'l}. \tag{2.10}$$

The symbols $b'\sim c^{\dagger}$ or $b'\sim c$ in (2.8) mean that b' is proportional to a creation or annihilation operator. Using (2.7) in (2.5) we immediately obtain the important relation

$$G^{(0)\pm}(t,t')=W^{(0)}(t,t^{<})\,F^{(0)\pm}(t^{<})\,W^{(0)}(t^{<},t')^{\mathrm{T}}, \tag{2.11}$$

where $t^{<}=\min\{t,t'\}$ and the operator product is defined by the generalized matrix multiplication

$$(AB)_{ll'}(bb')=\sum_{\bar{l}}\sum_{\bar{b}} A_{l\bar{l}}(b\bar{b}^{\dagger})\,B_{\bar{l}l'}(\bar{b}b'). \tag{2.12}$$

The replacement of H_0 by the full Hamiltonian H in (2.11) would lead to the generalized Kadanoff–Baym ansatz [1, 17]

$$G^{\pm}(t,t')=W(t,t^{<})\,F^{\pm}(t^{<})\,W(t^{<},t')^{\mathrm{T}}, \tag{2.13}$$

where

$$W(t,t';bb')=\begin{cases}G^{B}(t,t';bb') & \text{if} \quad b'\sim c^{\dagger}\\ G^{B}(t,t';bb')^{\mathrm{T}} & \text{if} \quad b'\sim c\end{cases} \tag{2.14}$$

and

$$G^{B}_{ll'}(t,t';bb')=\langle b_l(t)_{\mathrm{H}}\,b'_{l'}(t')_{\mathrm{H}}\rangle_{\rho(t_0)} \tag{2.15}$$

is the retarded or advanced Green's function for the nonequilibrium case. Although (2.13) is not correct, the central result in our first work I was that (2.13) can be used exactly for all unperturbed Green's functions in the perturbation series of $F^{\pm}$.

Simultaneously one has to disregard certain reducible graphs and only the so-called PSD2-diagrams remain (see Section 4 of I). To specify this let us first write down the perturbation series for the time derivative of $F^{\pm}$ according to I(2.60),

$$\frac{\partial}{\partial t} F^{\pm}_{ll'}(t; bb') + i[\varepsilon_1(t), F^{\pm}_1(t)]^{(\bullet)}_{ll'}(bb')$$
$$= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{G_{nm}} \sum_{(i_1, \ldots, i_n)} \int_{t_0}^{t} dt_{i_2} \int_{t_0}^{t_{i_2}} dt_{i_3} \cdots \int_{t_0}^{t_{i_{n-1}}} dt_{i_n}$$
$$\times \sum_{\{qn\}} z_G \prod_{i=1}^{n} c_i^{(0)} \bar{v}_i \prod_{j=1}^{m} g_j \Bigg|_{t_{i_1}=t}; \tag{2.16}$$

G_{nm} is a graph with n vertices and m correlations, $t_1, \ldots, t_n$ are the times at the vertices ordered according to $t_{i_1} > t_{i_2} > \cdots > t_{i_n}$, $\sum_{\{qn\}}$ denotes the sum over all quantum numbers, and z_G is the prefactor of the graph G_{nm} (see I(2.51)). The factor $c_i^{(0)}$ for the vertex i is defined by

$$c_i^{(0)} = \prod_{\substack{s \\ t_i = \tau_s = \tau'_s}} G_s^{(0)+} \Bigg\{ \prod_{\substack{s \\ t_i = \tau_s < \tau'_s}} G_s^{(0)+} \prod_{\substack{s \\ t_i = \tau'_s < \tau_s}} G_s^{(0)-} - (+ \leftrightarrow -) \Bigg\}, \tag{2.17}$$

where $G_s^{(0)\pm}$ are the unperturbed Green's functions at the line s:

$$\begin{aligned} G_s^{(0)+} &= \mp \langle b_{l'_s}^{(s)'}(\tau'_s)\, b_{l_s}^{(s)}(\tau_s) \rangle_{\rho(t_0)} \\ G_s^{(0)-} &= \langle b_{l_s}^{(s)}(\tau_s)\, b_{l'_s}^{(s)'}(\tau'_s) \rangle_{\rho(t_0)}. \end{aligned} \tag{2.18}$$

The matrix elements $\bar{v}_i$ and the correlations g_j are given by (see also I(2.36))

$$\bar{v}_i = \bar{v}(l_1 \cdots l_r; b_{l_1}^{1\dagger} \cdots b_{l_r}^{r\dagger}) \tag{2.19}$$

$$g_j = g(t_1 \cdots t_r; l_1 \cdots l_r; b^1 \cdots b^r) = \overbrace{b_{l_1}^1(t_1) \cdots b_{l_r}^r(t_r)}, \tag{2.20}$$

where $b^1 \cdots b^r$ are the operators at the vertex i or at the correlation j, respectively. The exact order of the creation and annihilation operators in (2.18), (2.19), and (2.20) is given by the general Feynman rules set up in Section 2 of I. Using (2.7), g_j can also be written as

$$g_j = \sum_{\bar{l}_1 \cdots \bar{l}_r} \sum_{\bar{b}^1 \cdots \bar{b}^r} \prod_{i=1}^{r} W_{l_i \bar{l}_i}^{(0)}(t_i, t_0; b^i \bar{b}^{i\dagger})\, g(t_0; \bar{l}_1 \cdots \bar{l}_r; \bar{b}^1 \cdots \bar{b}^r), \tag{2.21}$$

where

$$g(t_0; l_1 \cdots l_r; b^1 \cdots b^r) = \overbrace{b_{l_1}^1 \cdots b_{l_r}^r} \tag{2.22}$$

are the correlations at the initial time.

Finally, the driving term $[\varepsilon_1(t), F_{\mathrm{I}}^{\pm}(t)]^{(*)}$ in (2.16) is given by

$$[\varepsilon_1(t), F_{\mathrm{I}}^{\pm}(t)]^{(*)}(bb') = \begin{cases} [\varepsilon_1(t), F_{\mathrm{I}}^{\pm}(t)]_-(bb') & \text{if } b \sim c, b' \sim c^{\dagger} \\ [\varepsilon_1(t), F_{\mathrm{I}}^{\pm}(t)]_{\mp}(bb') & \text{if } b, b' \sim c, \\ \pm[\varepsilon_1(t), F_{\mathrm{I}}^{\pm}(t)]_{\mp}(bb') & \text{if } b, b' \sim c^{\dagger} \\ \pm[\varepsilon_1(t), F_{\mathrm{I}}^{\pm}(t)]_-(bb') & \text{if } b \sim c^{\dagger}, b' \sim c, \end{cases} \tag{2.23}$$

where we have used the generalized matrix multiplication (2.12).

Applying the renormalization methods of I to (2.16), one arrives at the following exact transport equation (see I(4.12))

$$\frac{\partial}{\partial t} F_{ll'}^{\pm}(t; bb') + i[\bar{\varepsilon}_1(t), F_{\mathrm{I}}^{\pm}(t)]_{ll'}^{(*)}(bb')$$
$$= \sum_{n=1}^{\infty} \sum_{\substack{m=0 \\ (n,m)\neq(1,0)}}^{\infty} \sum_{G_{nm}}' \sum_{(i_1,\ldots,i_n)}^{(2)} \int_{t_0}^{t} dt_{i_2} \int_{t_0}^{t_{i_2}} dt_{i_3} \cdots \int_{t_0}^{t_{i_{n-1}}} dt_{i_n}$$
$$\times \sum_{\{qn\}} z_G \prod_{i=1}^{n} c_i^{(2)} \bar{v}_i \prod_{j=1}^{m} g_j^{(2)} \bigg|_{t_{i_1}=t}. \tag{2.24}$$

The coefficients $c_i^{(2)}$ and $g_j^{(2)}$ are given by (2.17) and (2.21) if we drop the superscript $^{(0)}$ and use the generalized Kadanoff–Baym ansatz (2.13) for $G_{\gamma}^{\pm}$. Consequently the prime at the summation index and the sum $\sum^{(2)}$ indicate that we are only considering PSD2-diagrams without one-point self-energy insertions (see Section 4 of I for the precise definitions). The replacement of $\varepsilon_1(t)$ by $\bar{\varepsilon}_1(t)$ in the driving term, where

$$\bar{\varepsilon}_{l'l}(t; b'^{\dagger}b^{\dagger}) = \varepsilon_{l'l}(t; b'^{\dagger}b^{\dagger}) + \bar{v}_{l'l}^{\mathrm{HF}}(t; b'^{\dagger}b^{\dagger}), \tag{2.25}$$

corresponds to a renormalization of the unperturbed Hamiltonian by the Hartree–Fock contributions (see I(3.8))

$$\bar{v}_{l'l}^{\mathrm{HF}}(t; b'^{\dagger}b^{\dagger}) = \left\langle \frac{\partial^2 V}{\partial b_l \, \partial b'_{l'}}(t)_{\mathrm{H}} \right\rangle_{\rho(t_0)}. \tag{2.26}$$

In the next section we will use Eq. (2.24) as a starting point for the derivation of a general transport equation in weakly inhomogeneous quantum systems. All lowest order diagrams are explicitly calculated in Section 4 for a quantum solid in high external fields.

3. Weakly Inhomogeneous Quantum Systems

To treat the case of weak inhomogeneities it is most convenient to work in the momentum representation. Thus we choose $l = \mathbf{l}, \alpha$, where $\mathbf{l}$ denotes the quasimomentum of the particle and α are the remaining quantum numbers like spin, polarisation, band index. Here we intend to develop the more general theory for one-particle states which include the potential of a static lattice since this is necessary for a treatment of quantum solids. The continuum case without a lattice is then simply obtained by the limit $a \to 0$, where a is the lattice constant.

To define a weakly inhomogeneous system we have to study translations in space which, for an arbitrary lattice vector $\mathbf{a_n}$, are generated by the unitary operators [9] (we set $\hbar = 1$)

$$e^{i\mathbf{P}^{qs}\mathbf{a_n}}. \tag{3.1}$$

Here $\mathbf{P}^{qs}$ denotes the total quasimomentum operator

$$\mathbf{P}^{qs} = \sum_{l} \sum_{b'b \sim c^\dagger c} \mathbf{l} b'_l b_l \tag{3.2}$$

and $\mathbf{a_n}$ is defined by

$$\mathbf{a_n} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 + n_3 \mathbf{a}_3, \qquad n_i \in Z, \tag{3.3}$$

with $\{\mathbf{a}_i\}_{i=1,2,3}$ being the lattice basis vectors.

From (3.2) one obtains immediately the transformation law for a single creation or annihilation operator

$$e^{i\mathbf{P}^{qs}\mathbf{a_n}} b_l \, e^{-i\mathbf{P}^{qs}\mathbf{a_n}} = e^{i\mathbf{l}^*(b)\,\mathbf{a_n}} b_l, \tag{3.4}$$

where

$$\mathbf{l}^*(b) = \begin{cases} \mathbf{l} & \text{if} \quad b \sim c^\dagger \\ -\mathbf{l} & \text{if} \quad b \sim c. \end{cases} \tag{3.5}$$

For an arbitrary operator A we will use the Fourier-decomposition

$$A = \sum_{\mathbf{q}} A^{\mathbf{q}} \tag{3.6}$$

with

$$e^{i\mathbf{P}^{qs}\mathbf{a_n}} A^{\mathbf{q}} \, e^{-i\mathbf{P}^{qs}\mathbf{a_n}} = e^{i\mathbf{q}\mathbf{a_n}} A^{\mathbf{q}}. \tag{3.7}$$

This can always be achieved by applying (3.4) to the general representation (2.3) of A.

The Hamiltonian (2.1) is written in the form

$$H(t)=H_0+V+H_{ex}(t), \tag{3.8}$$

where H_0+V describes the time independent Hamiltonian of the equilibrium system and $H_{ex}(t)$ includes all external fields. Since H_0+V is translationally invariant we have

$$e^{i\mathbf{P}^{qs}\mathbf{a}_n}(H_0+V)\,e^{-i\mathbf{P}^{qs}\mathbf{a}_n}=H_0+V \tag{3.9}$$

which is also clear from

$$H_0=\sum_l \sum_{b'b\sim c^+c} \varepsilon_l^{(0)} b_l' b_l \tag{3.10}$$

and from the quasimomentum conservation for the matrix elements $\bar{v}(\cdots)$ in (2.3)

$$\bar{v}(l_1\cdots l_r;b^1\cdots b^r)=\sum_{\mathbf{K}} \delta_{\sum_i \mathbf{l}_i^*,\mathbf{K}}\bar{v}(l_1\cdots l_r;b^1\cdots b^r), \tag{3.11}$$

where $\mathbf{K}$ are the vectors of the reciprocal lattice defined by

$$e^{i\mathbf{K}\mathbf{a}_n}=1. \tag{3.12}$$

The property (3.9) implies that all Fourier components $(H_0+V)^{\mathbf{q}}$ are identical to zero if $\mathbf{q}\neq 0$; i.e., H_0+V describes a homogeneous system. However, the external fields, as well as the initial state, may not be translationally invariant in general. The corresponding Fourier components $H_{ex}(t)^{\mathbf{q}}$ and $\rho(t_0)^{\mathbf{q}}$ are assumed to be only relevant for

$$|\mathbf{q}|\leqslant q_0=\frac{2\pi}{\lambda_0}, \tag{3.13}$$

where λ_0 characterizes the wavelength of the inhomogeneities in our system. Since H_0+V commutes with $e^{i\mathbf{P}^{qs}\mathbf{a}_n}$, the same property (3.13) then will also hold for the Fourier components $U(t,t')_{\mathrm{H}}^{\mathbf{q}}$ of the time evolution operator $U(t,t')_{\mathrm{H}}$ of $H(t)$.

Now, let us denote by $A(l_1\cdots l_r;b^1\cdots b^r)=A_{\alpha_1\cdots\alpha_r}(\mathbf{l}_1\cdots\mathbf{l}_r;b^1\cdots b^r)$ any of the functions $F_{ll'}^{\pm}(t;bb')$, $W_{ll'}(t,t';bb')$, $\bar{v}_{ll'}^{\mathrm{HF}}(t;bb')$, $\bar{v}(l_1\cdots l_r;b^1\cdots b^r)$ or $g(t_0;l_1\cdots l_r;b^1\cdots b^r)$ which can occur in our general transport equation (2.24). By using the property (3.13) for $\rho(t_0)^{\mathbf{q}}$ and $U(t,t')_{\mathrm{H}}^{\mathbf{q}}$ together with (3.4) and (3.7), it is straightforward to see that $A(l_1\cdots l_r;b^1\cdots b^r)$ gives a significant contribution only if

$$\sum_{i=1}^{r}\mathbf{l}_i^*(b_i)=-\mathbf{q}-\mathbf{K},\qquad |\mathbf{q}|\leqslant q_0, \tag{3.14}$$

for a proper vector $\mathbf{K}$ of the reciprocal lattice. In the special case $A=\bar{v}$, we have $\mathbf{q}=0$ due to (3.11).

From (3.14) we see that the function $A(l_1 \cdots l_r; b^1 \cdots b^r)$ is very sensitive to changes in the variable $\sum_{i=1}^r \mathbf{l}_i^*$. However, for constant $\sum_{i=1}^r \mathbf{l}_i^*$, it will only vary essentially over a region of order $1/\lambda_f$ or $1/\lambda_V$. Thus, for weak inhomogeneities, that is, $q_0 \sim 1/\lambda_0 \ll 1/\lambda_F$, $1/\lambda_V$ (see (1.2)), we have in good approximation

$$A_{\alpha_1 \cdots \alpha_r}(\mathbf{l}_1 + \mathbf{q}_1, ..., \mathbf{l}_r + \mathbf{q}_r; b^1 \cdots b^r) \cong A_{\alpha_1 \cdots \alpha_r}(\mathbf{l}_1 \cdots \mathbf{l}_r; b^1 \cdots b^r) \tag{3.15}$$

if all $|\mathbf{q}_i| \leqslant q_0$ and $\sum_{i=1}^r \mathbf{q}_i^{(*)} = 0$.

The separation of the variables $\mathbf{l}_1 \cdots \mathbf{l}_r$ in slowly and strongly varying ones is best expressed by introducing the so-called Wigner or phase space transformation A^{ph} of the function A. These new quantities are defined in the variables

$$\mathbf{q} = -\sum_{i=1}^{r} \mathbf{l}_i^*(b^i) + \mathbf{K}, \qquad \tilde{\mathbf{l}}_i = \mathbf{l}_i + \frac{1}{r}\mathbf{q}^*(b^i) \tag{3.16}$$

which are only well defined if $|\mathbf{q}| \sim q_0$ is very small compared to the radius of the first Brioullin zone (1.BZ). However, this is equivalent to $\lambda_0 \gg a$, which corresponds to our condition (1.2) for weak inhomogeneities. Thus, for nearly all variables $\mathbf{l}_i$ in the 1.BZ (up to a negligible region of order q_0), the new variables $\tilde{\mathbf{l}}_i$ will also lie in the 1.BZ.

The generalized phase space transformation is now defined by

$$A^{\text{ph}}_{\alpha_1 \cdots \alpha_r}(\mathbf{q}; \tilde{\mathbf{l}}_1 \cdots \tilde{\mathbf{l}}_r; b^1 \cdots b^r) = \sum_{\mathbf{K}} \delta_{\sum_i \tilde{\mathbf{l}}_i^*(b^i), \mathbf{K}} A_{\alpha_1 \cdots \alpha_r}(\mathbf{l}_1 \cdots \mathbf{l}_r; b^1 \cdots b^r) \tag{3.17}$$

which, for the special case $r = 2$, reduces to

$$A^{\text{ph}}_{\alpha_1\alpha_2}(\mathbf{q}; \tilde{\mathbf{l}}_1 \tilde{\mathbf{l}}_2; b^1 b^2) = \delta_{\tilde{\mathbf{l}}_1^*(b^1) + \tilde{\mathbf{l}}_2^*(b^2), 0} A_{\alpha_1\alpha_2}(\tilde{\mathbf{l}}_1 - \tfrac{1}{2}\mathbf{q}^*(b^1), \tilde{\mathbf{l}}_2 - \tfrac{1}{2}\mathbf{q}^*(b^2); b^1 b^2). \tag{3.18}$$

The conventional Wigner transformation [22] corresponds to $b^1 b^2 = cc^\dagger$,

$$A^{\text{ph}}_{\alpha_1\alpha_2}(\mathbf{q}; \tilde{\mathbf{l}}_1 \tilde{\mathbf{l}}_2; cc^\dagger) = \delta_{\tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2} A_{\alpha_1\alpha_2}(\tilde{\mathbf{l}}_1 + \tfrac{1}{2}\mathbf{q}, \tilde{\mathbf{l}}_2 - \tfrac{1}{2}\mathbf{q}; cc^\dagger). \tag{3.19}$$

Equations (3.14), (3.15), and (3.17) imply the following properties for the Wigner function A^{ph},

$$A^{\text{ph}}_{\alpha_1 \cdots \alpha_r}(\mathbf{q}; \tilde{\mathbf{l}}_1 \cdots \tilde{\mathbf{l}}_r; b^1 \cdots b^r) \sim \sum_{\mathbf{K}} \delta_{\sum_i \mathbf{l}_i^*, \mathbf{K}} \tag{3.20}$$

$$A^{\text{ph}}_{\alpha_1 \cdots \alpha_r}(\mathbf{q}; \tilde{\mathbf{l}}_1 \cdots \tilde{\mathbf{l}}_r; b^1 \cdots b^r) \cong 0 \qquad \text{if} \quad |\mathbf{q}| \gg q_0 \tag{3.21}$$

and

$$A^{\text{ph}}_{\alpha_1 \cdots \alpha_r}(\mathbf{q}; \tilde{\mathbf{l}}_1 + \mathbf{q}_1, ..., \tilde{\mathbf{l}}_r + \mathbf{q}_r; b^1 \cdots b^r) \cong A^{\text{ph}}_{\alpha_1 \cdots \alpha_r}(\mathbf{q}; \tilde{\mathbf{l}}_1 \cdots \tilde{\mathbf{l}}_r; b^1 \cdots b^r) \tag{3.22}$$

if $|\mathbf{q}_i| \sim q_0$ and $\sum_{i=1}^r \mathbf{q}_i^*(b^i) = 0$.

Furthermore, we will define the lattice Fourier transformation of A^{ph} by

$$A^{\text{ph}}_{\alpha_1\cdots\alpha_r}(\mathbf{a}_n;\bar{\mathbf{l}}_1\cdots\bar{\mathbf{l}}_r;b^1\cdots b^r)=\sum_{\mathbf{q}} e^{i\mathbf{q}\mathbf{a}_n}A^{\text{ph}}_{\alpha_1\cdots\alpha_r}(\mathbf{q};\bar{\mathbf{l}}_1\cdots\bar{\mathbf{l}}_r;b^1\cdots b^r). \tag{3.23}$$

According to (3.21) the spatial variation of the function $A^{\text{ph}}_{\alpha_1\cdots\alpha_r}(\mathbf{a}_n;\dots)$ is given by $1/q_0\sim\lambda_0\gg a$. Thus it does not vary essentially over the lattice constant and the discret variable $\mathbf{a}_n$ can be replaced by the continuum variable $\mathbf{x}$.

All quantum numbers in our general transport equation (2.24) (except the two external ones) are contracted according to the generalized matrix multiplication (2.12). Therefore, in order to study the Wigner transformation of (2.24), we first have to consider the phase space transformation of the general operator product

$$\begin{aligned}(AB)&(l_1\cdots l_n;b^1\cdots b^n)\\&=\sum_{\bar{l}_1\cdots\bar{l}_s}\sum_{\bar{b}^1\cdots\bar{b}^s}A(l_1\cdots l_r\bar{l}_1\cdots\bar{l}_s;b^1\cdots b^r\bar{b}^{1\dagger}\cdots\bar{b}^{s\dagger})\\&\quad\times B(\bar{l}_1\cdots\bar{l}_s l_{r+1}\cdots l_n;\bar{b}^1\cdots\bar{b}^s b^{r+1}\cdots b^n)\end{aligned} \tag{3.24}$$

From (3.17) we immediately obtain

$$\begin{aligned}(AB)&^{\text{ph}}_{\alpha_1\cdots\alpha_n}(\mathbf{q};\bar{\mathbf{l}}_1\cdots\bar{\mathbf{l}}_n;b^1\cdots b^n)\\&=\sum_{\mathbf{q}_1\mathbf{q}_2}\delta_{\mathbf{q},\mathbf{q}_1+\mathbf{q}_2}\sum_{\bar{\alpha}_1\cdots\bar{\alpha}_s}\sum_{\bar{l}_1\cdots\bar{l}_s}\sum_{\bar{b}^1\cdots\bar{b}^s}A^{\text{ph}}_{\alpha_1\cdots\alpha_r\bar{\alpha}_1\cdots\bar{\alpha}_s}\\&\quad\times\Bigg(\mathbf{q}_1;\bar{\mathbf{l}}_1-\frac{1}{n}\mathbf{q}^*(b^1)+\frac{1}{r+s}\mathbf{q}_1^*(b^1),\dots,\bar{\mathbf{l}}_r-\frac{1}{n}\mathbf{q}^*(b^r)+\frac{1}{r+s}\mathbf{q}_1^*(b^r),\\&\quad\bar{\mathbf{l}}_1+\frac{1}{r+s}\mathbf{q}_1^*(\bar{b}^{1\dagger}),\dots,\bar{\mathbf{l}}_s+\frac{1}{r+s}\mathbf{q}_1^*(\bar{b}^{s\dagger});b^1\cdots b^r\bar{b}^{1\dagger}\cdots\bar{b}^{s\dagger}\Bigg)\\&\quad\times B^{\text{ph}}_{\bar{\alpha}_1\cdots\bar{\alpha}_s\alpha_{r+1}\cdots\alpha_n}\Bigg(\mathbf{q}_2;\bar{\mathbf{l}}_1+\frac{1}{s+n-r}\mathbf{q}_2^*(\bar{b}^1),\dots,\bar{\mathbf{l}}_s+\frac{1}{s+n-r}\mathbf{q}_2^*(\bar{b}^s),\\&\quad\bar{\mathbf{l}}_{r+1}-\frac{1}{n}\mathbf{q}^*(b^{r+1})+\frac{1}{s+n-r}\mathbf{q}_2^*(b^{r+1}),\dots,\\&\quad\bar{\mathbf{l}}_n-\frac{1}{n}\mathbf{q}^*(b^n)+\frac{1}{s+n-r}\mathbf{q}_2^*(b^n);\bar{b}^1\cdots\bar{b}^s b^{r+1}\cdots b^n\Bigg).\end{aligned} \tag{3.25}$$

Transforming the summation variables $\bar{\mathbf{l}}_1\cdots\bar{\mathbf{l}}_s$ according to

$$\bar{\mathbf{l}}_i\to\bar{\mathbf{l}}_i+\frac{1}{s}\mathbf{q}_1^*(\bar{b}^i)-\frac{r}{sn}\mathbf{q}^*(\bar{b}^i), \tag{3.26}$$

using $\mathbf{q}_1+\mathbf{q}_2=\mathbf{q}$, (3.22), and (3.23), we obtain in lowest order approximation

$$\begin{aligned}(AB)^{\text{ph}}_{\alpha_1\cdots\alpha_n}&(\mathbf{x};\tilde{\mathbf{l}}_1\cdots\tilde{\mathbf{l}}_n;b^1\cdots b^n)\\ &\cong\sum_{\bar{l}_1\cdots\bar{l}_s}\sum_{\bar{b}^1\cdots\bar{b}^s}A^{\text{ph}}_{\alpha_1\cdots\alpha_r\bar{\alpha}_1\cdots\bar{\alpha}_s}(\mathbf{x};\tilde{\mathbf{l}}_1\cdots\tilde{\mathbf{l}}_r\bar{\mathbf{l}}_1\cdots\bar{\mathbf{l}}_s;b^1\cdots b^r\bar{b}^{1\dagger}\cdots\bar{b}^{s\dagger})\\ &\quad\times B^{\text{ph}}_{\bar{\alpha}_1\cdots\bar{\alpha}_s\alpha_{r+1}\cdots\alpha_n}(\mathbf{x};\bar{l}_1\cdots\bar{l}_s\tilde{\mathbf{l}}_{r+1}\cdots\tilde{\mathbf{l}}_n;\bar{b}^1\cdots\bar{b}^s b^{r+1}\cdots b^n)\end{aligned}\tag{3.27}$$

or in operator form,

$$(AB)^{\text{ph}}(\mathbf{x})=A^{\text{ph}}(\mathbf{x})\,B^{\text{ph}}(\mathbf{x}),\tag{3.28}$$

where all quantities

$$[A^{\text{ph}}(\mathbf{x})]_{\alpha_1\cdots\alpha_r}(\tilde{\mathbf{l}}_1\cdots\tilde{\mathbf{l}}_r;b^1\cdots b^r)\doteq A^{\text{ph}}_{\alpha_1\cdots\alpha_r}(\mathbf{x};\tilde{\mathbf{l}}_1\cdots\tilde{\mathbf{l}}_r;b^1\cdots b^r)\tag{3.29}$$

are homogeneous operators in the sense that

$$\sum_{i=1}^{r}\mathbf{l}_i^*(b^i)=\mathbf{K},\tag{3.30}$$

according to (3.19).

Iterating this procedure for the r.h.s. of Eq. (2.24), we see that the whole phase space transformed collision integral can be written as if the whole system obeys an exact quasimomentum conservation (up to a vector of the reciprocal lattice) at each line, vertex, and correlation bubble. On the other hand, the inhomogeneity is simply expressed by a single parameter $\mathbf{x}$ and we obtain

$$\begin{aligned}\frac{\partial}{\partial t}&F^{\pm\text{ph}}_{\alpha\alpha'}(\mathbf{x};\mathbf{ll}';t;bb')+i[\bar{\varepsilon}_1(t),F^{\pm}_1(t)]^{(*)\,\text{ph}}_{\alpha\alpha'}(\mathbf{x};\mathbf{ll}';bb')\\ &=\sum_{n=1}^{\infty}\sum_{\substack{m=0\\(n,m)\neq(1,0)}}^{\infty}\sum_{G_{nm}}{}'\sum^{(2)}_{(i_1,\ldots,i_n)}\int_{t_0}^{t}dt_{i_2}\int_{t_0}^{t_{i_2}}dt_{i_3}\cdots\int_{t_0}^{t_{i_{n-1}}}dt_{i_n}\\ &\quad\times\sum^{(H)}_{\{qn\}}z_G\prod_{i=1}^{n}c_i^{(2)\,\text{ph}}(\mathbf{x})\,\bar{v}_i\prod_{j=1}^{m}g_j^{(2)\,\text{ph}}(\mathbf{x})\bigg|_{t_{i_1}=t},\end{aligned}\tag{3.31}$$

where due to (3.18)

$$\mathbf{l}^*(b)+\mathbf{l}'^*(b')=0\tag{3.32}$$

and $\sum^{(H)}_{\{qn\}}$ means that all wave vectors have to be chosen like in the homogeneous case. $c_i^{(2)\,\text{ph}}(\mathbf{x})$ and $g_j^{(2)\,\text{ph}}(\mathbf{x})$ are given by (see (2.17), (2.21), and (2.13))

$$c_i^{(2)\,\text{ph}}(\mathbf{x})=\prod_{\substack{s\\t_i=\tau_s=\tau'_s}}\tilde{G}_s^{+\text{ph}}(\mathbf{x})\left\{\prod_{\substack{s\\t_i=\tau_s<\tau'_s}}\tilde{G}_s^{+\text{ph}}(\mathbf{x})\prod_{\substack{s\\t_i=\tau'_s<\tau_s}}\tilde{G}_s^{-\text{ph}}(\mathbf{x})-(+\leftrightarrow-)\right\}\tag{3.33}$$

and

$$g_j^{(2)\,\mathrm{ph}}(\mathbf{x}) = \sum_{\bar{l}_1 \cdots \bar{l}_r} \sum_{\bar{b}^1 \cdots \bar{b}^r} \prod_{i=1}^{r} W^{\mathrm{ph}}_{\alpha_i \bar{\alpha}_i}(\mathbf{x}; l_i \bar{l}_i; t_i t_0; b^i \bar{b}^{i\dagger}) \times g^{\mathrm{ph}}_{\bar{\alpha}_1 \cdots \bar{\alpha}_r}(\mathbf{x}; \bar{\mathbf{l}}_1 \cdots \bar{\mathbf{l}}_r; \bar{b}^1 \cdots \bar{b}^r), \tag{3.34}$$

where

$$\tilde{G}^{\pm\mathrm{ph}}(\mathbf{x}; tt') = W^{\mathrm{ph}}(\mathbf{x}; tt^<)\, F^{\pm\mathrm{ph}}(\mathbf{x}; t^<)\, W^{\mathrm{ph}}(\mathbf{x}; t^< t')^{\mathrm{T}} \tag{3.35}$$

corresponds to the generalized Kadanoff–Baym ansatz of $G^{\pm\mathrm{ph}}(\mathbf{x}; t, t')$.

For the evaluation of the driving term on the l.h.s. of Eq. (3.31) we will go beyond the approximation (3.27). This is necessary since the leading terms can cancel each other if $[\bar{\varepsilon}_1(t), F_1^{\pm}(t)]^{(*)}$, given by (2.23), is identical to the commutator.

For this purpose, let us first eliminate one of the variables $\tilde{\mathbf{l}}_1$ and $\tilde{\mathbf{l}}_2$ in (3.18) by defining

$$A^{\mathrm{ph}}_{\alpha\alpha'}(\mathbf{q}, \mathbf{l}; bb') = A^{\mathrm{ph}}_{\alpha\alpha'}(\mathbf{q}, -\mathbf{l}^*(b), \mathbf{l}^*(b'); bb'). \tag{3.36}$$

Using this function in (3.25) we obtain exactly

$$(AB)^{\mathrm{ph}}_{\alpha\alpha'}(\mathbf{q}, \mathbf{l}; bb') = \sum_{\mathbf{q}_1 \mathbf{q}_2} \delta_{\mathbf{q}, \mathbf{q}_1 + \mathbf{q}_2} \sum_{\bar{\alpha}} \sum_{\bar{b}} A^{\mathrm{ph}}_{\alpha\bar{\alpha}}(\mathbf{q}_1, \mathbf{l} + \tfrac{1}{2}\mathbf{q}_2; b\bar{b}^\dagger)\, B^{\mathrm{ph}}_{\bar{\alpha}\alpha'}(\mathbf{q}_2, \mathbf{l} - \tfrac{1}{2}\mathbf{q}_1; \bar{b}b') \tag{3.37}$$

or

$$(AB)^{\mathrm{ph}}_{\alpha\alpha'}(\mathbf{q}, \mathbf{l}; bb') = \sum_{\mathbf{q}_1 \mathbf{q}_2} \delta_{\mathbf{q}, \mathbf{q}_1 + \mathbf{q}_2} \sum_{\bar{\alpha}} \sum_{\bar{b}} e^{(1/2)(\mathbf{q}_2(\partial^A/\partial \mathbf{l}) - \mathbf{q}_1(\partial^B/\partial \mathbf{l}))} A^{\mathrm{ph}}_{\alpha\bar{\alpha}}(\mathbf{q}_1, \mathbf{l}; b\bar{b}^\dagger)\, B^{\mathrm{ph}}_{\bar{\alpha}\alpha'}(\mathbf{q}_2, \mathbf{l}; \bar{b}b') \tag{3.38}$$

where $\partial^A/\partial \mathbf{l}$ acts on A^{ph} and $\partial^B/\partial \mathbf{l}$ on B^{ph}. The Fourier transformation of (3.38) gives

$$(AB)^{\mathrm{ph}}_{\alpha\alpha'}(\mathbf{x}, \mathbf{l}; bb') = \sum_{\bar{\alpha}} \sum_{\bar{b}} e^{(1/2i)((\partial^B/\partial \mathbf{x})(\partial^A/\partial \mathbf{l}) - (\partial^A/\partial \mathbf{x})(\partial^B/\partial \mathbf{l}))} A^{\mathrm{ph}}_{\alpha\bar{\alpha}}(\mathbf{x}, \mathbf{l}; b\bar{b}^\dagger)\, B^{\mathrm{ph}}_{\bar{\alpha}\alpha'}(\mathbf{x}, \mathbf{l}; \bar{b}b') \tag{3.39}$$

and the driving term up to first order can be written as

$$i[\bar{\varepsilon}_1(t), F_1^{\pm}(t)]^{(*)\,\mathrm{ph}} = i[\bar{\varepsilon}_1^{\mathrm{ph}}(t), F_1^{\pm\mathrm{ph}}(t)]^{(*)} + \frac{1}{2}\left[\frac{\partial \bar{\varepsilon}_1^{\mathrm{ph}}}{\partial \mathbf{l}}(t), \frac{\partial F_1^{\pm\mathrm{ph}}}{\partial \mathbf{x}}(t)\right]^{(**)} - \frac{1}{2}\left[\frac{\partial \bar{\varepsilon}_1^{\mathrm{ph}}}{\partial \mathbf{x}}(t), \frac{\partial F_1^{\pm\mathrm{ph}}}{\partial \mathbf{l}}(t)\right]^{(**)}, \tag{3.40}$$

where $[\cdots]^{(**)}$ is defined in the same way as $[\cdots]^{(*)}$ if one interchanges commutators with anticommutators. The generalized matrix multiplication in (3.40) is defined by

$$(A^{\text{ph}}B^{\text{ph}})_{\alpha\alpha'}(\mathbf{x},\mathbf{l};bb')=\sum_{\bar{\alpha}}\sum_{\bar{b}}A^{\text{ph}}_{\alpha\bar{\alpha}}(\mathbf{x},\mathbf{l};b\bar{b}^{\dagger})\,B^{\text{ph}}_{\bar{\alpha}\alpha'}(\mathbf{x},\mathbf{l};bb'). \tag{3.41}$$

Equation (3.31), together with (3.40), is the final result of this section. It constitutes a set of nonlinear transport equations for all single-time Green's functions in an arbitrary weakly inhomogeneous quantum many-body system. Except for a correction to the driving term it has locally exactly the same structure as the original transport equation (2.24). One simply has to replace all quantities by their corresponding generalized phase space transformation (3.23) with fixed parameter $\mathbf{x}$ and relations for the wave vectors like in a homogeneous system. Moreover, for given $\mathbf{x}$, the whole equation is independent of the representation for the remaining variables. Especially in Section 4 we will see that the conventional transport equations of solid state theory are obtained if one evaluates (3.31) with local and instantenous eigenfunctions of the unperturbed Hamiltonian $\varepsilon_1^{\text{ph}}(\mathbf{x}, t)$ in Wigner space including the external fields. Finally, let us mention that our transport equation (3.31) does not involve any gradient expansion in t and is therefore valid for arbitrarily time dependent external fields.

4. Quantum Solids

In this section we will apply our general transport equation (3.31) to a weakly inhomogeneous quantum solid in high electromagnetic fields. Thereby all possible interactions between Bloch electrons, phonons, and impurities are taken into account. Thus, if the external fields are described by the electromagnetic potentials $\mathbf{A}(\mathbf{x}, t)$ and $\varphi(\mathbf{x}, t)$, the Hamiltonian is given by

$$H(t)=H_0+V+H_{\text{ex}}(t) \tag{4.1}$$

with

$$H_0=H_0^{\text{e}}+H_0^{\text{p}} \tag{4.2}$$

$$V=V_{\text{ee}}+V_{\text{ep}}+V_{\text{ei}}+V_{\text{pp}}+V_{\text{pi}} \tag{4.3}$$

$$H_{\text{ex}}(t)=\sum_{j=1}^{N_e}\frac{1}{2m}\left[\left(\hat{\mathbf{p}}_j-\frac{e}{c}\mathbf{A}(\hat{\mathbf{x}}_j,t)\right)^2\right]+e\varphi(\hat{\mathbf{x}}_j,t). \tag{4.4}$$

Here the super- or subscripts e, p, and i refer to the electrons, phonons, and impurities, respectively. Thus H_0^{e} is the unperturbed Hamiltonian of the electrons,

V_{ee} describes the electron–electron interaction, etc. The space and momentum operators of the jth electron are given by $\hat{\mathbf{x}}_j$ and $\hat{\mathbf{p}}_j$.

For the unperturbed Hamiltonians we can write

$$H_0^e = \sum_k \varepsilon_k^{(0)} c_k^\dagger c_k \tag{4.5}$$

$$H_0^p = \sum_q \omega_q^{(0)} a_q^\dagger a_q, \tag{4.6}$$

where $k = \mathbf{k}$, $\alpha = \mathbf{k}, l, \sigma$ characterizes the quasimomentum, band index, and spin of the Bloch electron, and $q = \mathbf{q}, s$ describes the quasimomentum and polarisation of the phonon. Using this basis, the external Hamiltonian and the interaction terms read

$$H_{ex}(t) = \sum_{kk'} \varepsilon_{k'k}^{ex}(t)\, c_{k'}^\dagger c_k \tag{4.7}$$

with

$$\varepsilon_{k'k}^{ex}(t) = \langle k' |\, \frac{1}{2m}\left[\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t)\right)^2 - \hat{\mathbf{p}}^2\right] + e\varphi(\mathbf{x}, t)\, | k \rangle \tag{4.8}$$

and

$$V_{ee} = \frac{1}{2!\,2!} \sum_{k_1 k_2 k_1' k_2'} \bar{v}_{k_1' k_2', k_1 k_2} c_{k_1'}^\dagger c_{k_2'}^\dagger c_{k_2} c_{k_1} \tag{4.9}$$

$$V_{ep} = \sum_{kk'q} \bar{v}_{k',k}^q c_{k'}^\dagger c_k (a_q + a_{-q}^\dagger) \tag{4.10}$$

$$V_{ei} = \sum_{kk'} \bar{v}_{k',k} c_{k'}^\dagger c_k \tag{4.11}$$

$$V_{pp} = \frac{1}{3!} \sum_{q_1 q_2 q_3} \bar{v}^{q_1 q_2 q_3} N\{(a_{q_1} + a_{-q_1}^\dagger)(a_{q_2} + a_{-q_2}^\dagger)(a_{q_3} + a_{-q_3}^\dagger)\} \tag{4.12}$$

$$V_{pi} = \frac{1}{2!} \sum_{q_1 q_2} \bar{w}_1^{q_1 q_2} N\{(a_{q_1} + a_{-q_1}^\dagger)(a_{q_2} + a_{-q_2}^\dagger)\} + \frac{1}{2!} \sum_{q_1 q_2} \bar{w}_2^{q_1 q_2} N\{(a_{q_1} - a_{-q_1}^\dagger)(a_{q_2} - a_{-q_2}^\dagger)\}. \tag{4.13}$$

Explicit expressions for all matrix elements can be found in standard textbooks on solid state theory [9, 10, 24]. The second term of (4.13) accounts for the correction to the kinetic energy due to the presence of impurities [23]. The dependence of the matrix elements $\bar{v}_{k',k}$ and $\bar{w}_i^{q_1 q_2}$ ($i = 1, 2$) on the impurity positions $\mathbf{R}_j$ ($j = 1, ..., N_i$) is given by

$$\bar{v}_{k',k} = \sum_{\mathbf{q}} \sum_{j=1}^{N_i} e^{i\mathbf{q}\mathbf{R}_j} \tilde{v}_{k',k}(\mathbf{q}) \tag{4.14}$$

$$\bar{w}_i^{q_1 q_2} = \sum_{\mathbf{q}} \sum_{j=1}^{N_i} e^{i\mathbf{q}\mathbf{R}_j} \tilde{w}_i^{q_1 q_2}(\mathbf{q}), \tag{4.15}$$

where $\tilde{v}_{k',k}(\mathbf{q})$ and $\tilde{w}_i^{q_1 q_2}(\mathbf{q})$ fulfil the quasimomentum conservation

$$\tilde{v}_{k',k}(\mathbf{q}) \sim \sum_{\mathbf{K}} \delta_{\mathbf{k}' - \mathbf{k} + \mathbf{q}, \mathbf{K}} \tag{4.16}$$

$$\tilde{w}_i^{q_1 q_2}(\mathbf{q}) \sim \sum_{\mathbf{K}} \delta_{\mathbf{q} - \mathbf{q}_1 - \mathbf{q}_2, \mathbf{K}} \tag{4.17}$$

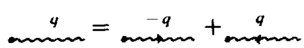
which is also valid for all the other matrix elements $\bar{v}_{k_1' k_2', k_1 k_2}$, $\bar{v}^q_{k',k}$, and $\bar{v}^{q_1 q_2 q_3}$, according to (3.11).

If we average the perturbation expansion (2.16) over the positions of the impurities, the exponential factors in (4.14) and (4.15) will lead to certain relationships between the vectors $\mathbf{q}$. For example, if we pick the same summation index out of s different matrix elements (4.14) or (4.15), we obtain

$$\left(\frac{1}{N_z}\right)^{N_i} \sum_{\mathbf{R}_1 \cdots \mathbf{R}_{N_i}} \sum_{j=1}^{N_i} e^{i\mathbf{q}_1\mathbf{R}_j} \cdot \ldots \cdot e^{i\mathbf{q}_s\mathbf{R}_j} = N_i \sum_{\mathbf{K}} \delta_{\mathbf{q}_1 + \cdots + \mathbf{q}_s, \mathbf{K}}, \tag{4.18}$$

where N_Z is the number of unit cells. Such a contribution is represented by a connection of s dashed lines at a cross (see Fig. 4.2), whereas $\tilde{v}_{k',k}(\mathbf{q})$ and $\tilde{w}_i^{q_1 q_2}(\mathbf{q})$ are associated with the vertices of Fig. 4.1. Adding these graphical elements to the general Feynman rules and using the same renormalization methods as in I, we will obtain the transport equation (2.24) in the same way for the impurity-averaged Green's functions.

Furthermore, due to the form of the interaction terms, we will use here a more compact notation for the phonon lines. For a vertex from $\tilde{v}_{k',k}(\mathbf{q})$ or $\bar{v}^{q_1 q_2 q_3}$ they are defined by



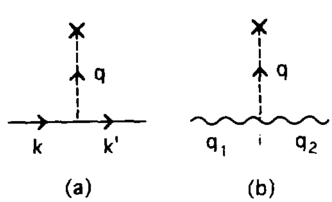


FIG. 4.1. Representation of the matrix elements $\tilde{v}_{k',k}(\mathbf{q})$ (a) and $\tilde{w}_i^{q_1 q_2}(\mathbf{q})$ (b).

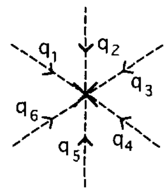
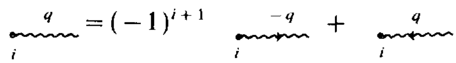


FIG. 4.2. Impurity averaging. The cross represents the contribution $N_i\,\delta_{\mathbf{q}_1+\cdots+\mathbf{q}_6,\,\mathbf{K}}$.

and for a vertex from $\tilde{w}_i^{q_1 q_2}(\mathbf{q})$ by



according to the form (4.13).

Now let us check if the conditions (3.13) and (1.2) for the Fourier components $H^{\mathbf{q}}_{\mathrm{ex}}(t)$ are valid. Otherwise we are not allowed to use the transport equation (3.31). Obviously, (3.13) and (1.2) are correct if the electromagnetic potentials are of the form

$$\varphi(\mathbf{x}, t) = \varphi_0\, e^{i(\mathbf{qx} - \omega t)}, \qquad \mathbf{A}(\mathbf{x}, t)\, \mathbf{A}_0\, e^{i(\mathbf{qx} - \omega t)} \tag{4.19}$$

with $1/|\mathbf{q}| \gg \lambda_F, \lambda_V, a$. However, they can as well be proportional to the coordinate $\mathbf{x}$ itsself as it is, e.g., the case for homogeneous fields

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \tfrac{1}{2}\mathbf{B} \wedge \mathbf{x} \\ \varphi(\mathbf{x}) &= -\mathbf{xE}. \end{aligned} \tag{4.20}$$

These functions are certainly not slowly varying and we have to consider the Fourier components $H^{\mathbf{q}}_{\mathrm{ex}}$ in more detail. From (3.6), (3.7), (4.7), and (4.8) we have

$$H^{\mathbf{q}}_{\mathrm{ex}} = \sum_{kk'} \delta_{\mathbf{k}',\,\mathbf{k}+\mathbf{q}} \langle \mathbf{k}+\mathbf{q}, \alpha' |\, \frac{1}{2m}\left[\left(\hat{\mathbf{p}} - \frac{e}{2c}\mathbf{B} \wedge \hat{\mathbf{x}}\right)^2 - \hat{\mathbf{p}}^2\right] - e\mathbf{xE}\, |\mathbf{k}\alpha\rangle\, c^\dagger_{k'} c_k. \tag{4.21}$$

Using

$$\langle \mathbf{k}'\alpha' |\, \hat{\mathbf{p}}\, |\mathbf{k}\alpha\rangle \sim \delta_{\mathbf{k},\,\mathbf{k}'} \tag{4.22}$$

and

$$\langle \mathbf{k}+\mathbf{q}, \alpha' |\, \hat{\mathbf{x}}\, |\mathbf{k}\alpha\rangle \sim 1/|\mathbf{q}|, \tag{4.23}$$

we see that $H^{\mathbf{q}}_{\mathrm{ex}}$ can be neglected if

$$\frac{eB}{c\,|\mathbf{q}|} \ll k_F \sim \frac{1}{\lambda_F}, \qquad eE\,\frac{1}{|\mathbf{q}|} \ll \varepsilon_F \tag{4.24}$$

which are exactly the conditions (1.3) and (1.4) for $|\mathbf{q}| \sim \lambda_F, \lambda_V$, or a. Consequently, (3.13) and (1.2) are also valid for electromagnetic potentials of the form (4.20).

In the next step we consider Eq. (3.31) for the electrons ($bb' = cc^\dagger$) and the phonons ($bb' = aa^\dagger$) separately,

$$\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t} + \left(\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t}\right)_D = \left(\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t}\right)_C \tag{4.25}$$

$$\frac{\partial n^{\pm}_{ss'}}{\partial t} + \left(\frac{\partial n^{\pm}_{ss'}}{\partial t}\right)_D = \left(\frac{\partial n^{\pm}_{ss'}}{\partial t}\right)_C, \tag{4.26}$$

where (see (3.36))

$$f^{\pm}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t) = F^{\pm\mathrm{ph}}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t; cc^\dagger) \tag{4.27}$$

$$n^{\pm}_{ss'}(\mathbf{x}, \mathbf{q}, t) = F^{\pm\mathrm{ph}}_{ss'}(\mathbf{x}, \mathbf{q}, t; aa^\dagger) \tag{4.28}$$

and the symbols D and C denote the driving term and the collision integral of (3.31). Depending on the various interaction terms, we can further write

$$\left(\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t}\right)_C = \left(\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{\mathrm{ee}} + \left(\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{\mathrm{ep}} + \left(\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{\mathrm{ei}} \tag{4.29}$$

$$\left(\frac{\partial n^{\pm}_{ss'}}{\partial t}\right)_C = \left(\frac{\partial n^{\pm}_{ss'}}{\partial t}\right)_{\mathrm{pp}} + \left(\frac{\partial n^{\pm}_{ss'}}{\partial t}\right)_{\mathrm{pe}} + \left(\frac{\partial n^{\pm}_{ss'}}{\partial t}\right)_{\mathrm{pi}}, \tag{4.30}$$

For the retarded or advanced Green's functions of the electrons and phonons we use the notation

$$G^{\mathrm{e}}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t, t') = G^{B\mathrm{ph}}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t, t'; cc^\dagger) \tag{4.31}$$

$$G^{\mathrm{p}}_{ss'}(\mathbf{x}, \mathbf{q}, t, t') = G^{B\mathrm{ph}}_{ss'}(\mathbf{x}, \mathbf{q}, t, t'; aa^\dagger). \tag{4.32}$$

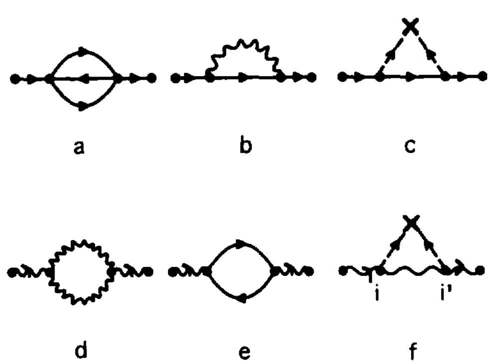


FIG. 4.3. Born approximation of the collision integral.

The lowest order diagrams of the collision integral in Born approximation are illustrated in Figs. 4.3a, b, c for the electrons and in Figs. 4.3d, e, f for the phonons. All higher orders in the interaction strength will be disregarded. Among them are also all contributions from the correlation parts and from phonon Green's functions with $bb'=aa$ or $bb'=a^\dagger a^\dagger$. This can explicitly be seen if the initial density matrix $\rho(t_0)$ is given by the grand canonical ensemble in equilibrium

$$\rho(t_0)=\rho_{\mathrm{eq}}=\frac{e^{-\beta K_{\mathrm{eq}}}}{\mathrm{Tr}\, e^{-\beta K_{\mathrm{eq}}}} \tag{4.33}$$

with $\beta=1/kT$ and $K_{\mathrm{eq}}=H_0+V-\mu\hat{N}$. As a consequence we have

$$F^{\pm}_{qq'}(t;aa),\; F^{\pm}_{qq'}(t;a^\dagger a^\dagger)\sim O(V) \tag{4.34}$$

since H_0 and H_{ex} conserve the phonon particle number, and

$$g(t_0;l_1\cdots l_r;b^1\cdots b^r)\sim O(V) \tag{4.35}$$

due to the factorization property I(2.41).

Using our general Feynman rules and the hermitian property for all potential matrix elements, we obtain the following expressions for the graphs of Fig. 4.3:

$$\begin{aligned}\left(\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{\mathrm{ee}}&(\mathbf{x},\mathbf{k},t)\\ &=\mathrm{Re}\int_{t_0}^{t}dt'[\langle\mathbf{k}\alpha|\,\mathrm{Tr}_2^{(\mathrm{e})}\bar{v}_{12}G^{\mathrm{e}}_1(\mathbf{x},t,t')\,f^{-}_1(\mathbf{x},t)\\ &\quad\times G^{\mathrm{e}}_2(\mathbf{x},t,t')\,f^{-}_2(\mathbf{x},t')\,\bar{v}_{12}f^{-}_1(\mathbf{x},t')\,G^{\mathrm{e}}_1(\mathbf{x},t',t)\\ &\quad\times f^{+}_2(\mathbf{x},t')\,G^{\mathrm{e}}_2(\mathbf{x},t',t)\,|\mathbf{k}\alpha'\rangle-(+\leftrightarrow-)]\end{aligned} \tag{4.36}$$

$$\begin{aligned}\left(\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{\mathrm{ep}}&(\mathbf{x},\mathbf{k},t)\\ &=-2\,\mathrm{Re}\int_{t_0}^{t}dt'\sum_{\mathbf{q}}\sum_{ss'}[\langle\mathbf{k}\alpha|\,\bar{v}_1^{-\mathbf{q}s}G^{\mathrm{e}}_1(\mathbf{x},t,t')\,f^{-}_1(\mathbf{x},t')\,\bar{v}_1^{\mathbf{q}s'}\\ &\quad\times f^{+}_1(\mathbf{x},t')\,G^{\mathrm{e}}_1(\mathbf{x},t',t)\,|\mathbf{k}\alpha'\rangle\cdot(\langle\mathbf{q}s'|\,n^{+}_1(\mathbf{x},t')\,G^{\mathrm{p}}_1(\mathbf{x},t',t)\,|\mathbf{q}s\rangle\\ &\quad+\langle-\mathbf{q}s|\,G^{\mathrm{p}}_1(\mathbf{x},t,t')\,n^{-}_1(\mathbf{x},t')\,|-\mathbf{q}s'\rangle)-(+\leftrightarrow-)]\end{aligned} \tag{4.37}$$

$$\begin{aligned}\left(\frac{\partial f^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{\mathrm{ei}}&(\mathbf{x},\mathbf{k},t)\\ &=-2\,\mathrm{Re}\int_{t_0}^{t}dt'N_i\sum_{\mathbf{q}_1\mathbf{q}_2}\sum_{\mathbf{K}}\delta_{\mathbf{q}_1+\mathbf{q}_2,\mathbf{K}}[\langle\mathbf{k}\alpha|\,\bar{v}_1(\mathbf{q}_1)\\ &\quad\times G^{\mathrm{e}}_1(\mathbf{x},t,t')\,f^{-}_1(\mathbf{x},t')\,\bar{v}_1(\mathbf{q}_2)\,f^{+}_1(\mathbf{x},t')\\ &\quad\times G^{\mathrm{e}}_1(\mathbf{x},t',t)\,|\mathbf{k}\alpha'\rangle-(+\leftrightarrow-)]\end{aligned} \tag{4.38}$$

$$\left(\frac{\partial n_{ss'}^{\pm}}{\partial t}\right)_{\mathrm{pp}}(\mathbf{x}, \mathbf{q}, t)$$

$$= -\mathrm{Re}\int_{t_0}^{t} dt' \sum_{\mathbf{q}_2\mathbf{q}_3} \sum_{\substack{s_1s_2s_3 \\ s_2's_3'}} \bar{v}^{\mathbf{q}s_1,\,\mathbf{q}_2s_2,\,\mathbf{q}_3s_3}\bar{v}^{-\mathbf{q}s,\,-\mathbf{q}_2s_2',\,-\mathbf{q}_3s_3'}$$
$$\times \{[\langle \mathbf{q}s_1 |\, n_1^+(\mathbf{x}, t')\, G_1^{\mathrm{p}}(\mathbf{x}, t', t)\, |\mathbf{q}s'\rangle (2\langle \mathbf{q}_2s_2 |\, n_1^+(\mathbf{x}, t')\, G_1^{\mathrm{p}}(\mathbf{x}, t', t)\, |\mathbf{q}_2s_2'\rangle$$
$$+ \langle -\mathbf{q}_2s_2' |\, G_1^{\mathrm{p}}(\mathbf{x}, t, t')\, n_1^-(\mathbf{x}, t')\, |-\mathbf{q}_2s_2'\rangle)$$
$$\times \langle -\mathbf{q}_3s_3' |\, G_1^{\mathrm{p}}(\mathbf{x}, t, t')\, n_1^-(\mathbf{x}, t')\, |-\mathbf{q}_3s_3\rangle - (+ \leftrightarrow -)]$$
$$+ [\langle \mathbf{q}s_1 |\, n_1^+(\mathbf{x}, t')\, G_1^{\mathrm{p}}(\mathbf{x}, t', t)\, |\mathbf{q}s'\rangle \langle \mathbf{q}_2s_2 |\, n_1^+(\mathbf{x}, t')\, G_1^{\mathrm{p}}(\mathbf{x}, t, t')\, |\mathbf{q}_2s_2'\rangle$$
$$\times \langle \mathbf{q}_3s_3 |\, n_1^+(\mathbf{x}, t')\, G_1^{\mathrm{p}}(\mathbf{x}, t', t)\, |qu3s_3'\rangle - (+ \leftrightarrow -)]\} \qquad (4.39)$$

$$\left(\frac{\partial n_{ss'}^{\pm}}{\partial t}\right)_{\mathrm{pe}}(\mathbf{x}, \mathbf{q}, t)$$

$$= 2\,\mathrm{Re}\int_{t_0}^{t} dt' \sum_{s_1} [\mathrm{Tr}_1^{(\mathrm{e})}\bar{v}_1^{-\mathbf{q}s} G_1^{\mathrm{e}}(\mathbf{x}, t, t')\, f_1^-(\mathbf{x}, t')$$
$$\times \bar{v}^{\mathbf{q}s_1} f_1^+(\mathbf{x}, t')\, G_1^{\mathrm{e}}(\mathbf{x}, t', t) \langle \mathbf{q}s_1 |\, n_1^+(\mathbf{x}, t')$$
$$\times G_1^{\mathrm{p}}(\mathbf{x}, t', t)\, |\mathbf{q}s'\rangle - (+ \leftrightarrow -)] \qquad (4.40)$$

$$\left(\frac{\partial n_{ss'}^{\pm}}{\partial t}\right)_{\mathrm{pi}}(\mathbf{x}, \mathbf{q}, t)$$

$$= -2\,\mathrm{Re}\int_{t_0}^{t} dt' N_i \sum_{\mathbf{q}_3\mathbf{q}_4} \sum_{\mathbf{K}} \delta_{\mathbf{q}_3+\mathbf{q}_4,\,\mathbf{K}} \sum_{\mathbf{q}_1} \sum_{s_1s_2s_2'}$$
$$\times \{[\tilde{w}_1^{\mathbf{q}_1s_1,\,-\mathbf{q}s}(\mathbf{q}_3) - \tilde{w}_2^{\mathbf{q}_1s_1,\,-\mathbf{q}s}(\mathbf{q}_3)] \cdot [\tilde{w}_1^{\mathbf{q}s_2,\,-\mathbf{q}_1s_2'}(\mathbf{q}_4) - \tilde{w}_2^{\mathbf{q}s_2,\,-\mathbf{q}_1s_2'}(\mathbf{q}_4)]$$
$$\times [\langle \mathbf{q}s_2 |\, n_1^+(\mathbf{x}, t')\, G_1^{\mathrm{p}}(\mathbf{x}, t', t)\, |\mathbf{q}s'\rangle$$
$$\times \langle \mathbf{q}_1s_1 |\, G_1^{\mathrm{p}}(\mathbf{x}, t, t')\, n_1^-(\mathbf{x}, t')\, |\mathbf{q}_1s_2'\rangle - (+ \leftrightarrow -)]$$
$$+ [\tilde{w}_1^{\mathbf{q}_1s_1,\,-\mathbf{q}s}(\mathbf{q}_3) + \tilde{w}_2^{\mathbf{q}_1s_1,\,-\mathbf{q}s}(\mathbf{q}_3)] \cdot [\tilde{w}_1^{\mathbf{q}s_2,\,-\mathbf{q}_1s_2'}(\mathbf{q}_4) + \tilde{w}_2^{\mathbf{q}s_2,\,-\mathbf{q}_1s_2'}(\mathbf{q}_4)]$$
$$\times [\langle \mathbf{q}s_2 |\, n_1^+(\mathbf{x}, t')\, G_1^{\mathrm{p}}(\mathbf{x}, t', t)\, |\mathbf{q}s'\rangle$$
$$\times \langle -\mathbf{q}_1s_2' |\, n_1^+(\mathbf{x}, t')\, G_1^{\mathrm{p}}(\mathbf{x}, t', t)\, |-\mathbf{q}_1s_1\rangle - (+ \leftrightarrow -)]\}. \qquad (4.41)$$

To achieve a compact form, we have used an operator notation in k- and q-space.

All operators G_1^{e}, G_1^{p}, $f_1^{\pm}$, and $n_1^{\pm}$ are diagonal in $\mathbf{k}$ and $\mathbf{q}$ but nondiagonal in α and s. The appearance of distribution functions $f_{\alpha\alpha'}^{\pm}$ and $n_{ss'}^{\pm}$ with different band- or polarisation indices cannot be found in standard textbooks on solid state theory. For small tunneling rates this effect has first been investigated in [11] for electron-impurity and electron-phonon interaction in homogeneous electric fields. Especially for high electric fields it becomes very important and will be considered in the following in analogy to [11] but for more general fields and interactions. For the phonons, however, we have

$$n^{\pm}_{ss'} \sim O(V) \qquad \text{for} \quad s \neq s' \tag{4.42}$$

since H_0 and $H_{ex}(t)$ are diagonal in the polarisation indices. Therefore, the distribution functions $n_{ss'}$ with $s \neq s'$ are neither important for the collision integral nor for the driving term.

The momentum transfer is described by the potential matrix elements which can also create Umklapp-processes or band- and polarisation transitions due to scattering. All the equations (4.36)–(4.41) are of non-Markovian character. They depend on the retarded or advanced Green's functions G^{e}_{1} and G^{p}_{1} which contain itself the external fields and account for intracollisional field effects and collisional broadening.

Now let us turn to the driving terms given by the general expression (3.40). The energy functions $\bar{\varepsilon}(t)$ can be calculated from (2.25) and (2.26),

$$\bar{\varepsilon}_{k'k}(t; cc^{\dagger}) = \delta_{k,k'}\varepsilon^{(0)}_{k} + \varepsilon^{ex}_{k'k}(t) + \bar{v}^{HF}_{k'k}(t; cc^{\dagger}) \tag{4.43}$$

$$\bar{\varepsilon}_{q'q}(t; aa^{\dagger}) = \delta_{q,q'}\omega^{(0)}_{q} + \bar{v}^{HF}_{q'q}(t; aa^{\dagger}) \tag{4.44}$$

with

$$\bar{v}^{HF}_{k'k}(t; cc^{\dagger}) = \sum_{k_1 k_1'} \bar{v}_{k'k_1', kk_1} \langle c^{\dagger}_{k_1'}(t)_H \, c_{k_1}(t)_H \rangle_{\rho(t_0)} + \sum_{q} \bar{v}^{q}_{k',k} \langle a_q(t)_H + a^{\dagger}_{-q}(t)_H \rangle_{\rho(t_0)} + N_i \, \delta_{\mathbf{k},\mathbf{k}'} \tilde{v}_{k'k}(0) \tag{4.45}$$

$$\bar{v}^{HF}_{q'q}(t; aa^{\dagger}) = \sum_{q_1} \bar{v}^{-q'qq_1} \langle a_{q_1}(t)_H \, a^{\dagger}_{-q_1}(t)_H \rangle_{\rho(t_0)} + N_i \, \delta_{\mathbf{q},\mathbf{q}'} \tilde{w}_1^{-q'q}(0) - N_i \, \delta_{\mathbf{q},\mathbf{q}'} \tilde{w}_2^{-q'q}(0). \tag{4.46}$$

Using the notation

$$\bar{\varepsilon}^{e}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t) = \bar{\varepsilon}^{ph}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t; cc^{\dagger}) \tag{4.47}$$

$$\bar{\varepsilon}^{p}_{s's}(\mathbf{x}, \mathbf{q}, t) = \bar{\varepsilon}^{ph}_{s's}(\mathbf{x}, \mathbf{q}, t; aa^{\dagger}) \tag{4.48}$$

for the phase-space transforms of the electron and phonon energy functions, we obtain from (3.27) and (4.8)

$$\varepsilon^{e}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t) = \delta_{\alpha\alpha'}\varepsilon^{(0)}_{\alpha}(\mathbf{k}) + \frac{1}{2m}\left[\left(\mathbf{p}_{\alpha'\alpha}(\mathbf{k}) - \frac{\varepsilon}{c}\mathbf{A}(\mathbf{x}, t)^2 - (\mathbf{p}_{\alpha'\alpha}(\mathbf{k})\right)^2\right] + e\varphi(\mathbf{x}, t) - \sum_{\mathbf{k}_1} \sum_{\alpha_1 \alpha_1'} \bar{v}_{\mathbf{k}\alpha', \mathbf{k}_1\alpha_1'; \mathbf{k}\alpha, \mathbf{k}_1\alpha_1} f^{+}_{\alpha_1\alpha_1'}(\mathbf{x}, \mathbf{k}_1, t) + N_i \tilde{v}_{\mathbf{k}\alpha', \mathbf{k}\alpha}(0) \tag{4.49}$$

$$\bar{\varepsilon}^{p}_{s's}(\mathbf{q}) = \delta_{ss'}\omega^{(0)}_{s}(\mathbf{q}) + N_i \tilde{w}_1^{-\mathbf{q}s', \mathbf{q}s}(0) - N_i \tilde{w}_2^{-\mathbf{q}s', \mathbf{q}s}(0), \tag{4.50}$$

where $\mathbf{p}_{\alpha'\alpha}(\mathbf{k}) = \langle \mathbf{k}\alpha' | \hat{\mathbf{p}} | \mathbf{k}\alpha \rangle$. The terms containing $\langle a_q + a^\dagger_{-q} \rangle$ have been omitted here since they lead to phonons of zero momentum and are of higher order according to

$$\langle a_q(t)_{\mathrm{H}} \rangle_{\rho(t_0)}, \langle a_q^\dagger(t)_{\mathrm{H}} \rangle_{\rho(t_0)} \sim O(V). \tag{4.51}$$

Furthermore we have used (see Appendix A)

$$\begin{aligned} \mathbf{A}^{\mathrm{ph}}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t) &\cong \delta_{\alpha\alpha'} \mathbf{A}(\mathbf{x}, t) \\ \varphi^{\mathrm{ph}}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t) &\cong \delta_{\alpha\alpha'} \varphi(\mathbf{x}, t) \end{aligned} \tag{4.52}$$

and

$$\langle \mathbf{k}'\alpha' | \hat{\mathbf{p}} | \mathbf{k}\alpha \rangle = \delta_{\mathbf{k}\mathbf{k}'} \mathbf{p}_{\alpha'\alpha}(\mathbf{k}) \tag{4.53}$$

In operator form, Eq. (4.49) reads

$$\bar{\varepsilon}^{\mathrm{e}}_1(\mathbf{x}, t) = \varepsilon^{\mathrm{e}}_1(\mathbf{x}, t) + \varepsilon^{\mathrm{eHF}}_1(\mathbf{x}, t) \tag{4.54}$$

with

$$\varepsilon^{\mathrm{e}}_1(\mathbf{x}, t) = \frac{1}{2m}\left(\hat{\mathbf{p}}_1 - \frac{e}{c}\mathbf{A}(\mathbf{x}, t)\right)^2 + V_{\mathrm{B}}(\hat{\mathbf{x}}) + e\varphi(\mathbf{x}, t) \tag{4.55}$$

and

$$\varepsilon^{\mathrm{eHF}}_1(\mathbf{x}, t) = -\mathrm{Tr}^{(\mathrm{e})}_2 \bar{v}_{12} f^+_2(\mathbf{x}, t) + N_i \bar{v}_1(0), \tag{4.56}$$

where V_B denotes the periodic Bloch potential. Thus, $\bar{\varepsilon}^{\mathrm{e}}_1$ has the form of a local Hamiltonian renormalized by Hartree–Fock contributions.

Let us consider the driving term for the electrons first. Using (3.40), together with (2.23), the l.h.s. of (4.25) can be written as

$$\begin{aligned} &\frac{\partial f^\pm_1}{\partial t}(\mathbf{x}, t) + \left(\frac{\partial f^\pm_1}{\partial t}\right)_D (\mathbf{x}, t) \\ &= \frac{\partial f^\pm_1}{\partial t}(\mathbf{x}, t) + i[\bar{\varepsilon}^{\mathrm{e}}_1(\mathbf{x}, t) f^\pm_1(\mathbf{x}, t) - f^\pm_1(\mathbf{x}, t)\bar{\varepsilon}^{\mathrm{e}}_1(\mathbf{x}, t)] \\ &\quad + \frac{1}{2}\left[\frac{\partial \bar{\varepsilon}^{\mathrm{e}}_1}{\partial \mathbf{k}}(\mathbf{x}, t)\frac{\partial f^\pm_1}{\partial \mathbf{x}}(\mathbf{x}, t) - \frac{\partial \bar{\varepsilon}^{\mathrm{e}}_1}{\partial \mathbf{x}}(\mathbf{x}, t)\frac{\partial f^\pm_1}{\partial \mathbf{k}}(\mathbf{x}, t)\right] \\ &\quad + \frac{1}{2}\left[\frac{\partial f^\pm_1}{\partial \mathbf{x}}(\mathbf{x}, t)\frac{\partial \bar{\varepsilon}^{\mathrm{e}}_1}{\partial \mathbf{k}}(\mathbf{x}, t) - \frac{\partial f^\pm_1}{\partial \mathbf{k}}(\mathbf{x}, t)\frac{\partial \bar{\varepsilon}^{\mathrm{e}}_1}{\partial \mathbf{x}}(\mathbf{x}, t)\right], \end{aligned} \tag{4.57}$$

where

$$\langle \mathbf{k}'\alpha' | \frac{\partial \bar{\varepsilon}^{\mathrm{e}}_1}{\partial \mathbf{k}}(\mathbf{x}, t) | \mathbf{k}\alpha \rangle = \delta_{\mathbf{k},\mathbf{k}'} \frac{\partial}{\partial \mathbf{k}} \varepsilon^{\mathrm{e}}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t). \tag{4.58}$$

This operator equation will now be evaluated for a certain basis set of complete and orthonormal one-particle wave functions. We choose here local and instantaneous eigenfunctions of $\varepsilon_1^e(\mathbf{x}, t)$ defined by

$$|\mathbf{k}\alpha\mathbf{x}t\rangle = e^{i(e/c)\mathbf{A}(\mathbf{x},t)\hat{\mathbf{x}}} \left|\mathbf{k} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t), \alpha\right\rangle \tag{4.59}$$

for fixed parameters $\mathbf{x}$ and t. Some straightforward properties are

$$\langle \mathbf{k}'\alpha'\mathbf{x}t \mid \mathbf{k}\alpha\mathbf{x}t\rangle \sim \delta_{\mathbf{k}\mathbf{k}'}\,\delta_{\alpha\alpha'} \tag{4.60}$$

$$\langle \mathbf{k}'\alpha'\mathbf{x}t \mid \mathbf{k}\alpha\rangle \sim \delta_{\mathbf{k}\mathbf{k}'} \tag{4.61}$$

$$\varepsilon_1^e(\mathbf{x}, t)\,|\mathbf{k}\alpha\mathbf{x}t\rangle = \left[\varepsilon_\alpha^{(0)}\left(\mathbf{k} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t) + e\varphi(\mathbf{x}, t)\right)\right]|\mathbf{k}\alpha\mathbf{x}t\rangle. \tag{4.62}$$

The distribution functions and energies in this new basis are

$$\tilde{f}^{\pm}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t) = \langle \mathbf{k}\alpha\mathbf{x}t|\, f_1^{\pm}(\mathbf{x}, t)\,|\mathbf{k}\alpha'\mathbf{x}t\rangle \tag{4.63}$$

$$\tilde{\bar{\varepsilon}}^{e}_{l'\alpha}(\mathbf{x}, \mathbf{k}, t) = \langle \mathbf{k}\alpha\mathbf{x}t|\, \bar{\varepsilon}_1^e(\mathbf{x}, t)\,|\mathbf{k}\alpha\mathbf{x}t\rangle. \tag{4.64}$$

Explicitly, we have for the energy function

$$\tilde{\bar{\varepsilon}}^{e}_{l'\alpha}(\mathbf{x}, \mathbf{k}, t) = \delta_{\alpha\alpha'}\tilde{\varepsilon}^{e}_{\alpha}(\mathbf{x}, \mathbf{k}, t) + \tilde{\varepsilon}^{e\mathrm{HF}}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t) \tag{4.65}$$

with

$$\tilde{\varepsilon}^{e}_{\alpha}(\mathbf{x}, \mathbf{k}, t) = \varepsilon_\alpha^{(0)}\left(\mathbf{k} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t) + e\varphi(\mathbf{x}, t)\right) \tag{4.66}$$

$$\begin{aligned}\tilde{\varepsilon}^{e\mathrm{HF}}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t) = &-\sum_{\mathbf{k}_1}\sum_{\alpha_1\alpha_1'}\left\langle \mathbf{k} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t), \alpha'; \mathbf{k}_1 - \frac{e}{c}\mathbf{A}(\mathbf{x}, t), \alpha_1'\right|\bar{v}_{12}\\ &\times\left|\mathbf{k} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t), \alpha; \mathbf{k}_1 - \frac{e}{c}\mathbf{A}(\mathbf{x}, t), \alpha_1\right\rangle \tilde{f}^{+}_{\alpha_1\alpha_1'}(\mathbf{x}, \mathbf{k}_1, t)\\ &+ N_i\left\langle \mathbf{k} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t), \alpha'\right|\tilde{v}_1(0)\left|\mathbf{k} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t), \alpha\right\rangle.\end{aligned} \tag{4.67}$$

The time derivative of f is transformed according to

$$\begin{aligned}\langle \mathbf{k}\alpha\mathbf{x}t|\,\frac{\partial f_1^{\pm}}{\partial t}(\mathbf{x}, t)\,|\mathbf{k}\alpha'\mathbf{x}t\rangle = &\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha'}}{\partial t}(\mathbf{x}, \mathbf{k}, t)\\ &-\frac{e}{c}\dot{\mathbf{A}}(\mathbf{x}, t)\sum_{\alpha_1}\left[\mathbf{u}_{\alpha\alpha_1}(\mathbf{x}, \mathbf{k}, t)\,\tilde{f}^{\pm}_{\alpha_1\alpha'}(\mathbf{x}, \mathbf{k}, t)\right.\\ &\left.-\tilde{f}^{\pm}_{\alpha\alpha_1}(\mathbf{x}, \mathbf{k}, t)\,\mathbf{u}_{\alpha_1\alpha'}(\mathbf{x}, \mathbf{k}, t)\right]\end{aligned} \tag{4.68}$$

with

$$\mathbf{u}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t) = \int d^3x' u^*_{\mathbf{k}-(e/c)\mathbf{A}(\mathbf{x},t),\alpha}(\mathbf{x}') \frac{\partial}{\partial \mathbf{k}} u_{\mathbf{k}-(e/c)\mathbf{A}(\mathbf{x},t),\alpha'}(\mathbf{x}'). \tag{4.69}$$

Thus, in the new basis, the transport equation (4.25) takes the form (we omit the arguments $\mathbf{x}$, $\mathbf{k}$, and t)

$$\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha'}}{\partial t} + \left(\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{D_1} + \left(\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{D_2} = \left(\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha'}}{\partial t}\right)_c, \tag{4.70}$$

where $(\cdots)_{D_2}$ corresponds to the third and fourth term on the r.h.s. of (4.57) and $(\cdots)_{D_1}$ is defined by

$$\left(\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{D_1} = i \sum_{\alpha_1} (\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha_1} \tilde{f}^{\pm}_{\alpha_1\alpha'} - \tilde{f}^{\pm}_{\alpha\alpha_1} \tilde{\varepsilon}^{\mathrm{T}}_{\alpha_1\alpha'}) \tag{4.71}$$

with

$$\tilde{\varepsilon}^{\mathrm{T}}_{\alpha'\alpha} = \delta_{\alpha\alpha'} \tilde{\varepsilon}^{\mathrm{e}}_{\alpha} + \tilde{\varepsilon}^{\mathrm{eHF}}_{\alpha'\alpha} + i \frac{e}{c} \dot{\mathbf{A}} \mathbf{u}_{\alpha'\alpha}. \tag{4.72}$$

For $\alpha = \alpha'$, Eq. (4.71) gives

$$\left(\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha}}{\partial t}\right)_{D_1} = 2\,\mathrm{Re}\left\{ i \sum_{\substack{\alpha' \\ \alpha' \neq \alpha}} \tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha'} \tilde{f}^{\pm}_{\alpha'\alpha} \right\}, \tag{4.73}$$

whereas, for $\alpha \neq \alpha'$, we have

$$\begin{aligned} \left(\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{D_1} &= i(\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha} - \tilde{\varepsilon}^{\mathrm{T}}_{\alpha'\alpha'}) \tilde{f}^{\pm}_{\alpha\alpha'} + i\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha'} (\tilde{f}^{\pm}_{\alpha'\alpha'} - \tilde{f}^{\pm}_{\alpha\alpha}) \\ &\quad + i \sum_{\bar{\alpha} \neq \alpha, \alpha'} (\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\bar{\alpha}} \tilde{f}^{\pm}_{\bar{\alpha}\alpha'} - \tilde{f}^{\pm}_{\alpha\bar{\alpha}} \tilde{\varepsilon}^{\mathrm{T}}_{\bar{\alpha}\alpha'}). \end{aligned} \tag{4.74}$$

Neglecting the higher order contributions coming from the terms $(\cdots)_{D_2}$ and $(\cdots)_C$ in (4.70) for $\alpha \neq \alpha'$, we can determine the time evolution of $\tilde{f}^{\pm}_{\alpha\alpha'}$ $(\alpha \neq \alpha')$ approximately by

$$\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha'}}{\partial t} + \left(\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha'}}{\partial t}\right)_{D_1} = 0. \tag{4.75}$$

Note that such a procedure is not true for the case $\alpha = \alpha'$ since the largest terms $\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha} \tilde{f}^{\pm}_{\alpha\alpha}$ cancel each other in (4.71). Thus, $(\cdots)_{D_1}$, $(\cdots)_{D_2}$ and $(\cdots)_C$ can be of the same order of magnitude for $\alpha = \alpha'$. Now, since

$$\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha'} \ll \tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha}, \tilde{\varepsilon}^{\mathrm{T}}_{\alpha'\alpha'} \qquad \text{for} \quad \alpha \neq \alpha' \tag{4.76}$$

due to (4.72) and our condition (1.4), we can use a perturbation expansion in $\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha'}$ $(\alpha \neq \alpha')$ to solve (4.75). In lowest order we obtain

$$\tilde{f}^{\pm}_{\alpha\alpha'}(t) = \tilde{f}^{\pm}_{\alpha\alpha'}(t_0)\, e^{-i\int_{t_0}^{t} dt' [\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha}(t') - \tilde{\varepsilon}^{\mathrm{T}}_{\alpha'\alpha'}(t')]}$$
$$+ i \int_{t_0}^{t} dt' \tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha'}(t') [\tilde{f}^{\pm}_{\alpha\alpha}(t') - \tilde{f}^{\pm}_{\alpha'\alpha'}(t')]\, e^{-i\int_{t'}^{t} d\tau [\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha}(\tau) - \tilde{\varepsilon}^{\mathrm{T}}_{\alpha'\alpha'}(\tau)]} \qquad (4.77)$$

and (4.73) becomes

$$\left(\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha}}{\partial t}\right)_{D_1} = 2\,\mathrm{Re} \sum_{\substack{\alpha' \\ \alpha' \neq \alpha}} \Bigg\{ i\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha'}(t)\, \tilde{f}^{\pm}_{\alpha'\alpha}(t_0)\, e^{i\int_{t_0}^{t} dt' [\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha}(t') - \tilde{\varepsilon}^{\mathrm{T}}_{\alpha'\alpha'}(t')]}$$
$$- \tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha'}(t) \int_{t_0}^{t} dt' \tilde{\varepsilon}^{\mathrm{T}}_{\alpha'\alpha}(t') [\tilde{f}^{\pm}_{\alpha'\alpha'}(t') - \tilde{f}^{\pm}_{\alpha\alpha}(t')]\, e^{i\int_{t'}^{t} d\tau [\tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha}(\tau) - \tilde{\varepsilon}^{\mathrm{T}}_{\alpha'\alpha'}(\tau)]} \Bigg\} \qquad (4.78)$$

which, except for the first term, is identical to the result obtained in [11]. $\tilde{f}^{\pm}_{\alpha\alpha'}(t_0)$ can be calculated from the initial density matrix (4.33) and is proportional to the interaction strength,

$$\tilde{f}^{\pm}_{\alpha\alpha'}(t_0) = f^{\pm}_{\alpha\alpha'}(t_0) \sim O(V) \qquad \text{for} \quad \alpha \neq \alpha'. \qquad (4.79)$$

Furthermore, from (4.76), (4.77), and (4.79), we see that

$$\tilde{f}^{\pm}_{\alpha\alpha'} \ll \tilde{f}^{\pm}_{\alpha\alpha}, \tilde{f}^{\pm}_{\alpha'\alpha'} \qquad \text{for} \quad \alpha \neq \alpha', \qquad (4.80)$$

a property which is not valid in the representation $|\mathbf{k}\alpha\rangle$.

Using (4.76) and (4.80), we can now neglect all distribution functions $\tilde{f}^{\pm}_{\alpha\alpha'}$ and energies $\tilde{\tilde{\varepsilon}}^{\mathrm{e}}_{\alpha\alpha'}$ with $\alpha \neq \alpha'$ occurring in the collision integral $(\partial \tilde{f}^{\pm}_{\alpha\alpha}/\partial t)_C$ or in the second part $(\partial \tilde{f}^{\pm}_{\alpha\alpha}/\partial t)_{D_2}$ of the driving term. In this approximation, the transformation laws for $\partial A^{\mathrm{ph}}/\partial \mathbf{x}$ and $(\partial A^{\mathrm{ph}}/\partial \mathbf{k})$ $(A = f^{\pm}, \bar{\varepsilon}^{\mathrm{e}})$ become

$$\begin{aligned} \langle \mathbf{k}\alpha\mathbf{x}t|\, \frac{\partial A^{\mathrm{ph}}}{\partial \mathbf{x}}(\mathbf{x}, t)\, |\mathbf{k}\alpha\mathbf{x}t\rangle &\cong \frac{\partial \tilde{A}^{\mathrm{ph}}_{\alpha\alpha}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{k}, t) \\ \langle \mathbf{k}\alpha\mathbf{x}t|\, \frac{\partial A^{\mathrm{ph}}}{\partial \mathbf{k}}(\mathbf{x}, t)\, |\mathbf{k}\alpha\mathbf{x}t\rangle &\cong \frac{\partial \tilde{A}^{\mathrm{ph}}_{\alpha\alpha}}{\partial \mathbf{k}}(\mathbf{x}, \mathbf{k}, t), \end{aligned} \qquad (4.81)$$

where we have used (4.58)–(4.61). Thus we obtain from (4.57)

$$\left(\frac{\partial \tilde{f}^{\pm}_{\alpha\alpha}}{\partial t}\right)_{D_2} = \frac{\partial \tilde{\tilde{\varepsilon}}^{\mathrm{e}}_{\alpha\alpha}}{\partial \mathbf{k}} \frac{\partial \tilde{f}^{\pm}_{\alpha\alpha}}{\partial \mathbf{x}} - \frac{\partial \tilde{\tilde{\varepsilon}}^{\mathrm{e}}_{\alpha\alpha}}{\partial \mathbf{x}} \frac{\partial \tilde{f}^{\pm}_{\alpha\alpha}}{\partial \mathbf{k}} \qquad (4.82)$$

and the l.h.s. of our transport equation (4.70) is complete. However, (4.82) has not yet the desired form, since it still contains an explicit dependence on the vector potential **A** instead of the electric and magnetic fields. Thus, in the last step, we use the transformation

$$\phi^{\pm}_{\alpha\alpha'}(\mathbf{x}, \mathbf{p}, t) = \tilde{f}^{\pm}_{\alpha\alpha'}\left(\mathbf{x}, \mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{x}, t), t\right) \tag{4.83}$$

$$\sigma^{e}_{\alpha\alpha'}(\mathbf{x}, \mathbf{p}, t) = \tilde{\tilde{\varepsilon}}^{e}_{\alpha\alpha'}\left(\mathbf{x}, \mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{x}, t), t\right) \tag{4.84}$$

and obtain after some manipulations the following final result from (4.70), (4.78), and (4.82),

$$\frac{\partial \phi^{\pm}_{\alpha}}{\partial t} + \left(\frac{\partial \phi^{\pm}_{\alpha}}{\partial t}\right)_{D_1} + \left(\frac{\partial \phi^{\pm}_{\alpha}}{\partial t}\right)_{D_2} = \left(\frac{\partial \phi^{\pm}_{\alpha}}{\partial t}\right)_{C} \tag{4.85}$$

with $\phi_\alpha \equiv \phi_{\alpha\alpha}$ and

$$\begin{aligned}\left(\frac{\partial \phi^{\pm}_{\alpha}}{\partial t}\right)_{D_1} = 2\,\mathrm{Re} \sum_{\substack{\alpha' \\ \alpha' \neq \alpha}} \Big\{ & i\sigma^{T}_{\alpha\alpha'}(\mathbf{x}, \mathbf{p}, t)\, \phi^{\pm}_{\alpha'\alpha}(\mathbf{p}(\mathbf{x}tt_0), t_0) \\ & \times e^{-i\int_{t_0}^{t} dt'[\sigma^{T}_{\alpha'}(\mathbf{x}, \mathbf{p}(\mathbf{x}tt'), t') - \sigma^{T}_{\alpha}(\mathbf{x}, \mathbf{p}(\mathbf{x}tt'), t')]} \\ & - \sigma^{T}_{\alpha\alpha'} \int_{t_0}^{t} dt' \sigma^{T}_{\alpha'\alpha}(\mathbf{x}, \mathbf{p}(\mathbf{x}tt'), t') \\ & \times [\phi^{\pm}_{\alpha'}(\mathbf{x}, \mathbf{p}(\mathbf{x}tt'), t') - \phi^{\pm}_{\alpha}(\mathbf{x}, \mathbf{p}(\mathbf{x}tt'), t')] \\ & \times e^{-i\int_{t'}^{t} d\tau[\sigma^{T}_{\alpha'}(\mathbf{x}, \mathbf{p}(\mathbf{x}t\tau), \tau) - \sigma^{T}_{\alpha}(\mathbf{x}, \mathbf{p}(\mathbf{x}t\tau), \tau)]} \Big\}\end{aligned} \tag{4.86}$$

$$\begin{aligned}\left(\frac{\partial \phi^{\pm}_{\alpha}}{\partial t}\right)_{D_2}(\mathbf{x}, \mathbf{p}, t) = {} & \mathbf{v}^{T}_{\alpha}(\mathbf{x}, \mathbf{p}, t) \frac{\partial \phi^{\pm}_{\alpha}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{p}, t) \\ & + e\left[\mathbf{E}^{T}(\mathbf{x}, \mathbf{p}, t) + \frac{1}{c}\mathbf{v}^{T}_{\alpha}(\mathbf{x}, \mathbf{p}, t) \wedge B(\mathbf{x}, t)\right] \frac{\partial \phi^{\pm}_{\alpha}}{\partial \mathbf{p}}(\mathbf{x}, \mathbf{p}, t),\end{aligned} \tag{4.87}$$

where $\sigma \equiv \sigma_{\alpha\alpha}$ and

$$\mathbf{p}(\mathbf{x}tt') = \mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{x}, t) - \frac{e}{c}\mathbf{A}(\mathbf{x}, t') \tag{4.88}$$

$$\sigma^{T}_{\alpha\alpha'}(\mathbf{x}, \mathbf{p}, t) = \sigma^{e}_{\alpha\alpha'}(\mathbf{x}, \mathbf{p}, t) + i\frac{e}{c}\dot{\mathbf{A}}(\mathbf{x}, t)\, \tilde{\mathbf{u}}_{\alpha\alpha'}(\mathbf{p}) \tag{4.89}$$

$$\sigma^{e}_{\alpha\alpha'}(\mathbf{x}, \mathbf{p}, t) = \delta_{\alpha\alpha'}[\varepsilon^{(0)}_{\alpha}(\mathbf{p}) + e\varphi(\mathbf{x}, t)] + \sigma^{eHF}_{\alpha\alpha'}(\mathbf{x}, \mathbf{p}, t) \tag{4.90}$$

$$\sigma^{eHF}_{\alpha\alpha'}(\mathbf{x}, \mathbf{p}, t) = -\sum_{\mathbf{p}_1\alpha_1} \bar{v}_{\mathbf{p}\alpha,\mathbf{p}_1\alpha_1;\mathbf{p}\alpha',\mathbf{p}_1\alpha_1} \phi^{+}_{\alpha_1}(\mathbf{x}, \mathbf{p}_1, t) + N_i \tilde{v}_{\mathbf{p}\alpha',\mathbf{p}\alpha}(0) \tag{4.91}$$

$$\tilde{\mathbf{u}}_{\alpha\alpha'}(\mathbf{p}) = \int d^3x\, u^*_{\mathbf{p}\alpha}(\mathbf{x}) \frac{\partial}{\partial \mathbf{p}} u_{\mathbf{p}\alpha'}(\mathbf{x}) \tag{4.92}$$

$$\begin{aligned} \mathbf{v}^{\mathrm{T}}_\alpha(\mathbf{x}, \mathbf{p}, t) &= \frac{\partial \sigma^{\mathrm{e}}_\alpha}{\partial \mathbf{p}}(\mathbf{x}, \mathbf{p}, t) \\ &= \frac{\partial \varepsilon^{(0)}_\alpha}{\partial \mathbf{p}}(\mathbf{p}) - \sum_{\mathbf{p}'} \frac{\partial}{\partial \mathbf{p}} \bar{v}_{pp', pp'} \phi^+_{\alpha'}(\mathbf{x}, \mathbf{p}', t) + N_i \frac{\partial}{\partial \mathbf{p}} \bar{v}_{p,p}(0) \end{aligned} \tag{4.93}$$

$$\begin{aligned} \mathbf{E}^{\mathrm{T}}(\mathbf{x}, \mathbf{p}, t) &= \mathbf{E}(\mathbf{x}, t) - \frac{1}{e} \frac{\partial \sigma^{\mathrm{HF}}_\alpha}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{p}, t) \\ &= \mathbf{E}(\mathbf{x}, t) - \frac{1}{e} \sum_{p'} \bar{v}_{pp', pp'} \frac{\partial \phi^+_{\alpha'}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{p}', t). \end{aligned} \tag{4.94}$$

For the evaluation of the collision integrals (4.36), (4.37), and (4.38), we use the basis states (4.59) at different times t or t' and define

$$\tilde{G}^{\mathrm{e}}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t, t') = \langle \mathbf{k}\alpha\mathbf{x}t |\, G^{\mathrm{e}}(\mathbf{x}, t, t')\, | \mathbf{k}\alpha'\mathbf{x}t' \rangle. \tag{4.95}$$

Neglecting the collisional broadening by using the lowest order result (see Appendix B)

$$\tilde{\tilde{G}}^{\mathrm{e}(0)}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t, t') = \delta_{\alpha\alpha'}\, e^{-i\int_{t'}^{t} d\tau\, \tilde{\varepsilon}^{\mathrm{T}}_{\alpha\alpha}(\mathbf{x}, \mathbf{k}, \tau)} \tag{4.96}$$

$$\bar{G}^{\mathrm{p}(0)}_{ss'}(\mathbf{x}, \mathbf{q}, t, t') = \delta_{ss'}\, e^{-i\bar{\varepsilon}^{\mathrm{p}}_{ss}(\mathbf{q})(t - t')} \tag{4.97}$$

and performing the transformations (4.83) and (4.84), we obtain in a straigtforward manner

$$\left(\frac{\partial \phi^\pm_\alpha}{\partial t}\right)_C = \left(\frac{\partial \phi^\pm_\alpha}{\partial t}\right)_{\mathrm{ee}} + \left(\frac{\partial \phi^\pm_\alpha}{\partial t}\right)_{\mathrm{ep}} + \left(\frac{\partial \phi^\pm_\alpha}{\partial t}\right)_{\mathrm{ei}} \tag{4.98}$$

with

$$\begin{aligned} \left(\frac{\partial \phi^\pm_\alpha}{\partial t}\right)_{\mathrm{ee}}(\mathbf{x}, \mathbf{p}, t) = &-\mathrm{Re} \int_{t_0}^{t} dt' \sum_{p_2 p'_1 p'_2} \bar{v}_{pp_2, p'_1 p'_2} \bar{v}_{p'_1 p'_2, pp_2}(\mathbf{x}tt') \\ &\times e^{-i\int_{t'}^{t} d\tau [\sigma^{\mathrm{T}}_{\alpha'_1}(\mathbf{x}, \mathbf{p}'_1(\mathbf{x}t\tau), \tau) + \sigma^{\mathrm{T}}_{\alpha'_2}(\mathbf{x}, \mathbf{p}'_2(\mathbf{x}t\tau), \tau) - \sigma^{\mathrm{T}}_{\alpha}(\mathbf{x}, \mathbf{p}(\mathbf{x}t\tau), \tau) - \sigma^{\mathrm{T}}_{\alpha_2}(\mathbf{x}, \mathbf{p}(\mathbf{x}t\tau), \tau)]} \\ &\times [\phi^+_{\alpha'_1}(\mathbf{x}, \mathbf{p}'_1(\mathbf{x}tt'), t')\, \phi^+_{\alpha'_2}(\mathbf{x}, \mathbf{p}'_2(\mathbf{x}tt'), t') \\ &\times \phi_\alpha(\mathbf{x}, \mathbf{p}(\mathbf{x}tt'), t')\, \phi^-_{\alpha_2}(\mathbf{x}, \mathbf{p}_2(\mathbf{x}tt'), t') - (+ \leftrightarrow -)] \end{aligned} \tag{4.99}$$

$$\begin{aligned} \left(\frac{\partial \phi^\pm_\alpha}{\partial t}\right)_{\mathrm{ep}}(\mathbf{x}, \mathbf{p}, t) = &\, 2\,\mathrm{Re} \int_{t_0}^{t} dt' \sum_{p'q} \bar{v}^{q}_{p,p'} \bar{v}^{-q}_{p',p}(\mathbf{x}tt') \\ &\times e^{-i\int_{t'}^{t} d\tau [\sigma^{\mathrm{T}}_{\alpha'}(\mathbf{x}, \mathbf{p}'(\mathbf{x}t\tau), \tau) - \sigma^{\mathrm{T}}_{\alpha}(\mathbf{x}, \mathbf{p}(\mathbf{x}t\tau), \tau)]} \\ &\{[e^{i\bar{\varepsilon}^{\mathrm{p}}_s(-\mathbf{q})(t - t')} n^-_s(\mathbf{x}, -\mathbf{q}, t') + e^{-i\bar{\varepsilon}^{\mathrm{p}}_s(\mathbf{q})(t - t')} n^+_s(\mathbf{x}, \mathbf{q}, t')] \\ &\times \phi^+_{\alpha'}(\mathbf{x}, \mathbf{p}'(\mathbf{x}tt'), t')\, \phi^-_\alpha(\mathbf{x}, \mathbf{p}(\mathbf{x}tt'), t') - (+ \leftrightarrow -)\} \end{aligned} \tag{4.100}$$

$$\left(\frac{\partial \phi_\alpha^\pm}{\partial t}\right)_{\text{ei}} (\mathbf{x}, \mathbf{p}, t) = 2\,\text{Re} \int_{t_0}^{t} dt' N_i \sum_{\mathbf{q}_1 \mathbf{q}_2} \sum_{\mathbf{K}} \delta_{\mathbf{q}_1 + \mathbf{q}_2, \mathbf{K}} \sum_{p'} \times \tilde{v}_{p,p'}(\mathbf{q}_1)\, \tilde{v}_{p',p}(\mathbf{q}_2; \mathbf{x}tt')\, e^{-i \int_{t'}^{t} d\tau [\sigma_{\alpha'}^{\mathrm{T}}(\mathbf{x}, \mathbf{p}'(\mathbf{x}t\tau), \tau) - \sigma_\alpha^{\mathrm{T}}(\mathbf{x}, \mathbf{p}(\mathbf{x}t\tau), \tau)]} \times [\phi_{\alpha'}^{+}(\mathbf{x}, \mathbf{p}'(\mathbf{x}tt'), t') - \phi_\alpha^{+}(\mathbf{x}, \mathbf{p}(\mathbf{x}tt'), t')], \tag{4.101}$$

where $\bar{\varepsilon}_s^{\mathrm{p}} \equiv \bar{\varepsilon}_{ss}^{\mathrm{p}}$ and

$$\begin{aligned} \bar{v}_{p_1' p_2', p p_2}(\mathbf{x}tt') &= \langle \mathbf{p}_1'(\mathbf{x}tt')\,\alpha_1', \mathbf{p}_2'(\mathbf{x}tt')\,\alpha_2' |\, \bar{v}_{12}\, | \mathbf{p}(\mathbf{x}tt')\,\alpha, \mathbf{p}_2(\mathbf{x}tt')\,\alpha_2 \rangle \\ \bar{v}_{p',p}^{q}(\mathbf{x}tt') &= \bar{v}^{q}_{\mathbf{p}'(\mathbf{x}tt')\,\alpha', \mathbf{p}(\mathbf{x}tt')\,\alpha} \\ \tilde{v}_{p',p}(\mathbf{q}_2; \mathbf{x}tt') &= \tilde{v}_{\mathbf{p}'(\mathbf{x}tt')\,\alpha', \mathbf{p}(\mathbf{x}tt')\,\alpha}(\mathbf{q}_2). \end{aligned} \tag{4.102}$$

Finally, for the expectation value of the local current operator

$$\hat{\mathbf{j}}(\mathbf{x}, t) = \frac{e}{2m} \sum_{i=1}^{N_e} \left[\hat{\mathbf{p}}_i - \frac{e}{c} \mathbf{A}(\hat{\mathbf{x}}_i, t), \delta(\hat{\mathbf{x}}_i - \mathbf{x}) \right]_+ \tag{4.103}$$

we obtain (see Appendix C)

$$\overline{\langle \hat{\mathbf{j}}(\mathbf{x}, t) \rangle_{\rho(t)}} = -\frac{e}{m} \frac{1}{\Omega} \sum_{\mathbf{p}} \sum_{\alpha} \frac{\partial \varepsilon_\alpha^{(0)}}{\partial \mathbf{p}}(\mathbf{p})\, \phi_\alpha^{+}(\mathbf{x}, \mathbf{p}, t), \tag{4.104}$$

where the bar on $\langle \cdots \rangle$ denotes an average over the unit cell around $\mathbf{x}$.

The transport equation for the phonons can easily be obtained from (4.26), (4.30), (4.39)–(4.41), (3.40), and (4.50). Thereby we will neglect all contributions from $n_{ss'}^{\pm}$ with $s \neq s'$ (see (4.42)) and from Green's functions with $bb' = aa$ or $a^\dagger a^\dagger$ (see (4.34)). For G^p we will use the lowest order result (4.97). Thus, with $n_s^{\pm} \equiv n_{ss}^{\pm}$, we obtain the final result for the phonons

$$\frac{\partial n_s^\pm}{\partial t} + \left(\frac{\partial n_s^\pm}{\partial t}\right)_D = \left(\frac{\partial n_s^\pm}{\partial t}\right)_{\text{pp}} + \left(\frac{\partial n_s^\pm}{\partial t}\right)_{\text{pe}} + \left(\frac{\partial n_s^\pm}{\partial t}\right)_{\text{pi}}, \tag{4.105}$$

where

$$\left(\frac{\partial n_s^\pm}{\partial t}\right)_D (\mathbf{x}, \mathbf{q}, t) = \mathbf{v}_s^{\mathrm{T}}(\mathbf{q}) \frac{\partial n_s^\pm}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{q}, t) \tag{4.106}$$

$$\mathbf{v}_s^{\mathrm{T}}(\mathbf{q}) = \frac{\partial \omega_s^{(0)}}{\partial \mathbf{q}}(\mathbf{q}) + N_i \frac{\partial}{\partial \mathbf{q}} [\tilde{w}_1^{-q,q}(0) - \tilde{w}_2^{-q,q}(0)] \tag{4.107}$$

and

$$\left(\frac{\partial n_s^\pm}{\partial t}\right)_{\text{pp}} (\mathbf{x}, \mathbf{q}, t) = \text{Re} \int_{t_0}^{t} dt' \sum_{\mathbf{q}_2 \mathbf{q}_3} |\bar{v}^{q q_2 q_3}|^2\, e^{i[\bar{\varepsilon}_s^{\mathrm{p}}(\mathbf{q}) - \bar{\varepsilon}_{s_2}^{\mathrm{p}}(-\mathbf{q}_2)](t-t')} \times \{ n_s^{-}(\mathbf{x}, \mathbf{q}, t')\, n_s^{+}(\mathbf{x}, -\mathbf{q}_2, t') [2 e^{i\bar{\varepsilon}_{s_3}^{\mathrm{p}}(\mathbf{q})(t-t')} n_{s_3}^{-}(\mathbf{x}, \mathbf{q}_3, t') + e^{-i\bar{\varepsilon}_{s_3}^{\mathrm{p}}(-\mathbf{q}_3)(t-t')} n_{s_3}^{+}(\mathbf{x}, -\mathbf{q}_3, t') - (+ \leftrightarrow -) \} \tag{4.108}$$

$$\left(\frac{\partial n_s^{\pm}}{\partial t}\right)_{\text{pe}}(\mathbf{x}, \mathbf{q}, t) = 2\,\text{Re} \int_{t_0}^{t} dt' \sum_{pp'} \bar{v}^{q}_{p',p} \bar{v}^{-q}_{p,p'}(\mathbf{x}tt')$$
$$\times e^{-i\int_{t'}^{t} d\tau [\sigma^{\text{T}}_{\alpha'}(\mathbf{x}, \mathbf{p}'(\mathbf{x}t\tau), \tau) - \sigma^{\text{T}}_{\alpha}(\mathbf{x}, \mathbf{p}(\mathbf{x}t\tau), \tau)]}\, e^{i\bar{\varepsilon}^{\text{P}}_s(\mathbf{q})(t-t')}$$
$$\times [\phi^{-}_{\alpha'}(\mathbf{x}, \mathbf{p}'(\mathbf{x}tt'), t')\, \phi^{+}_{\alpha}(\mathbf{x}, \mathbf{p}(\mathbf{x}tt'), t')$$
$$\times n^{+}_{s}(\mathbf{x}, \mathbf{q}, t') - (+ \leftrightarrow -)] \tag{4.109}$$

$$\left(\frac{\partial n_s^{\pm}}{\partial t}\right)_{\text{pi}}(\mathbf{x}, \mathbf{p}, t) = 2\,\text{Re} \int_{t_0}^{t} dt' N_i \sum_{\mathbf{q}_1\mathbf{q}_2} \sum_{\mathbf{K}} \delta_{\mathbf{q}_1+\mathbf{q}_2, \mathbf{K}}$$
$$\times |\tilde{w}_1^{q', -q}(\mathbf{q}_1) + \tilde{w}_2^{q', -q}(\mathbf{q}_2)|^2\, e^{i[\bar{\varepsilon}^{\text{P}}_s(\mathbf{q}) - \bar{\varepsilon}^{\text{P}}_{s'}(\mathbf{q}')](t-t')}$$
$$\times [n^{+}_{s'}(\mathbf{x}, \mathbf{q}', t') - n^{+}_{s}(\mathbf{x}, \mathbf{q}, t')]. \tag{4.110}$$

The second terms of (4.39) and (4.41) have been omitted here since they do not fulfil the energy conservation if we take the Markov limit of (4.108) and (4.110). This limit is given by

$$\text{Re} \int_{t_0}^{t} dt'\, e^{-i\Delta\varepsilon(t-t')} h(t') = \text{Re} \int_{0}^{t-t_0} dt'\, e^{-i\Delta\varepsilon t'} h(t-t')$$
$$\cong \text{Re} \int_{0}^{\infty} dt'\, e^{-i\Delta\varepsilon t'} h(t) = \pi\delta(\Delta\varepsilon)\, h(t) \tag{4.111}$$

and leads to the golden rule in (4.99)–(4.101) and (4.108)–(4.110) if we neglect the influence of external fields and Hartree–Fock potentials on the collision integrals.

5. Conclusions

The main issue of this paper was to set up a general theory for single-time Green's functions in an arbitrary weakly inhomogeneous quantum system. As an application we have studied a quantum solid in moderately high electric and magnetic fields (see the condition (1.4a), (1.4b)). The final transport equations are given by (3.31), (4.85), and (4.105), and the electric current can be calculated from (4.104). No restrictions with respect to the time dependence of the external fields, the band structure, the statistics, and the kind of interaction have been made during the calculation. Furthermore, the fields and the initial density matrix can be inhomogeneous if they do not vary on a microscopic scale (see the condition (1.2)).

Other treatments of the same topic do not have such a general range of validity [2–5, 11–21]. First, they are all restricted to homogeneous fields and to initial states given by the unperturbed equilibrium density matrix. Furthermore, within the Keldysh formalism of double-time or single-time Green's functions, one often applies a gradient expansion in time [2, 3, 18–20] or one treats the case of free electrons in homogeneous electric fields [16, 17, 21]. Methods based on other non-equilibrium transport theories are restricted to Boltzmann statistics or to certain kinds of interactions [11–14].

Thus, the second part of this paper was an attempt to present a complete and microscopic theory of Bloch electrons and phonons in moderately high, weakly inhomogeneous, and arbitrarily time-dependent electric and magnetic fields.

APPENDIX A

In this appendix we derive Eq. (4.53). If $A(\hat{\mathbf{x}}, t)$ is an arbitrary but weakly inhomogeneous operator we can write

$$A(\hat{\mathbf{x}}, t) = \sum_q e^{i\mathbf{q}\hat{\mathbf{x}}} A(\mathbf{q}, t) \tag{A.1}$$

with $|\mathbf{q}| \ll a^{-1}$. For A^{ph} we obtain, due to (3.19) and (3.23),

$$A^{\text{ph}}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t) = \sum_{\mathbf{q}\mathbf{q}'} e^{i\mathbf{q}\mathbf{x}} A(\mathbf{q}', t) \langle \mathbf{k} + \tfrac{1}{2}\mathbf{q}, \alpha' | e^{i\mathbf{q}'\hat{\mathbf{x}}} | \mathbf{k} - \tfrac{1}{2}\mathbf{q}, \alpha \rangle. \tag{A.2}$$

Using the Bloch functions

$$\langle \mathbf{x} \mid \mathbf{k}\alpha \rangle = e^{i\mathbf{k}\mathbf{x}} u_{\mathbf{k}\alpha}(\mathbf{x}), \tag{A.3}$$

we can write (A.2) in the form

$$\begin{aligned} A^{\text{ph}}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t) &= \sum_{\mathbf{q}\mathbf{q}'} e^{i\mathbf{q}\mathbf{x}} A(\mathbf{q}', t) \int d^3x\, e^{i(\mathbf{q}' - \mathbf{q})\mathbf{x}'} u^*_{\mathbf{k} + (1/2)\mathbf{q}, \alpha'}(\mathbf{x}')\, u_{\mathbf{k} - (1/2)\mathbf{q}, \alpha}(\mathbf{x}') \\ &= \sum_{\mathbf{q}} e^{i\mathbf{q}\mathbf{x}} A(\mathbf{q}, t) \int d^3x'\, u^*_{\mathbf{k} + (1/2)\mathbf{q}, \alpha'}(\mathbf{x}')\, u_{\mathbf{k} - (1/2)\mathbf{q}, \alpha}(\mathbf{x}'), \end{aligned} \tag{A.4}$$

where, in the second step, we have used the periodicity of $u_{\mathbf{k}\alpha}$. Now, since $|\mathbf{k}| \sim a^{-1}$ and $|\mathbf{q}| \ll a^{-1}$, we can expand

$$u_{\mathbf{k} + (1/2)\mathbf{q}, \alpha}(\mathbf{x}) = u_{\mathbf{k}\alpha}(\mathbf{x}) + O(|\mathbf{q}|\, a) \tag{A.5}$$

and obtain, together with the orthogonality relation $\langle \mathbf{k}\alpha' \mid \mathbf{k}\alpha \rangle = \delta_{\alpha\alpha'}$, for the lowest order of (A.4),

$$A^{\text{ph}}_{\alpha'\alpha}(\mathbf{x}, \mathbf{k}, t) = \sum_{\mathbf{q}} e^{i\mathbf{q}\mathbf{x}} A(\mathbf{q}, t)\, \delta_{\alpha\alpha'} = \delta_{\alpha\alpha'} A(\mathbf{x}, t) \tag{A.6}$$

which is equivalent to (4.53).

APPENDIX B

In this appendix we derive Eqs. (4.96) and (4.97). Replacing $H(t)$ in (2.15) by

$$\bar{H}_0(t) = \tfrac{1}{2} \sum_{ll'} \sum_{bb'} \bar{\varepsilon}_{l'l}(t; b'^{\dagger} b^{\dagger})\, N(b'_{l'} b_l) \tag{B.1}$$

and using I(A.6), we arrive at the following differential equation for the retarded or advanced Green's function in lowest order including Hartree–Fock contributions,

$$\frac{\partial}{\partial t} \bar{G}^{B(0)}_{ll'}(t, t'; bb')$$
$$= -i \sum_{\bar{l}} \sum_{\bar{b}} \bar{\varepsilon}_{l\bar{l}}(t; b\bar{b}^{\dagger})\, \bar{G}^{B(0)}_{\bar{l}l'}(t, t'; \bar{b}b') \begin{cases} 1 & \text{for } b \sim c \\ \pm 1 & \text{for } b \sim c^{\dagger}. \end{cases} \tag{B.2}$$

Applying this equation to the Bloch electrons and phonons, together with a phase-space transformation according to (3.27), (4.31), (4.32), (4.47), and (4.48), we obtain in operator form

$$\frac{\partial}{\partial t} \bar{G}^{e(0)}(\mathbf{x}, t, t') = -i\bar{\varepsilon}^{e}(\mathbf{x}, t)\, \bar{G}^{e(0)}(\mathbf{x}, t, t') \tag{B.3}$$

$$\frac{\partial}{\partial t} \bar{G}^{p(0)}(\mathbf{x}, t, t') = -i\bar{\varepsilon}^{p}(\mathbf{x}, t)\, \bar{G}^{p(0)}(\mathbf{x}, t, t') \tag{B.4}$$

From (4.50) we obtain immediately the solution of (B.4),

$$\bar{G}^{p(0)}_{ss'}(\mathbf{x}, \mathbf{q}, t, t') \cong \delta_{ss'}\, e^{-i\bar{\varepsilon}^{p}_{ss}(\mathbf{q})(t-t')}, \tag{B.5}$$

where we have neglected all contributions coming from $s \neq s'$ since $\bar{\varepsilon}^{p}_{ss'} \ll \bar{\varepsilon}^{p}_{ss}, \bar{\varepsilon}^{p}_{s's'}$.
Using (4.95), (4.64), (4.72), and

$$\langle \mathbf{k}\alpha\mathbf{x}t|\, \frac{\partial}{\partial t} \bar{G}^{e(0)}(\mathbf{x}, t, t')\, |\mathbf{k}\alpha'\mathbf{x}t\rangle$$
$$= \frac{\partial}{\partial t} \bar{\bar{G}}^{e(0)}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t, t') - \frac{e}{c} \dot{\mathbf{A}}(\mathbf{x}, t) \sum_{\alpha_1} \mathbf{u}_{\alpha\alpha_1}(\mathbf{x}, \mathbf{k}, t)\, \bar{\bar{G}}^{e(0)}_{\alpha_1\alpha'}(\mathbf{x}, \mathbf{k}, t, t') \tag{B.6}$$

(compare with (4.68)), we can express (B.3) in the basis functions (4.59)

$$\frac{\partial}{\partial t} \bar{\bar{G}}^{e(0)}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t, t') = -i \sum_{\alpha_1} \bar{\varepsilon}^{T}_{\alpha\alpha_1}(\mathbf{x}, \mathbf{k}, t)\, \bar{\bar{G}}^{e(0)}_{\alpha_1\alpha'}(\mathbf{x}, \mathbf{k}, t, t') \tag{B.7}$$

with the lowest order solution (see also (4.76), (4.77))

$$\bar{\bar{G}}^{e(0)}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t, t') \cong \delta_{\alpha\alpha'}\, e^{-i\int_{t'}^{t} d\tau\, \bar{\varepsilon}^{T}_{\alpha\alpha}(\mathbf{x}, \mathbf{k}, \tau)}. \tag{B.8}$$

APPENDIX C

In this appendix we derive Eq. (4.104). First of all, the expectation value of (4.103) can be written as

$$\langle \hat{\mathbf{j}}(\mathbf{x}, t) \rangle_{\rho(t)} = -\frac{e}{c} \operatorname{Re} \sum_{kk'} \langle k' | \left[\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A}(\mathbf{x}, t) \right] \delta(\hat{\mathbf{x}} - \mathbf{x}) \, |k\rangle \, F^{+}_{kk'}(t; cc^{\dagger}) \qquad (\text{C.1})$$

or with (4.27) and (3.19)

$$\langle \hat{\mathbf{j}}(\mathbf{x}, t) \rangle_{\rho(t)} = -\frac{e}{c} \operatorname{Re} \sum_{\mathbf{k}, \mathbf{q}} \sum_{\alpha\alpha'} \left\langle \mathbf{k} - \frac{\mathbf{q}}{2}, \alpha' \right| \left[\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A}(\mathbf{x}, t) \right]$$

$$\delta(\hat{\mathbf{x}} - \mathbf{x}) \left| \mathbf{k} - \frac{\mathbf{q}}{2}, \alpha \right\rangle f^{+}_{\alpha\alpha'}(\mathbf{q}, \mathbf{k}, t) \qquad (\text{C.2})$$

Using (4.53), together with

$$\mathbf{p}_{\alpha'\alpha}\left(\mathbf{k} - \frac{\mathbf{q}}{2} \right) = \mathbf{p}_{\alpha'\alpha}(\mathbf{k}) + O(|\mathbf{q}| \, a) \qquad (\text{C.3})$$

and (see (A.3) and (A.5))

$$\left\langle \mathbf{k} - \frac{\mathbf{q}}{2}, \alpha_1 \right| \delta(\hat{\mathbf{x}} - \mathbf{x}) \left| \mathbf{k} + \frac{\mathbf{q}}{2}, \alpha \right\rangle$$

$$= e^{i\mathbf{q}\mathbf{x}} u^{*}_{\mathbf{k}-\mathbf{q}/2, \alpha_1}(\mathbf{x}) \, u_{\mathbf{k}+\mathbf{q}/2, \alpha}(\mathbf{x}) \cong e^{i\mathbf{q}\mathbf{x}} u^{*}_{\mathbf{k}\alpha_1}(\mathbf{x}) \, u_{\mathbf{k}\alpha}(\mathbf{x}), \qquad (\text{C.4})$$

Eq. (C.2) can be transformed to

$$\langle \hat{\mathbf{j}}(\mathbf{x}, t) \rangle_{\rho(t)} = -\frac{e}{c} \operatorname{Re} \sum_{\mathbf{k}} \sum_{\alpha\alpha'} \sum_{\alpha_1}$$

$$\times \langle \mathbf{k}\alpha' | \, \hat{\mathbf{p}} - \frac{e}{c} \mathbf{A}(\mathbf{x}, t) \, | \mathbf{k}\alpha_1 \rangle \, u^{*}_{\mathbf{k}\alpha_1}(\mathbf{x}) \, u_{\mathbf{k}\alpha}(\mathbf{x}) \, f^{+}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t) \qquad (\text{C.5})$$

and the average over a unit cell yields

$$\overline{\langle \hat{\mathbf{j}}(\mathbf{x}, t) \rangle_{\rho(t)}} = -\frac{e}{m\Omega} \operatorname{Tr}^{(e)}_{1} \left[\hat{\mathbf{p}}_1 - \frac{e}{c} \mathbf{A}(\mathbf{x}, t) \right] f^{+}_{1}(\mathbf{x}, t), \qquad (\text{C.6})$$

where we have used the periodicity property of $u_{\mathbf{k}\alpha}(\mathbf{x})$,

$$\overline{u^{*}_{\mathbf{k}\alpha_1}(\mathbf{x}) \, u_{\mathbf{k}\alpha}(\mathbf{x})} = \frac{1}{\Omega} \langle \mathbf{k}\alpha_1 \mid \mathbf{k}\alpha \rangle = \frac{1}{\Omega} \delta_{\alpha_1, \alpha}, \qquad (\text{C.7})$$

and the fact that $\mathbf{A}(\mathbf{x}, t)$ and $f^{+}_{\alpha\alpha'}(\mathbf{x}, \mathbf{k}, t)$ are slowly varying functions in $\mathbf{x}$.

Equation (C.6) can as well be written in the basis (4.59) and we obtain from (4.59), (4.63), and (4.83)

$$\overline{\langle \hat{\mathbf{j}}(\mathbf{x}, t) \rangle_{\rho(t)}} = -\frac{e}{m}\frac{1}{\Omega}\sum_{\mathbf{p}}\sum_{\alpha\alpha'} \langle \mathbf{p}\alpha' | \hat{\mathbf{p}} | \mathbf{p}\alpha \rangle \phi^{+}_{\alpha\alpha'} \tag{C.8}$$

or with (4.80) and

$$\langle \mathbf{p}\alpha' | \hat{\mathbf{p}} | \mathbf{p}\alpha \rangle = \delta_{\alpha\alpha'} \frac{\partial \varepsilon^{(0)}_{\alpha}}{\partial \mathbf{p}}(\mathbf{p}) + [\varepsilon^{(0)}_{\alpha}(\mathbf{p}) - \varepsilon^{(0)}_{\alpha'}(\mathbf{p})]\, \tilde{\mathbf{u}}_{\alpha'\alpha}(\mathbf{p}) \tag{C.9}$$

(see (4.92) for the definition of $\tilde{\mathbf{u}}_{\alpha'\alpha}$) in lowest order,

$$\overline{\langle \hat{\mathbf{j}}(\mathbf{x}, t) \rangle_{\rho(t)}} = -\frac{e}{m}\frac{1}{\Omega}\sum_{\mathbf{p}}\sum_{\alpha} \frac{\partial \varepsilon^{(0)}_{\alpha}}{\partial \mathbf{p}}(\mathbf{p})\, \phi^{+}_{\alpha}(\mathbf{x}, \mathbf{p}, t). \tag{C.10}$$

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References

1. H. Schoeller, to be published.
2. J. Rammer and H. Smith, *Rev. Mod. Phys.* **58** (1986), 323.
3. J. Rammer, Lic. Scient. thesis, University of Copenhagen, 1984.
4. F. A. Buot, *Phys. Rev. B* **10** (1974), 3700.
5. F. A. Buot, *J. Appl. Phys.* **68** (1990), 3418.
6. J. Zak, *in* "Solid State Physics," (H. Ehrenreich, F. Seitz, and D. Turnbull, Eds.), Vol. 27, p. 1, Academic Press, New York, 1972.
7. G. H. Wannier, *Rev. Mod. Phys.* **34** (1962), 645.
8. L. M. Roth, *J. Phys. Chem. Solids* **23** (1962), 433.
9. N. W. Ashcroft and N. D. Mermin, "Solid State Physics," Holt, Rinehart, & Winston, New York, 1976.
10. J. M. Ziman, "Electrons and Phonons," Oxford Univ. Press, London, 1960.
11. J. B. Krieger and G. J. Iafrate, *Phys. Rev. B* **35** (1987), 9644.
12. J. R. Barker, *J. Phys. C* **6** (1973), 2663.
13. J. R. Barker, *in* "Physics of Non-Linear Transport in Semiconductors, Proceedings, NATO-ASI," (D. K. Ferry, J. R. Barker, and C. Jacoboni, Eds.), p. 126, Plenum, New York, 1980.
14. N. Pottier and D. Calecki, *Physica A* **110** (1982), 471.
15. L. Reggiani, *Physica B* **134** (1985), 123.
16. F. S. Khan, J. H. Davies, and J. W. Wilkins, *Phys. Rev. B* **36** (1987), 2578.
17. P. Lipavsky, V. Spicka, and B. Velicky, *Phys. Rev. B* **34** (1986), 6933.
18. W. Hänsch and G. D. Mahan, *Phys. Rev. B* **38** (1983), 1886.
19. W. Hänsch and G. D. Mahan, *Phys. Rev. B* **38** (1983), 1902.

20. G. D. Mahan, *Phys. Rep.* **145** (1987), 251.
21. R. Bertoncini, A. M. Kriman, and D. K. Ferry, *J. Phys. (Condens. Matter)* **2** (1990), 5991.
22. S. R. de Groot, "La Transformation de Weyl et la Fonction de Wigner: Une Forme Alternative de la Mécanique Quantique," Presses Univ. Montréal, Montréal, 1974.
23. R. J. Hardy, *Phys. Rev.* **132** (1963), 168.
24. R. E. Peierls, "Quantum Theory of Solids," Oxford Univ. (Clarendon), Oxford, 1955.