Chapter 3

Air damping

For a conventional machine, the damping effects caused by the surrounding air can in general be ignored. This is because the energy dissipation rate caused by the air damping is much smaller than the energy supplied to the system if the moving speed of the mechanical parts is not excessively high. However, in the development of micro mechanical devices, estimating the damping effects of the system is one of the most important steps in the design process, since they determine the dynamic performance of the devices.

As air damping is related to the surface area of the moving parts, air damping may become very important for micro-mechanical devices and systems in determining their dynamic performance due to the large surface area to volume ratio of the moving parts. For some micromechanical devices, the energy consumed by air damping must be minimized so that the motion of mechanical parts can be maximized with a limited energy supply. For other situations, air damping has to be controlled so that the system energy is consumed by the air damping at a proper rate to ensure that the system has an optimum dynamic performance.

In this chapter, the basic concept of air damping is introduced and different air damping mechanisms as well as the damping effects for some typical micro structures will be discussed.

§3.1. Viscous flow of a fluid

§3.1.1. Viscosity of a fluid

(1) The coefficient of viscosity of a fluid

Although a fluid at rest cannot permanently resist the attempt of a shear stress to change its shape, viscous force appears to oppose the relative motion between different layers of the fluid. Viscosity is thus an internal friction between adjacent layers moving with different velocities.

The internal shear force in a steady flow of a viscous fluid is proportional to the velocity gradient. If the flow is in the x-direction and the speed of the flow is distributed in the y-direction, i.e., the flow velocity in the x-direction, u, is a function of y, the shear force τ_{yx} is:

$$\tau_{yx} = \mu \frac{du(y)}{dy} \tag{3.1}$$

where μ is the coefficient of viscosity of the fluid. For a gas, the coefficient of viscosity is a constant for a steady flow. For many pure liquids, the coefficient of viscosity is also a constant. These liquids are called Newtonian liquids.

According to Eq. (3.1), the coefficient of viscosity has a unit of Pa·sec or Pa·s. At room temperature (20°C), air has a coefficient of viscosity of 1.8×10^{-5} Pa·s and the coefficient of viscosity of water is 1.0×10^{-3} Pa·s.

(2) The mechanism of viscosity

Though both liquid and gas show viscosity, they have different properties due to different mechanisms.

For a steady liquid, the relative positions of adjacent molecules in the same layer are basically stable, but the relative positions of molecules in adjacent layers of a laminar flow change due to the flow. Fig. 3.1 shows the change of the relative position between molecules A and B in adjacent layers with different flow velocities, where the molecule A has a higher velocity than molecule B. The approach of A and B is accompanied by a decrease of intermolecular potential energy and an increase in molecular kinetic energy. While the molecular kinetic energy becomes disordered, a temporary bond is formed. The external force must do work if the molecules are later to be separated. The work done by the external force becomes random energy.

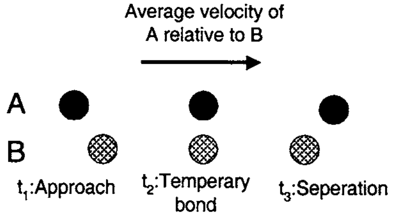


Fig. 3.1. Mechanism of viscosity in liquid

According to the mechanism described for a liquid, a temperature increase means that the molecules have a greater thermal speed, which in turn allows a smaller time in which the molecular energy can be disordered (i.e., less energy is needed to de-bond the molecular pair later). Therefore, the viscosity of most liquids decreases with temperature.

For gases, the thermal motion of a molecule is much larger than its drift motion related to the flow of the gas. In Fig. 3.2, the molecule A with a smaller drift velocity moving up across the boundary CD (due to the thermal motion) acquires a larger drift velocity, i.e., gains drift momentum, and experiences a force to the right. This means that the molecule has exerted a force to the left on the upper layer, which tends to retard the faster layer.

Similarly, the molecule B in the faster layer moving down across the boundary CD (due to the thermal motion) exerts a force to the right on the slower layer into which it moves.

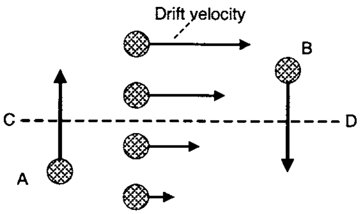


Fig. 3.2. Mechanism of viscosity in gas

(3) The temperature dependence of viscosity of gas

Due to the mechanism described above, a temperature increase means that molecules have a greater thermal speed, which increases the rate at which they cross the layers. Therefore, the viscosity of a gas increases with temperature. A quantitative analysis by a simple model based on the kinetic theory of gas [1] predicts that:

$$\mu = \frac{1}{3}\rho\bar{v}\lambda \tag{3.2}$$

where ρ is the density, $\bar{v}$ is the average velocity of the molecules and λ is the mean free path of the molecules. According to the Kinetic Theory of gas, $\bar{v}$, λ and ρ are:

$$\bar{v} = \sqrt{\frac{8RT}{\pi M_m}}, \ \lambda = \frac{1}{\pi\sqrt{2}nd^2} \text{ and } \rho = n\frac{M_m}{N_{av}}$$

respectively, where R is the Universal Molar constant (R=8.31 kg·m^2/sec^2/°K), M_m the molar mass, d the effective molecular diameter of the gas, T the absolute temperature and N_{av} the Avogadro constant ($N_{av} = 6.0247\times10^{23}\ /\ mol$). Therefore, we have:

$$\mu = \frac{2\sqrt{R}}{\pi n d^2}\sqrt{M_m T} \tag{3.3}$$

Eq. (3.3) suggests that μ is independent of pressure, *P*. Maxwell confirmed experimentally that this result is true over a wide range of pressure, provided that the pressure is not too small. Eq. (3.3) also indicates that μ increases in direct proportion to $\sqrt{M_m}$ and $\sqrt{T}$. Experiments have confirmed that μ increases with temperature but the power slightly exceeds 1/2.

The temperature and molecular dependence of μ can be expressed by an empirical relation known as Sutherland Equation [2]:

$$\mu = \mu_o \frac{1+T_S/T_o}{1+T_S/T}\sqrt{\frac{T}{T_o}} \tag{3.4}$$

where T_o=273.16K, μ_o is the coefficient of viscosity at T_o and T_S is a constant. μ_o and T_S are dependent on the specific gas considered.

μ_o and T_S for some gases are listed in Table 3.1.

Table 3. 1. μ_o and T_S for some gases

gas	air	N_2	H_2	CO_2
$\mu_o/10^{-6}$ (Pa·*S*)	17.2	16.6	8.40	13.8
T_S/°*K*	124	104	71	254

Usually, the coefficient of viscosity of liquid is much more sensitive to temperature than that of gas. The data for the coefficient of viscosity of water under one atmosphere are listed in Table 3.2. For comparison, the data for air are also listed.

Table 3.2. Temperature dependence of coefficient of viscosity for water and air (in 10^{-3} Pa·S for water and in 10^{-6} Pa·S for air)

t/°C	0	10	20	30	40	50	60	70	80	90	100
H_2O	1.79	1.30	1.02	0.80	0.65	0.55	0.47	0.41	0.36	0.32	0.28
air	17.2	17.8	18.1	18.7	19.2	19.6	20.1	20.4	21.0	21.6	21.8

§3.1.2. Viscous flow

(1) Equations for viscous flow

Consider a cubic element in a fluid as shown in Fig. 3.3. There are six shearing force components on its surface caused by the velocity gradient of the flow: $\tau_{xy}(x_o)$, $\tau_{xy}(x_o+dx), \tau_{yz}(y_o), \tau_{yz}(y_o+dy), \tau_{zx}(z_o), \tau_{zx}(z_o+dz)$.

There are also six normal force components on its surface caused by pressure: $P(x)dydz$, $P(x+dx)dydz$, $P(y)dxdz$, $P(y+dy)dxdz$, $P(z)dxdy$, and $P(z+dz)dxdy$.

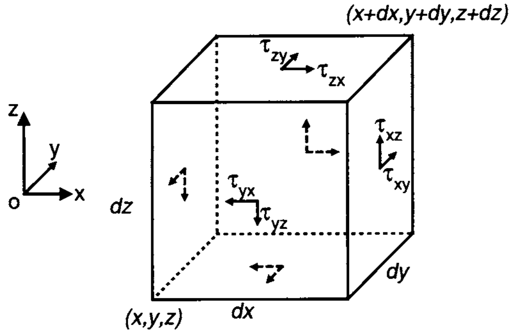


Fig. 3.3. Shearing stresses on the surfaces of an element cube

For a steady flow, assuming the weight of the fluid is negligible, the force balance for the cube in the z-direction is:

$$[P(z)-P(z+dz)]dxdy+[\tau_{xz}(x+dx)-\tau_{xz}(x)]dydz$$
$$+[\tau_{yz}(y+dy)-\tau_{yz}(y)]dxdz=0$$

Therefore, we have:

$$\frac{\partial P}{\partial z}=\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}$$

As $\tau_{xz}=\mu\frac{\partial w}{\partial x}$, $\tau_{yz}=\mu\frac{\partial w}{\partial y}$,

$$\frac{\partial P}{\partial z}=\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}) \tag{3.5}$$

where w is the velocity component in z-direction. For the same reason, we have:

$$\frac{\partial P}{\partial y}=\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}) \tag{3.6}$$

and

$$\frac{\partial P}{\partial x}=\mu(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}) \tag{3.7}$$

where u and v are velocity components in the x- and y-directions, respectively. Eqs. (3.5), (3.6) and (3.7) are equations for viscous flow of a fluid caused by a pressure, P.

(2) Flow in a pipe

Let the length of the pipe be L and the radius of the circular cross section equal to a, and $L>>a$, as shown in Fig. 3.4. If z-axis is taken along the centroid of the pipe, Eq. (3.5) is the only equation to be used to decide the flow.

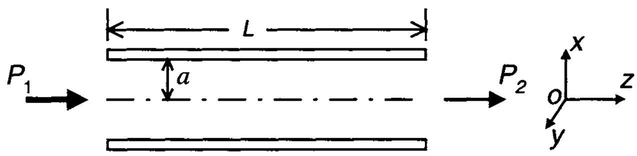


Fig. 3.4. Fluid flow in a long pipe

As the length of pipe, L, is much larger than its radius, a, the flow in the pipe is in the z-direction and the velocity distribution is symmetric against the z-axis. By using polar coordinates in the x-y plane and putting the origin at the center of the cross section of the pipe, Eq. (3.5) can be written as:

$$\frac{\partial P}{\partial z} = \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} w(r)\right)$$

By integration:

$$r\frac{\partial}{\partial r} w(r) = \frac{1}{2\mu}\frac{\partial P}{\partial z} r^2 + C_1$$

As $\frac{\partial}{\partial r} w(r) = 0$ at r=0 due to the symmetric distribution, C_1=0. By a second integration:

$$w(r) = \frac{1}{4\mu}\frac{\partial P}{\partial z} r^2 + C_2$$

According to the boundary condition of $w(a) = 0$, we find:

$$w(r) = -\frac{1}{4\mu}\frac{\partial P}{\partial z}(a^2 - r^2) \tag{3.8}$$

The negative sign indicates that the velocity is in the opposite direction of the pressure gradient. If the pressure difference between the two ends of the pipe is P, i.e. $\frac{\partial P}{\partial z} = \frac{P}{L}$, we have:

$$w(r) = -\frac{1}{4\mu}\frac{P}{L}(a^2 - r^2)$$

The flow rate, i.e., the volume of fluid passing through the pipe per unit time, is:

$$Q = \int_0^a |w(r)| 2\pi r dr$$

By simple calculation:

$$Q = \frac{\pi a^4}{8\mu}\frac{P}{L} \tag{3.9}$$

and the average velocity of the flow is:

$$\overline{w} = \frac{Q}{\pi a^2} = \frac{a^2}{8\mu}\frac{P}{L} \tag{3.10}$$

(3) Reynolds' Number

The flow pattern described in the above is an orderly flow that is called streamline flow or laminar flow. Streamline flow occurs only when the speed of the flow is small. The flow will become turbulent if the speed of the flow exceeds a certain limit. The criterion for turbulence is usually given by the value of the Reynolds' number, Re. Reynolds' number, Re, is a dimensionless number that, for a tube, takes the form of:

$$\mathrm{Re} = \frac{\overline{v}\rho d}{\mu}$$

where ρ is the specific mass of the fluid, $\overline{v}$ the velocity of the fluid and d the diameter of the tube.

Re is a convenient parameter for measuring the stability of flow. However, the critical value of Re that causes instability of fluid flow depends strongly on the shape of the tube and can only be determined by experiments. For tubes with circular cross-section, we have:

(a) Re$<$ 2200, the flow is laminar

(b) Re$\sim$ 2200, the flow is unstable

(c) Re$\geq$ 2200, the flow is turbulent

The Reynolds' number is also useful in measuring the stability of fluid flowing through a solid object inside the fluid (or, the moving of a solid object through a fluid at rest). In this case, the general form of the Reynolds' number is:

$$\mathrm{Re} = \frac{v\rho l}{\mu}$$

where ρ is the specific mass, μ the coefficient of viscosity of the fluid, v the relative speed between the object and the fluid at rest and l is a characteristic dimension of the object. For example, l is the diameter of a sphere and, for a column with a circular cross section moving through the fluid laterally, l is the diameter of the cross section, etc. The critical value of the Reynolds' number that causes instability depends on the shape of the object and can only be determined by experiments.

§3.1.3. Drag force on a moving object

Drag force will be applied on a body if the body is held steadily in a flow of fluid (or the body is dragged through a steady fluid) because there exists a velocity gradient between the boundary layer and the more distant points in the viscous fluid. As the analysis for the drag force is quite complicated, the drag forces for some simple body structures moving with small speeds through an infinitive viscous fluid are given here [3].

(1) Sphere with a radius r:

$$F = 6\pi\mu r v \tag{3.11}$$

(2) Circular dish with a radius of r, moving in its normal direction:

$$F = 16\mu r v \tag{3.12}$$

(3) Circular dish with a radius of r moving in its plane direction:

$$F = \frac{32}{3}\mu r v \tag{3.13}$$

where v is the speed of the circular dish relative to the distant fluid.

When Eqs. (3.11), (3.12) and (3.13) are compared, we can find that the dependence of drag forces on different cross sections or on the moving direction are not significant. All three drag forces for low speed motion can be written in the same form as:

$$F = 6\pi\alpha\mu r v \tag{3.14}$$

where the value of α for a sphere, a dish moving in its normal direction and a dish moving in its plane direction are α=1.0, 0.85 and 0.567, respectively. Note that drag forces are independent of the specific mass of the fluid, ρ (Stokes' law).

However, this conclusion is not true for higher moving speeds. The force working on a sphere with a radius, r, oscillating in a fluid is given by [3]:

$$F = -\beta_1 v - \beta_2 \frac{dv}{dt} \tag{3.15}$$

with

$$\beta_1 = 6\pi\mu r + 3\pi r^2 \sqrt{2\rho\mu\omega}$$

and

$$\beta_2 = \frac{2}{3}\pi\rho r^3 + 3\pi r^2 \sqrt{\frac{2\rho\mu}{\omega}}$$

where v is the relative moving velocity and ω the radial frequency of the motion. Note that both β_1 and β_2 are dependent on the specific mass of the fluid.

For even higher speeds, the flow may become turbulent. In a turbulent flow the drag force is proportional to the momentum change of the fluid, which, in turn, is proportional to the mass of fluid whose velocity is changed in a unit time and to the velocity change of the mass. Therefore, we have:

$$F \sim (\pi\rho r^2 v)(v)$$

or

$$F \sim \pi\rho r^2 v^2$$

It is now dependent on ρ and v but not on μ.

§3.1.4. The effects of air damping on micro-dynamics

As seen in *§3.1.3*, the drag force applied to a sphere moving in a viscous fluid at a speed of v is:

$$F = 6\pi\mu r v$$

where μ is the coefficient of viscosity of the fluid and r the radius of the sphere. The ratio between the drag force F and the mass of the body, M, is:

$$\frac{F}{M}=\frac{6\pi\mu rv}{\frac{4}{3}\pi r^3\rho}=\frac{4.5\mu v}{\rho r^2} \tag{3.16}$$

where ρ is the specific density of the body. It is obvious that for the same conditions, the smaller the dimension of the body the larger the effect of the drag force on the body. For example, for a silicon ball of radius $r = 1cm$ moving in air with a velocity of 1 cm/sec, F/M is 3.5×10^{-6} m/sec^2, while, for a silicon ball of radius 10 microns, F/M is 3.5 m/sec^2, one million times larger. Therefore, the drag force caused by the viscosity of the surrounding air (or other media) is usually negligible for conventional mechanical structure but it may play an important role for the motion of micro machines.

Now let us look at a practical example. The differential equation for a beam-mass (spring-mass) accelerometer is:

$$m\ddot{x} = -kx - c\dot{x}$$

where k is the spring constant of the beam and c is the coefficient of damping force caused by the surrounding medium such as air. A very important dynamic parameter of the accelerometer is the damping ratio of the system, ζ. The definition of ζ is:

$$\zeta=\frac{c}{2m\omega_o}=\frac{c}{2\sqrt{mk}}$$

where ω_0 is the free vibration frequency of the system. The damping ratio, ζ, for an accelerometer is usually required to be around 0.7 so that the system shows the best frequency response to an input signal (not shown in the equation). Quite often, the quality factor, Q, is used to characterize the mechanical system. For small damping, the relation between the quality factor and the damping ratio is: $Q=\frac{1}{2\zeta}$.

According to Eq. (3.14), the coefficient of damping force, c, is proportional to the dimensions of the mechanical structure and the coefficient of viscosity of the surrounding fluid. As m is quite large for an accelerometer made of conventional mechanical structures, ζ is usually very small in air. It is quite difficult to raise the damping ratio, ζ, to around 0.7 even if the structure is filled with oil of high viscosity. But for an accelerometer formed using a micromechanical structure, the damping ratio, ζ, can be easily raised to around 0.7 in air by using some mechanical structure to increase the damping force in a controlled way. The advantages of air damping as opposed oil damping include a much lower temperature coefficient and ease of packaging the device. The basic mechanisms of air damping for micromechanical structures are squeeze-film air damping and slide-film air damping. The basic principles and relations for these mechanisms will be described in §3.2 and §3.3.

Air damping is expected to be reduced to a minimum for a high Q factor in many micromechanical systems, such as in resonant sensors or gyroscopes (see Chapter 9). In these cases, air should be evacuated from a hermetically sealed package where the micromechanical structures are housed. The damping of microstructures in rare air will be discussed in §3.4.

§3.2. Squeeze-film air damping

§3.2.1. Basic equations for squeeze-film air damping

(1) Squeeze-film air damping

When a plate is placed in parallel to a wall and moving towards the wall, the air film between the plate and the wall is squeezed so that some of the air flows out of the gap. Therefore, an additional pressure Δp develops in the gap due to the viscous flow of the air, as shown in Fig. 3.5.

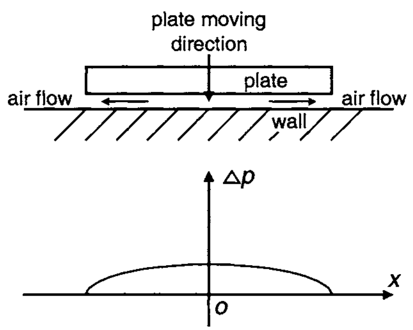


Fig. 3.5. Pressure built up by squeeze-film motion

On the contrary, when the plate is moving away from the wall, the pressure in the gap is reduced to keep the air flowing into the gap.

In both cases, the forces on the plate caused by the built-up pressure are always against the movement of the plate. The work done by the plate is consumed by the viscous flow of the air and transformed into heat. In other words, the air film acts as a damper and the damping is called squeeze-film air damping.

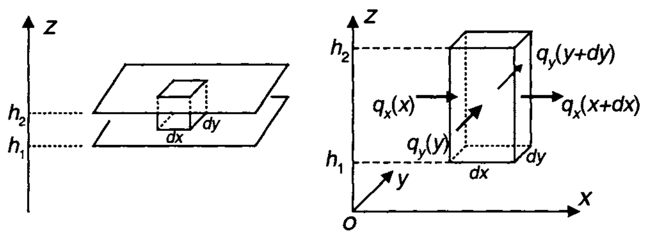
Obviously, the damping force of squeeze-film air damping is dependent on the gap distance; the smaller the gap, the larger the damping force. When the plate is very far away from the wall, the pressure build-up is negligible and the damping force will be reduced to the drag force discussed in §3.1.

Squeeze-film air damping is quite often used to increase the effect of air damping to an expected level for micro structures and the damping force can be controlled by the distance of the air gap.

(2) Basic equations

Suppose we have a pair of plates in parallel with the x-y plane of the Cartesian coordinates as shown in Fig. 3.6 and the dimensions of the plates are much larger than the distance between them so that the gas flow between the plates caused by the relative motion of the plates is lateral (in the x-and y-direction but not in the z-direction).

Let us consider a column element, $hdxdy$ (where $h = h_2 - h_1$), as shown in Fig. 3.6, where q_x is the flow rate in the x-direction per unit width of the y-direction and q_y is the flow rate in the y-direction per unit width of the x-direction.



(a) (b)

Fig. 3.6. Mass flow into and out of an elemental unit
(a) A column element between two plates, (b) the definitions of flow rates

The balance of mass flow for the column element requires:

$$(\rho q_x)_x dy - (\rho q_x)_{x+dx} dy + (\rho q_y)_y dx - (\rho q_y)_{y+dy} dx = \left(\frac{\partial \rho h_2}{\partial t} - \frac{\partial \rho h_1}{\partial t}\right) dxdy$$

By making use of the relations $(\rho q_x)_{x+dx} = (\rho q_x)_x + \dfrac{\partial(\rho q_x)}{\partial x} dx$, $(\rho q_y)_{y+dy} = (\rho q_y)_y + \dfrac{\partial(\rho q_y)}{\partial y} dy$ and h=h_2–h_1, we have:

$$\frac{\partial(\rho q_x)}{\partial x} + \frac{\partial(\rho q_y)}{\partial y} + \frac{\partial(\rho h)}{\partial t} = 0 \tag{3.17}$$

To find q_x and q_y for the equation, we first have to find the speed distribution in the z-direction. To do this we cut a section element from the column element between z and $z+dz$, as shown in Fig. 3.7. The force balance in the x-direction requires:

$$P(x)dydz + T_{zx}(z+dz)dxdy = P(x+dx)dydz + T_{zx}(z)dxdy$$

Therefore, we have:

$$\frac{\partial T_{zx}}{\partial z} = \frac{\partial P}{\partial x}$$

According to §3.1, we have:

$$T_{zx} = \mu \frac{\partial u}{\partial z}$$

where u is the component of velocity in the x-direction. Therefore we have:

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$$

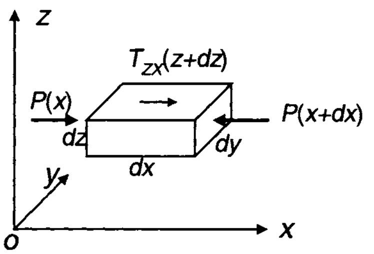


Fig. 3.7. Force balance on a section element

For a small gap, $P(x,y)$ is not a function of z. By integrating the equation twice we have:

$$u(z) = \frac{1}{2\mu}\frac{\partial P}{\partial x}z^2 + C_1\frac{1}{\mu}z + C_2 \tag{3.18}$$

If the plates do not move laterally and we put the origin of the coordinates on the bottom plate, the boundary conditions for Eq. (3.18) are:

$$u(0) = 0, \; u(h) = 0$$

Therefore:

$$u(z) = \frac{1}{2\mu}\frac{\partial P}{\partial x}z(z-h) \tag{3.19}$$

The flow rate in the x-direction for a unit width in the y-direction is:

$$q_x = \int_0^h u dz = -\frac{h^3}{12\mu}\left(\frac{\partial P}{\partial x}\right) \tag{3.20}$$

The negative sign in the equation indicates that the flow is in the direction with decreasing pressure.

Similarly:

$$q_y = -\frac{h^3}{12\mu}\left(\frac{\partial P}{\partial y}\right) \tag{3.21}$$

By substituting Eqs. (3.20) and (3.21) into (3.17), we find:

$$\frac{\partial}{\partial x}\left(\rho\frac{h^3}{\mu}\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho\frac{h^3}{\mu}\frac{\partial P}{\partial y}\right) = 12\frac{d(h\rho)}{dt} \tag{3.22}$$

Eq. (3.22) is referred to as Reynolds' equation. In the process of the derivation of Eq. (3.22) it has been assumed that the fluid behavior is governed by viscous forces which are large relative to momentum changes. Alternatively, Eq. (3.22) can also be derived from the much more complicated Navier-Stokes equation under the condition that the Modified Reynolds' Number for a squeeze film, R_S, is much smaller than unity [4, 5], i.e., the condition of:

$$R_S = \frac{\omega h^2 \rho}{\mu} << 1$$

where ω is the radial frequency of the oscillating plate. This condition is satisfied for typical silicon microstructures. For example, an air-filled accelerometer with an air film thickness of 25 microns, oscillating at a frequency of 1 kHz, would have a modified Reynolds' number of $R_S = 0.26$.

As h is assumed to be uniform in both the x- and y-directions, we have:

$$\frac{\partial}{\partial x}\left(\rho\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho\frac{\partial P}{\partial y}\right) = \frac{12\mu}{h^3}\frac{d(h\rho)}{dt}$$

For an isothermal film, the air density, ρ, is proportional to pressure P, i.e., $\rho = \frac{P}{P_o}\rho_o$. The above equation can also be written as:

$$\frac{\partial^2}{\partial x^2}P^2 + \frac{\partial^2}{\partial y^2}P^2 = \frac{24\mu}{h^3}\frac{d(hP)}{dt} \tag{3.23}$$

or

$$\nabla^2 P^2 = \frac{24\mu}{h^3}\frac{d(hP)}{dt}$$

Eq. (3.23) can be developed into:

$$\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2 + P\left(\frac{\partial^2}{\partial x^2}P + \frac{\partial^2}{\partial y^2}P\right) = \frac{12\mu}{h^3}\left(P\frac{dh}{dt} + h\frac{dP}{dt}\right) \tag{3.24}$$

Assuming that $h=h_o+\Delta h$ and $P=P_o+\Delta P$, for small motion distance of the plate, we have $\Delta h<<h_o$ and $\Delta P<<P_o$. Under these conditions, Eq. (3.24) can be approximated as:

$$P_o\left(\frac{\partial^2 \Delta P}{\partial x^2} + \frac{\partial^2 \Delta P}{\partial y^2}\right) = \frac{12\mu}{h^3}P_o h_o\left(\frac{1}{h_o}\frac{d\Delta h}{dt} + \frac{1}{P_o}\frac{d\Delta P}{dt}\right)$$

If $\frac{\Delta P}{P_o} << \frac{\Delta h}{h_o}$, we have:

$$\frac{\partial^2 \Delta P}{\partial x^2} + \frac{\partial^2 \Delta P}{\partial y^2} = \frac{12\mu}{h^3}\frac{d\Delta h}{dt} \tag{3.25}$$

or,

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{12\mu}{h^3}\frac{dh}{dt} \tag{3.26}$$

In Eq. (3.26), P is equivalent to ΔP and h is equivalent to Δh. For convenience, the P in Eq. (3.26) is sometimes read as ΔP, the variation of pressure. However, attention must be given to the difference in the boundary conditions for P and ΔP: $P = P_o$ and $\Delta P = 0$ at the periphery of the plate.

Before ending this section, let us discuss once more the condition for Eq. (3.26). Suppose that the typical dimension of the plate is l (e.g. the radius of a disk or the half width of a rectangle) and the motion of the plate is a sinusoidal vibration with an amplitude δ, i.e., $h = h_o + \delta \sin \omega t$. From Eq. (3.26), we can make a rough and ready estimation of ΔP:

$$\frac{\Delta P}{l^2} = \frac{12\mu}{h_o^3}\delta\omega\cos\omega t$$

or

$$\frac{\Delta P}{P_o} = \frac{12\mu l^2 \omega}{P_o h_o^2}\frac{\delta\cos\omega t}{h_o} = \sigma\frac{\delta\cos\omega t}{h_o}$$

where $\sigma \equiv \dfrac{12\mu l^2\omega}{P_o h_o^2}$ is referred to as "squeeze number". As $\delta\cos\omega t$ has the same order of magnitude as $\Delta h = \delta\sin\omega t$, we obtain:

$$\frac{\Delta P}{P_o} \cong \sigma\frac{\Delta h}{h_o}$$

Therefore, the condition for the validity of Eq. (3.26), i.e. $\dfrac{\Delta P}{P_o} << \dfrac{\Delta h}{h_o}$, is equivalent to σ<<1, or

$$\frac{l}{h_o} << \sqrt{\frac{P_o}{12\mu\omega}}$$

As an example, let us assume the conditions that $\omega=2\pi\times10^3$/sec, $P_o=10^5$ Pa (i.e., 1 *atm.*) and $\mu=1.8\times10^{-5}$ Pa·sec (for air at 20°C). The requirement for l for the validity of Eq. (3.26) is $l<<854\ h_o$ (e.g., l<<17mm for h_o=20μm). For the same conditions but with higher oscillating frequency such as $\omega=2\pi\times10^4$/sec, the condition becomes $l<<84.5h_o$ (e.g. l<<1.7mm for h_0=20μm).

§3.2.2. Long rectangular plate

(1) Damping force for parallel motion

Consider a pair of rectangular plates with length, L, much larger than width, B. The origin of the Cartesian coordinates is at the center of the lower plate and the x-axis is along the width direction, as shown in Fig. 3.8. The problem is virtually one dimensional. As mentioned in §3.2.1, P in Eq. (3.26) is read as $\Delta P(x)$. Therefore, the boundary conditions are:

$$P\left(\pm\frac{1}{2}B\right) = 0 \qquad (2.27)$$

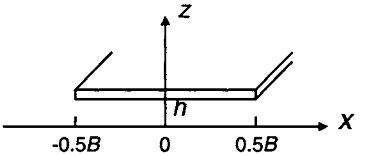


Fig. 3.8. Squeeze-film air damping of a long rectangular plate

As the problem is one dimensional, Eq. (3.26) is written as:

$$\frac{d^2P}{dx^2} = \frac{12\mu}{h^3}\frac{dh}{dt} \tag{3.28}$$

where h is the distance between the two plates. By integrating Eq. (3.28) twice, we obtain:

$$P(x) = \frac{6\mu}{h^3}\frac{dh}{dt}x^2 + C_1x + C_2$$

By using the boundary conditions, we obtain:

$$P(x) = -\frac{6\mu}{h^3}\left(\frac{B^2}{4} - x^2\right)\frac{dh}{dt} \tag{3.29}$$

$P(x)$ is positive when the air film is squeezed ($\frac{dh}{dt} < 0$), and vice versa. The maximum pressure build-up is at the center of the plate (x=0) where $P(0) = -\frac{3\mu B^2}{2h^3}\frac{dh}{dt}$. The distribution of the pressure build-up is shown in Fig. 3.9. The damping force F on the plate is:

$$F_{lr} = \int_{-\frac{B}{2}}^{\frac{B}{2}} P(x)Ldx = -\frac{\mu B^3 L}{h^3}\frac{dh}{dt} \equiv -\frac{\mu B^3 L}{h^3}\dot{h}$$

According to the definition of $F = -c\dot{x}$, the coefficient of damping force for a long rectangular plate is:

$$c_{lr} = \frac{\mu B^3 L}{h^3} \tag{3.30}$$

Note that Eq. (3.30) is only valid for rectangular plates whose length, L, is much larger than their width, B. For a rectangular plate with a comparable L and B, the squeeze-film air damping will be discussed in §3.2.4.

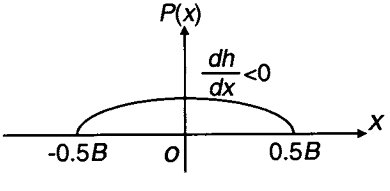


Fig. 3.9. Pressure distribution under a long rectangular plate

(2) Example

Suppose that the width of a pair of plates is B=2mm, the length of the plates is 10 mm, the gap between the two plates is h_o=20μm and the motion of the upper plate can be described by $h=h_o+\delta\sin\omega t$, where δ=1μm and $\omega=2\pi\times10^3$. The environment is air with a pressure of one atmosphere.

According to the condition described above, we have:

$$\frac{dh}{dt}=\delta\cdot\omega\cos\omega t$$

and

$$P(x)=-\frac{6\mu}{h^3}\left(\frac{B^2}{4}-x^2\right)\delta\cdot\omega\cos\omega t=-\frac{3\mu B^2}{2h^3}\left[1-\left(\frac{2x}{B}\right)^2\right]\delta\omega\cos\omega t$$

As the coefficient of viscosity of air is $\mu=1.8\times10^{-5}$ Pa·s (at 20°C), the pressure build-up at x=0 is:

$$P(0)=-85\cos\omega t\ (Pa)$$

Therefore, the maximum pressure built up by the squeeze film motion is 8.5×10^{-4} *atm*. As $\frac{\delta}{h}=0.05$, the result verifies that $\frac{\Delta P}{P_o}<<\frac{\delta}{h_o}$. The pressure is not easily to build up due to of the low viscosity nature of gas. This phenomenon is often described as "gas is incompressible".

According to Eq. (3.30) the coefficient of damping force is:

$$c_{lr}=0.182N/(m/\sec)$$

§3.2.3. Circular and annular plates

(1) Circular plate

For a circular plate moving against a wall, the equation for air damping can be written in a polar coordinate system as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}P(r)\right)=\frac{12\mu}{h^3}\frac{dh}{dt} \tag{3.31}$$

and the boundary conditions are:

$$P(a)=0,\ \frac{dP}{dr}(0)=0 \tag{3.32}$$

where a is the radius of the plate. By integrating Eq. (3.31) and using the boundary conditions in Eq. (3.32), we find:

$$P(r) = -\frac{3\mu}{h^3}\left(a^2 - r^2\right)\frac{dh}{dt} \tag{3.33}$$

The damping force on the circular plate is:

$$F_{cir} = \int_0^a P(r)2\pi r dr = -\frac{3\pi}{2h^3}\mu a^4 \frac{dh}{dt}$$

or

$$F_{cir} = -\frac{3}{2\pi}\frac{\mu A^2}{h^3}\frac{dh}{dt} = -0.4775\frac{\mu A^2}{h^3}\frac{dh}{dt} \tag{3.34}$$

where $A = \pi a^2$ is the area of the plates. The coefficient of damping force is:

$$c_{cir} = \frac{3\pi}{2h^3}\mu a^4 \tag{3.35}$$

(2) Annular plate

For an annular plate moving against a wall, the equation for air damping is the same as Eq. (3.31), but the boundary conditions are different. The boundary conditions are:

$P(a)=0$, $P(b)=0$

where a and b are the outer and inner radii of the annular plates as shown in Fig. 3.10. By solving Eq. (3.31) with the boundary conditions, the built-up pressure is:

$$P(r) = \left(-\frac{3\mu}{h^3}a^2\left(1 - \frac{r^2}{a^2}\right) + \frac{3\mu}{h^3}a^2\left(1 - \frac{b^2}{a^2}\right)\frac{\ln\frac{r}{a}}{\ln\frac{b}{a}}\right)\frac{dh}{dt} \tag{3.36}$$

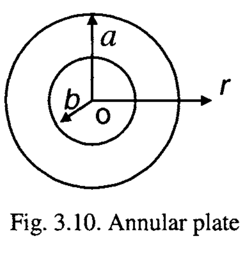


Fig. 3.10. Annular plate

If the ratio of b/a is denoted as β, we have the damping force for the annular plate:

$$F_{ann} = \int_b^a P(r) 2\pi\, r dr = -\frac{3\pi\mu a^4}{2h^3}\left(1-\beta^4+\frac{\left(1-\beta^2\right)^2}{\ln\beta}\right)\dot{h}$$

The force can be written as:

$$F_{ann} = -\frac{3\pi\mu a^4}{2h^3} K(\beta)\dot{h} = -\frac{3\mu A^2}{2\pi h^3} K(\beta)\dot{h}$$

where $A = \pi a^2$ and $K(\beta)$ is a function of $\beta = \dfrac{b}{a}$:

$$K(\beta) = 1-\beta^4+\frac{(1-\beta^2)^2}{\ln\beta}$$

The coefficient of damping force for an annular plate is:

$$c_{ann} = \frac{3\mu a^2 A}{2h^3} K(\beta) \tag{3.37}$$

§3.2.4. Rectangular plate

In this section, we will discuss the squeeze-film air damping for a rectangular plate in a general form. If the side lengths in the *x*- and *y*-directions of the plate are $B=2a$ and $L=2b$, respectively, as shown in Fig. 3.11 (here, *a* and *b* are comparable), the differential equation for pressure in the air film is Eq. (3.26) and the boundary conditions are:

$$P(\pm a, y) = 0,\ P(x, \pm b) = 0 \tag{3.38}$$

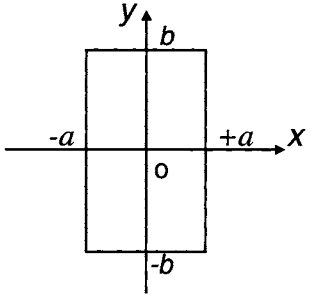


Fig. 3.11. Rectangular plate

The solution to Eq. (3.26) can be divided into two parts: $P = p_1 + p_2$, where p_1 is a specific solution to Eq. (3.26), i.e., p_1 is a solution to equation:

$$\frac{\partial^2 p_1}{\partial x^2}+\frac{\partial^2 p_1}{\partial y^2}=\frac{12\mu}{h^3}\dot{h} \tag{3.39}$$

and p_2 is a general solution to the Laplace equation:

$$\frac{\partial^2 p_2}{\partial x^2}+\frac{\partial^2 p_2}{\partial y^2}=0 \tag{3.40}$$

(1) Solution of p_1

Assuming that p_1 takes the form of:

$$p_1 = A + Bx + Cx^2$$

and meets the boundary conditions of $p_1(\pm a)=0$, we find:

$$p_1 = -\frac{6\mu}{h^3}\frac{dh}{dt}(a^2 - x^2) \tag{3.41}$$

(2) Boundary condition of p_2

From the definition of $P=p_1+p_2$ and the boundary conditions of $p(\pm a, y) = 0$, we have:

$$p_1(\pm a, y) + p_2(\pm a, y) = 0$$

According to Eq. (3.41), the boundary conditions for p_2 at $x = \pm a$ can be shown to be:

$$p_2(\pm a, y) = 0 \tag{3.42}$$

According to Eq. (3.38), the boundary conditions for P at $y = \pm b$ should be $P(x,\pm b) = 0$, i.e.,

$$p_1(x) + p_2(x,\pm b) = 0$$

Therefore, the boundary conditions for p_2 at $y = \pm b$ are:

$$p_2(x,\pm b) = -p_1(x) = \frac{6\mu}{h^3}\frac{dh}{dt}(a^2 - x^2) \tag{3.43}$$

The complete boundary conditions for p_2 are Eqs. (3.42) and (3.43).

(3) Solution of p_2

To find the solution of p_2, we separate the variables by assuming that:

$$p_2 = X(x)Y(y) \tag{3.44}$$

By substituting Eq. (3.44) into Eq. (3.40), we find:

$$X''(x)Y(y) + Y''(y)X(x) = 0$$

or

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

Therefore, we have two independent equations:

$$X''(x) - \lambda X(x) = 0$$

and

$$Y''(y) + \lambda Y(y) = 0$$

For $X(x)$, we assume that:

$$X(x) = A_1 \cos \alpha x + A_2 \sin \alpha x$$

As $X(\pm a)$=0, we have A_2=0 and $\alpha = \frac{2n\pi}{a}$ for n=1,3,5, etc., i.e.,

$$X(x) = A_1 \cos(\frac{2n\pi x}{a}) \tag{3.45}$$

For $Y(y)$, we assume:

$$Y(y) = C_1 \cosh(\gamma y) + C_2 \sinh(\gamma y)$$

where $\gamma = \frac{2n\pi}{a}$ and n=1,3,5, etc.. By using the boundary conditions for $Y(y)$, i.e., $Y(b) = Y(-b)$, we find C_2=0. Therefore, we have:

$$Y(y) = C_1 \cosh(\frac{2n\pi y}{a}) \tag{3.46}$$

By Eqs. (3.44), (3.45) and (3.46), we can write:

$$p_2(x, y) = \sum_{n=1,3,5,}^{\infty} a_n \cosh\frac{n\pi y}{2a} \cos\frac{n\pi x}{2a} \tag{3.47}$$

To satisfy the boundary conditions shown in Eq. (3.43), we have:

$$\sum_{n=1,3,5,}^{\infty} a_n \cosh\frac{n\pi b}{2a} \cos\frac{n\pi}{2a} x = \frac{6\mu}{h^3} \dot{h}(a^2 - x^2) \tag{3.48}$$

The constant a_n's are found to be:

$$a_n = \frac{\int_{-a}^{a} \frac{6\mu}{h^3}\dot{h}(a^2-x^2)\cos\frac{n\pi x}{2a}dx}{\cosh\frac{\pi nb}{2a}\int_{-a}^{a}\cos^2\frac{n\pi x}{2a}dx} = \frac{192\mu\dot{h}a^2}{n^3\pi^3h^3}\frac{\sin\frac{n\pi}{2}}{\cosh\frac{n\pi b}{2a}} \quad (n\text{=1,3,5, etc.}) \tag{3.49}$$

Therefore, we have:

$$p_2(x,y) = \frac{192\mu a^2}{h^3\pi^3}\dot{h}\sum_{n=1,3,5,}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3\cosh\frac{n\pi b}{2a}}\cosh\frac{n\pi y}{2a}\cos\frac{n\pi x}{2a}$$

The final solution for the pressure is:

$$P = p_1 + p_2$$

$$= -\frac{6\mu\dot{h}}{h^3}(a^2-x^2) + \frac{192\mu\dot{h}a^2}{h^3\pi^3}\sum_{n=1,3,5,}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3\cosh\frac{n\pi b}{2a}}\cosh\frac{n\pi y}{2a}\cos\frac{n\pi x}{2a} \tag{3.50}$$

(4) Damping force

The damping force exerted on the rectangular plate is:

$$F_{rec} = \int_{-a}^{a}dx\int_{-b}^{b}P(x,y)dy$$

$$= -\frac{16a^3b\mu}{h^3}\dot{h}\left\{1-\frac{192}{\pi^5}\left(\frac{a}{b}\right)\sum_{n=1,3,5,}^{\infty}\frac{1}{n^5}\tanh\left(\frac{n\pi b}{2a}\right)\right\} \equiv -\frac{\mu LB^3}{h^3}\dot{h}\beta\left(\frac{B}{L}\right) \tag{3.51}$$

where the factor $\beta\left(\frac{B}{L}\right)$ is a function of the aspect ratio $\frac{B}{L}$, i.e.,

$$\beta\left(\frac{B}{L}\right) = \left\{1-\frac{192}{\pi^5}\left(\frac{B}{L}\right)\sum_{n=1,3,5,}^{\infty}\frac{1}{n^5}\tanh\left(\frac{n\pi L}{2B}\right)\right\}$$

The dependence of $\beta\left(\frac{B}{L}\right)$ on $\frac{B}{L}$ is shown by the curve in Fig. 3.12. For a very long plate, β=1, and for a square plate (i.e., a=b), β=0.42. The coefficient of the damping force is:

$$c_{rec} = \frac{\mu LB^3}{h^3}\beta\left(\frac{B}{L}\right) \tag{3.52}$$

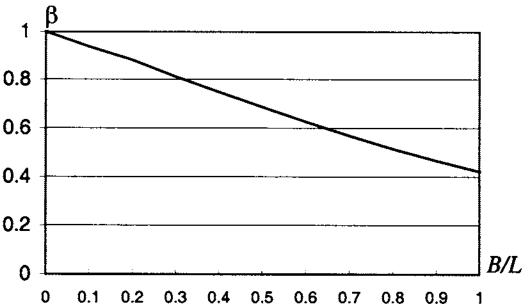


Fig. 3.12. The dependence of factor $\beta\left(\frac{B}{L}\right)$ on the aspect ratio $\frac{B}{L}$

§3.2.5. Perforated plate

Moving plates in microstructures are sometimes perforated to reduce the damping effect to a certain level for certain applications such as in accelerometers, microphones and so on. Therefore, estimation of the air damping force for a perforated plate is important in designing the devices.

Suppose that a plate is perforated with circular holes of radius b and the holes are uniformly distributed in a hexagon close-packed pattern as shown in Fig. 3.13, to allow highest hole density and uniformity. If the distance between the centers of two adjacent holes is $2a$, the area for a hexagonal cell including the hole is $A_1 = 2\sqrt{3}a^2$. If the area of the whole plate, A, is much larger than A_1, the damping force for the whole plate can be approximated as the sum of the damping force for all cells.

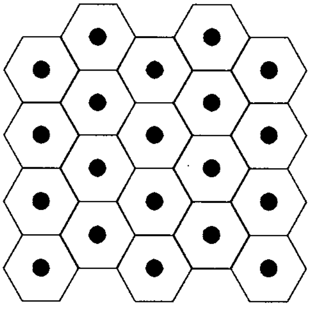


Fig. 3.13. Plate with hexagon close-packed perforation

To find the damping force on a hexagonal cell, the cell is approximated as an annulus with an outer radius of $a_1 = \sqrt{\frac{2\sqrt{3}}{\pi}}a \approx 1.05a$ and an inner radius of b. The equation for the pressure build-up caused by the parallel motion of the plate is the same as the isolated annulus discussed in §3.2.3:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}P(r) = \frac{12\mu}{h^3}\dot{h} \tag{3.31}$$

As air does not flow between the cells, the boundary conditions are:

$$P(b) = 0, \ \frac{\partial P}{\partial r}(a_1) = 0 \tag{3.53}$$

By solving Eqs. (3.31) and (3.53), we find:

$$P(r) = -\frac{3\mu a_1^2}{h^3}\dot{h}\left(2\ln\frac{r}{b} - \frac{r^2 - b^2}{a_1^2}\right)$$

The damping force for a cell is:

$$F_1 = \int_b^{a_1} -\frac{3\mu a_1^2}{h^3}\dot{h}\left(2\ln\frac{r}{b} - \frac{r^2 - b^2}{a_1^2}\right)2\pi r dr$$

$$= -\frac{\mu(\pi a_1^2)^2}{h^3}\dot{h}\frac{3}{\pi}\left[2\ln\frac{a_1}{b} - \left(1 - \frac{b^2}{a_1^2}\right)^2 - \frac{1}{2}\left(1 - \frac{b^4}{a_1^4}\right)\right]$$

By using the notation of $\beta = \frac{b}{a_1}$, we have:

$$F_1 = -\frac{3\mu A_1^2}{2\pi h^3}\dot{h}\left(4\beta^2 - \beta^4 - 4\ln\beta - 3\right)$$

or

$$F_1 = -\frac{3\mu}{2\pi h^3 n^2}\dot{h}\left(4\beta^2 - \beta^4 - 4\ln\beta - 3\right) \tag{3.54}$$

where n is the hole density of the plate (i.e. the hole number per unit area of the plate, $n = \frac{N}{A}$, where A is the total area of the plate and N is the total number of holes in the plate). The total damping force on the perforated plate is approximately:

$$F_p = \frac{A}{A_1}F_1 = -\frac{3\mu A}{2\pi n h^3}\dot{h}k(\beta) = -\frac{3\mu A^2}{2\pi h^3 N}\dot{h}k(\beta) \tag{3.55}$$

where $k(\beta)$ is:

$$k(\beta) \equiv 4\beta^2 - \beta^4 - 4\ln\beta - 3 \tag{3.56}$$

According to the definition of a_1, we have $\beta = \frac{b}{a_1} = \sqrt{\frac{\pi}{2\sqrt{3}}}\frac{b}{a} \cong 0.952\frac{b}{a}$. The dependence of k on b/a is shown by the curve in Fig. 3.14.

For a finite plate area A, the damping force found from Eq. (3.55) should be larger than its real value (especially for small holes), as the boundary conditions on the edges of the plate have not been considered. An empirical approach for a better approximation is that the resistive force given by Eq. (3.55) and the resistive force for a non-perforated plate of the same shape and size are considered to act in parallel. Suppose that the plate is a rectangular one. The squeeze-film damping force, F_{rec}, of a non-perforated rectangular plate with the same shape and area can be found using Eq. (3.51) in §3.2.4. Then the resultant damping force for the perforated plate, F_T, is given by the following relation:

$$F_T = \frac{F_p F_{rec}}{F_p + F_{rec}} \tag{3.57}$$

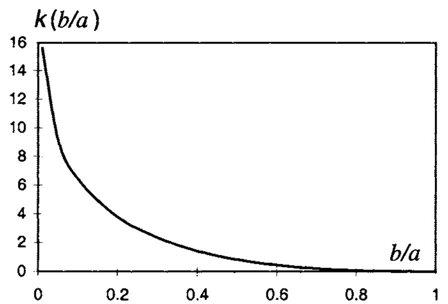


Fig. 3.14. The dependence of factor k on b/a

§3.2.6. Oscillating beams

Micro beams are widely used as resonators in micro mechanical sensors and actuators, especially in resonant type devices. Therefore, knowledge of air damping effects on oscillating micro beams is important. As the motion of the oscillating beam structure is not uniform, there is no closed form solution for beam vibration problems involving damping force or damping ratio. Though numerical analysis is possible, it is not convenient as a design tool. Therefore, some simplified analyses are introduced in this section to obtain approximate solutions in the closed form. In the discussion in this section, two working models will be considered: the squeeze film-damping [5] and the drag force damping [3].

(1) Squeeze-film damping

Suppose a beam is placed parallel to a wall and is clamped at one end or both ends and oscillates in its normal direction in air. Assume the length of the beam, L, is much larger than its width, B, and its width is much larger than its thickness, h. If the gap between the beam and the substrate, d_o, is small when compared to the beam width, the main mechanism of damping is squeeze-film damping. As the moving speed of the oscillating beam is not uniform and its distribution is dependent on the vibration mode, the air damping force can hardly be simplified by a lumped model. As the length of the beam is much larger than its width, the air flow caused by the vibration is mainly in a lateral direction. According to Eq. (3.30), the squeeze-film air damping force on a unit length (L=1) of beam is:

$$F_s = \frac{\mu B^3}{d_o^3}\dot{w}(x,t) \tag{3.58}$$

where μ the coefficient of viscosity of air and $\dot{w}(x,t)$ is the moving speed of the beam sector considered. With reference to Eq. (2.104) in §2.4.1, the equation for a forced vibration of the beam with squeeze-film air damping is:

$$\rho Bh\ddot{w}(x,t) + c_s\dot{w}(x,t) + EI\frac{d^4w(x,t)}{dx^4} = F(x)\sin\omega t \tag{3.59}$$

where ρ is the specific mass of the beam material, E is the Young modulus of the beam material, I is the moment of inertia of the beam, F is the amplitude of the external driven force and c_s is the coefficient of squeeze-film air damping force per unit length of the beam. According to Eq. (3.58), we have:

$$c_s = \frac{\mu B^3}{d_o^3} \tag{3.60}$$

$w(x,t)$ in Eq. (3.59) can be developed into the series:

$$w(x,t) = \sum_{n=1}^{\infty}\varphi_n(t)W_n(x) \tag{3.61}$$

where the $W_n(x)$'s are the shape-functions of the free vibration of the beam structure, i.e., they are solutions of the following equations:

$$EI\frac{d^4W_n(x)}{dx^4} = \rho Bh\omega_n^2 W_n(x) \tag{3.62}$$

Substituting Eq. (3.61) into Eq. (3.59), multiplying a specific $W_n(x)$, taking integration from 0 to L and making use of Eq. (3.62) and the orthogonal characteristics of the $W_n'(x)$s (i.e., $\int_0^L W_m(x)W_n(x)dx = 0$ for $m \neq n$), we obtain:

$$m_n\ddot{\varphi}_n(t) + c_{sn}\dot{\varphi}_n(t) + m_n\omega_n^2\varphi_n(t) = F_n \sin\omega t \tag{3.63}$$

where

$$m_n = \rho Bh\int_0^L W_n^2(x)dx,\ F_n = \int_0^L F(x)W_n(x)dx,\ c_{sn} = c_s\frac{m_n}{\rho Bh} = \frac{\mu B^2}{\rho h d_o^3}m_n$$

Eq. (3.63) can also be written as:

$$\ddot{\varphi}_n(t) + 2n_{sn}\dot{\varphi}_n(t) + \omega_n^2\varphi_n(t) = f_n \sin\omega t \tag{3.64}$$

where

$$n_{sn} = \frac{c_{sn}}{2m_n} = \frac{c_s}{2\rho Bh}$$

and

$$f_n = \frac{F_n}{m_n}$$

When the beam is oscillating at one of its natural vibration frequencies, ω_n, the corresponding damping ratio can be obtained directly from Eqs. (3.60) and (3.64) based on the discussion in §2.5:

$$\zeta_{sn} = \frac{c_s}{2\rho Bh\omega_n} = \frac{\mu B^2}{2\rho h\omega_n d_o^3} \tag{3.65}$$

The Q factor for this vibration mode is:

$$Q_{sn} = \frac{\rho h d_o^3\omega_n}{\mu B^2} \tag{3.66}$$

This is a simple but a very useful result indicating that the Q factor of a beam resonator is related to the geometries of the beam and the frequency of the vibration mode. It is especially interesting that the higher the vibration mode (corresponding to a higher frequency), the higher the Q factor.

According to Eqs. (3.63) and (3.64), the effectiveness of an external driving force for a specific vibration mode is determined by $f_n = \frac{1}{m_n}\int_0^L F(x)W_n(x)dx$.

(2) Drag force damping

If the beam is far away from a surrounding object (as in the case where a beam is driven into vibration by a piezoelectric method), the main damping mechanism is the drag force of air flow. As there is no closed form solution to the damping force for most mechanical structures including beams, a simplified dish-model is suggested in this section. Similar to the bead model [6], the model replaces the beam with a string of dishes as shown in Fig. 3.15. The diameter of the dishes is equal to the beam width, B, and the interference in air flow between neighboring dishes is negligible. According to Eq. (3.12), the air damping force on the *i'th* dish is:

$$F_{ai} = 8\mu B\dot{w}_i \qquad (3.67)$$

where w_i is the displacement of the i'th dish. Since the number of dishes per unit length of the beam is $1/B$, the damping force per unit length of the beam is:

$$F_{a1} = 8\mu\dot{w} \qquad (3.68)$$

The equation for a forced vibration of the beam with drag force damping is:

$$\rho bh\ddot{w}(x,t) + c_{a1}\dot{w}(x,t) + EI\frac{d^4 w(x,t)}{dx^4} = F(x,t) \qquad (3.69)$$

where c_{al} is the coefficient of air-flow damping force. According to Eq. (3.68), we have:

$$c_{a1} = 8\mu \qquad (3.70)$$

Following similar argumentation as made for squeeze film damping, the damping ratio for the *n'th* vibration mode is:

$$\zeta_{an} = \frac{c_{a1}}{2\rho Bh\omega_n} = \frac{4\mu}{\rho Bh\omega_n} \qquad (3.71)$$

where ω_n is the radial frequency of the n'th vibration mode. The Q factor for the vibration mode is:

$$Q_{sn} = \frac{\rho Bh\omega_n}{8\mu} \qquad (3.72)$$

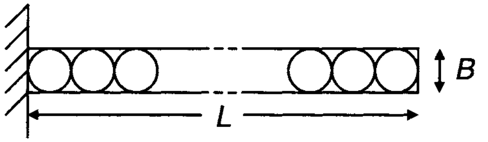


Fig. 3.15. Dish-string model for air damping of beam (top view)

In the above discussion on damping force, the dishes are considered as being isolated from one another and the dishes do not fill the beam area completely. As a matter of fact, no air flow is allowed between two neighboring dishes. Therefore, the damping force given by Eq. (3.68) is under-estimated. As a rough modification, the Q factor given by Eq. (3.72) is reduced by a factor that is found by comparing the squeeze-film damping force for a long rectangular plate with that for a circular plate. Using Eqs. (3.30) and (3.34), the factor is found to be $\frac{32}{3\pi}$. Therefore, the modified estimation for the Q factor is $Q_{sn} \doteq \frac{3\pi\rho B h\omega_n}{256\mu} \cong 0.037\frac{\rho B h\omega_n}{\mu}$.

§3.2.7. Effects caused by finite squeeze number

In §3.2.1, the basic equation for squeeze-film air damping was derived. For convenience, the equation is given again here:

$$\frac{\partial^2 P^2}{\partial x^2}+\frac{\partial^2 P^2}{\partial y^2}=\frac{24\mu}{h^3}\frac{\partial(hP)}{\partial t} \tag{3.23}$$

Under the condition that the squeeze number σ is negligible, (i.e., $\sigma=\frac{12\mu l^2\omega}{P_o h_o^2} << 1$, where l is the characteristic dimension of the plate, e.g., the radius of a disk or the half width of a strip or rectangle plate), Eq. (3.23) can be simplified to:

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}=\frac{12\mu}{h^3}\frac{dh}{dt} \tag{3.26}$$

Based on Eq. (3.26), discussions on squeeze-film air damping for some typical structures were carried out in §3.2.3 to §3.2.6.

However, if σ is small but not negligible, Eq. (3.23) instead of Eq. (3.26) must be used for squeeze-film air damping problems. We will discuss some of the effects caused by the finite value of σ in this section.

For simplicity of discussion, the following transformations are made to make the variables dimensionless: $\tilde{x} = \frac{x}{l}$, $\tilde{y} = \frac{y}{l} y$, $\tilde{h} = \frac{h}{h_o}$, $\tilde{P} = \frac{P}{P_o}$ and $\tilde{t} = \omega t$. With these dimensionless variables, Eq. (3.23) is written as:

$$\frac{\partial^2 \tilde{P}^2}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{P}^2}{\partial \tilde{y}^2} = 2\sigma \frac{1}{\tilde{h}^3} \frac{\partial\left(\tilde{h}\tilde{P}\right)}{\partial \tilde{t}} \tag{3.73}$$

$\tilde{h}$ is the varying dimensionless film thickness due to the oscillation of the damper plate, and it can be expressed as:

$$\tilde{h} = 1 + \varepsilon \sin \tilde{t} \tag{3.74}$$

For small σ, the pressure $\tilde{P}$ can be expressed as:

$$\tilde{P} = 1 + p_1\sigma + p_2\sigma^2 + O(\sigma^3) \tag{3.75}$$

Terms related to σ^3 and up will be neglected in this analysis.

Substituting Eq. (3.75) into Eq. (3.73) gives the following equations:

$$\frac{\partial^2 p_1}{\partial \tilde{x}^2} + \frac{\partial^2 p_1}{\partial \tilde{y}^2} = \frac{\dot{\tilde{h}}}{\tilde{h}^3} \tag{3.76}$$

and

$$\frac{\partial^2 p_2}{\partial \tilde{x}^2} + \frac{\partial^2 p_2}{\partial \tilde{y}^2} = \frac{1}{\tilde{h}^3} \frac{\partial}{\partial \tilde{t}}\left(\tilde{h} p_1\right) - \frac{1}{2}\left(\frac{\partial^2 p_1^2}{\partial \tilde{x}^2} + \frac{\partial^2 p_1^2}{\partial \tilde{y}^2}\right) \tag{3.77}$$

The boundary conditions for p_1 and p_2 are derived from the criteria that the pressure $\tilde{P}$ be unity on all outside boundaries and the pressure distribution has a zero slope at the symmetric points of the plate. It should be pointed out that the first order equation, i.e., Eq. (3.76) for p_1, is, in fact, the equation for "incompressible condition" related to very small compression numbers ($\sigma \ll 1$). The effects caused by the finite value of σ are represented by p_2. To demonstrate the effects caused by a finite p_2, only the resistive force on a long rectangular plate damper is considered. For plates with other shapes, readers are referred to reference [7].

For a long rectangular plate with a length L much larger than its width B, the problem can be treated as one-dimensional. According to the coordinates shown in Fig. 3.16, Eq. (3.76) can be simplified to:

$$\frac{\partial^2 p_1}{\partial \tilde{x}^2} = \frac{\dot{\tilde{h}}}{\tilde{h}^3} \tag{3.78}$$

where $l = B/2$. The boundary conditions for p_1 are:

$$p_1\left(\pm 1, \tilde{t}\right) = 0, \quad \frac{\partial p_1}{\partial \tilde{x}}\left(0, \tilde{t}\right) = 0 \tag{3.79}$$

The solution of p_1 is:

$$p_1\left(\tilde{x}, \tilde{t}\right) = \frac{\dot{\tilde{h}}}{2\tilde{h}^3}\left(\tilde{x}^2 - 1\right) \tag{3.80}$$

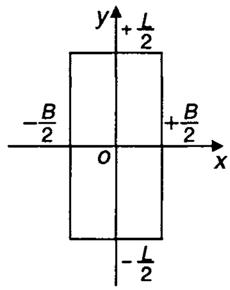


Fig. 3.16. A long rectangular plate and its coordinates

By substituting Eq. (3.80) into Eq. (3.77), we have:

$$\frac{\partial^2 p_2}{\partial \tilde{x}^2} = \frac{\ddot{\tilde{h}}}{2\tilde{h}^5}(\tilde{x}^2 - 1) - \frac{\dot{\tilde{h}}^2}{2\tilde{h}^6}(5\tilde{x}^2 - 3)$$

The boundary conditions for p_2 are:

$$p_2\left(\pm 1, \tilde{t}\right) = 0, \quad \frac{\partial p_2}{\partial \tilde{x}}\left(0, \tilde{t}\right) = 0$$

The solution to p_2 is:

$$p_2\left(\tilde{x}, \tilde{t}\right) = \frac{\ddot{\tilde{h}}}{24\tilde{h}^5}\left(\tilde{x}^4 - 6\tilde{x}^2 + 5\right) - \frac{\dot{\tilde{h}}^2}{24\tilde{h}^6}\left(5\tilde{x}^4 - 18\tilde{x}^2 + 13\right) \tag{3.81}$$

From the solution to p_1 and p_2, we have:

$$\tilde{P} = 1 + \frac{\dot{\tilde{h}}}{2\tilde{h}^3}\left(\tilde{x}^2 - 1\right)\sigma + \left[\frac{\ddot{\tilde{h}}}{24\tilde{h}^5}\left(\tilde{x}^4 - 6\tilde{x}^2 + 5\right) - \frac{h\dot{\tilde{h}}^2}{24\tilde{h}^6}\left(5\tilde{x}^4 - 18\tilde{x}^2 + 13\right)\right]\sigma^2 \tag{3.82}$$

The force on the plate is:

$$F = P_o LB\int_0^1 (\tilde{P}-1)d\tilde{x} = P_o LB\left[-\frac{1}{3}\frac{\dot{\tilde{h}}}{\tilde{h}^3}\sigma + \frac{2}{15}\frac{\ddot{\tilde{h}}}{\tilde{h}^5}\sigma^2 - \frac{1}{3}\frac{\dot{\tilde{h}}^2}{\tilde{h}^6}\sigma^2\right]$$

$$= P_o LB\left[-\frac{1}{3}\frac{h_o^2\dot{h}}{h^3\omega}\sigma + \frac{2}{15}\frac{h_o^4\ddot{h}}{h^5\omega^2}\sigma^2 - \frac{1}{3}\frac{h_o^4\dot{h}^2}{h^6\omega^2}\sigma^2\right] \quad (3.83)$$

Based on Eq. (3.83), two conditions are considered:

(1) Small amplitude

By using the small amplitude condition of $\varepsilon<<1$ so that $h \approx h_o$ and using the relations:

$$\dot{h} = h_o\varepsilon\omega\cos\omega t, \quad \ddot{h} = -h_o\varepsilon\omega^2\sin\omega t$$

Eq. (3.83) can be approximated as:

$$F = P_o LB\left[-\frac{1}{3}\sigma\varepsilon\cos\omega t - \frac{2}{15}\sigma^2\varepsilon\sin\omega t - \frac{1}{3}\sigma^2\varepsilon^2\cos^2\omega t\right]$$

$$= P_o LB\left[-\frac{1}{3}\sigma\varepsilon\cos\omega t - \frac{2}{15}\sigma^2\varepsilon\sin\omega t - \frac{1}{6}\sigma^2\varepsilon^2 - \frac{1}{6}\sigma^2\varepsilon^2\cos 2\omega t\right] \quad (3.84)$$

$$\equiv F_D + F_k + F_R + F_{2\omega}$$

The meaning of the four terms are explained as follows:

(a) Damping Effect

The first term in Eq. (3.84) is:

$$F_D = -\frac{1}{3}P_o LB\sigma\varepsilon\cos\omega t = -\frac{\mu LB^3}{h_o^3}h_o\varepsilon\omega\cos\omega t = -\frac{\mu LB^3}{h_o^3}\dot{h}$$

This is exactly the same result as given in Eq. (3.30), i.e., it is the result of the first order approximation.

The energy loss of the system due to this damping force in one cycle is:

$$\Delta E = \int_0^T -\frac{\mu LB^3}{h_o^3}h_o\varepsilon\omega\cos\omega t\dot{h}dt$$

$$= \int_0^{2\pi} -\frac{\mu LB^3\omega}{h_o^3}h_o^2\varepsilon^2\cos^2\omega t d\omega t = -\frac{\mu LB^3\omega}{2h_o}\varepsilon^2$$

Therefore, we have:

$$Q = \frac{2\pi E_T}{\Delta E} = \frac{2\pi \frac{1}{2} k(h_o \varepsilon)^2}{\frac{\mu L B^3 \omega}{2h_o} \varepsilon^2} = \frac{2\pi k h_o^3}{\mu L B^3 \omega}$$

where E_T is the total energy of the system and k is the elastic constant of the mechanical spring supporting the plate.

(b) Elastic Effect

The second term in Eq. (3.84) is:

$$F_k = -\frac{2}{15} P_o L B \sigma^2 \varepsilon \sin \omega t = -\frac{2}{15} P_o L B \frac{9\mu^2 \omega^2 B^4}{h_o^5} h_o \varepsilon \sin \omega t$$

$$= -\frac{6}{5} L B \frac{\mu^2 \omega^2 B^4}{P_o h_o^5} \Delta h$$

Obviously, F_k has the same nature as the elastic recovery force of a spring. F_k is significant at high squeeze numbers when the air flow into and out of the gap fails to keep up with the motion of plate. Therefore, this effect is referred to as the "elastic effect" of the air film. Since the trapped air acts as a spring it does not cause energy losses.

(c) Rectification Effect

The third term F_R in Eq. (3.84) is a constant force stemmed from the quadratic term of p_2 in Eq. (3.81). Therefore, it is referred to as the rectification force due to the nonlinear relationship between the pressure and the displacement.

(d) The force of higher harmonics

The last term in Eq. (3.84), $F_{2\omega}$, represents a force component whose frequency is twice as large as that of the oscillating plate. This component is also caused by the quadratic term of p_2.

(2)Large amplitude condition

If the amplitude of the oscillating plate is not negligible, then the value of $h = h_o(1 + \varepsilon \sin \omega t)$, instead of $h \approx h_o$, has to be used in Eq. (3.83). In this case, Eq. (3.83) can be developed into the form of:

$$F = A_o + A_1 \cos \omega t + B_1 \sin \omega t + A_2 \cos 2\omega t + B_2 \sin 2\omega t + \cdots \qquad (3.85)$$

where A_o is the rectification force, and A_1 and B_1 are the amplitudes of the damping and elastic forces, respectively. They can be found using the following expressions:

$$A_o = \frac{1}{2\pi}\int_0^{2\pi} F \cdot d\omega t = P_o LB \frac{\varepsilon^2\left(4+3\varepsilon^2\right)\sigma^2}{24\left(1-\varepsilon^2\right)^{9/2}} \tag{3.86}$$

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} F\cos\omega t d\omega t = -P_o LB \frac{\varepsilon\sigma}{3\left(1-\varepsilon^2\right)^{3/2}} \tag{3.87}$$

and

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} F\sin\omega t d\omega t = -P_o LB \frac{\left(\frac{3}{4}\varepsilon^4+6\varepsilon^2+2\right)\sigma^2\varepsilon}{15\left(1-\varepsilon^2\right)^{9/2}} \tag{3.88}$$

Eqs. (3.86) to (3.88) show that the rectification force, the damping force and the elastic forces increase with an increase in the amplitude indicated by ε.

From Eq. (3.87) the damping force is:

$$F_D = A_1\cos\omega t = -\frac{\mu LB^3}{h_o^3}\frac{1}{\left(1-\varepsilon^2\right)^{3/2}}\dot{h} \tag{3.89}$$

The coefficient of damping force is:

$$c = \frac{\mu LB^3}{h_o^3}\frac{1}{\left(1-\varepsilon^2\right)^{3/2}} \tag{3.90}$$

Obviously, for small amplitudes, the coefficient is the same as that given by Eq. (3.30).

§3.3. Slide-film air damping

§3.3.1. Basic equations for slide-film air damping

(1) Model of slide-film air damping

Micromechanical devices fabricated by surface micromachining technology feature thin layer movable plates (about 2 μm thick) suspended over a substrate by flexures placed a small distance apart. This basic structure facilitates the lateral motion of the plates for such applications as resonators, actuators, accelerometers, etc. As the dimensions of the moving plates are usually much larger than their thickness and their distance from the substrate, the viscous damping by the ambient air plays a major role in energy dissipation of the dynamic system; the air film behaves as a slide-film damper to the moving structure. To investigate the basic features of slide-film damping, a simplified mechanical model is considered: an infinitive plate, immersed in an incompressible viscous fluid, moving in a lateral direction at a constant distance from the substrate [8]. The model is schematically illustrated in Fig. 3.17.

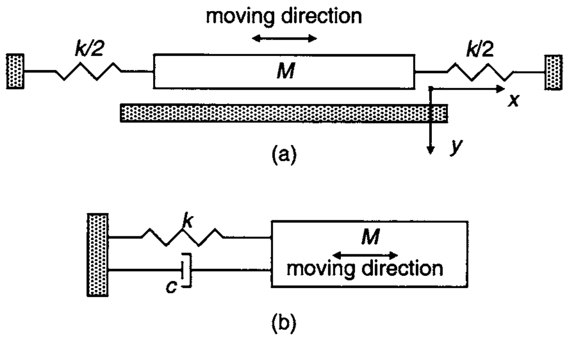


Fig. 3.17. Schematic model for a slide-film air damping
(a) schematic structure (b) a lumped damping model

(2) Basic equations

The general equation for the steady flow of an incompressible fluid is the well-known Navier-Stokes equation [3]:

$$\rho\left[\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right]=\vec{F}-\nabla p+\mu\nabla^2\vec{v} \tag{3.91}$$

where ρ is the density of the fluid, $\vec{v}$ is the velocity of the fluid, $\vec{F}$ is the external force, p is the pressure of the fluid, μ is the coefficient of viscosity of the fluid, and ∇ and ∇^2 denote gradient and Laplace operators, respectively. Suppose that the plate is in the x-y plane of a coordinate system and the movement is in x-direction. In this case, there is no external force or pressure gradient and $v_x \equiv u >> v_y$ and v_z, Eq. (3.91) becomes:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} \tag{3.92}$$

For an infinite plate, the second term on the left side of Eq. (3.92) vanishes, resulting in:

$$\frac{\partial u}{\partial t}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} \tag{3.93}$$

The boundary conditions for Eq. (3.93) are that u equals zero at the substrate surface and u equals the velocity of the moving plate near the surface of the plate.

For a plate with a finite area, the second term in Eq. (3.92) will not be zero. Now let us discuss the conditions for the approximation of Eq. (3.93).

Suppose that the motion of a finite plate with reference to its balanced position is a simple harmonic oscillation:

$$x(t)=a_o \sin\omega t$$

where a_o is the amplitude of the simple harmonic oscillation. Therefore, we have:

$$u(t)=a_o\omega\cos\omega t \equiv u_o\cos\omega t$$

and:

$$\frac{\partial u}{\partial t}=-u_o\omega\sin\omega t \tag{3.94}$$

where $u_o=a_o\omega$. If the typical dimension of the plate is l, we have:

$$u\frac{\partial u}{\partial x}\approx\frac{u_o^2}{l}=\frac{a_o^2\omega^2}{l} \tag{3.95}$$

and

$$\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}\approx\frac{\mu}{\rho}\frac{a_o\omega}{d^2} \tag{3.96}$$

where d is the distance between the substrate and the plate.

Therefore, the approximation conditions for Eq. (3.93) are:

a) $\left|\frac{\partial u}{\partial t}\right| >> \left|u\frac{\partial u}{\partial x}\right|$. This requires small amplitude, i.e., $a_o << l$,

b) $\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} >> u\frac{\partial u}{\partial x}$. Using Eqs. (3.95) and (3.96), this condition becomes $l >> \frac{\rho\omega a_o d^2}{2\mu}$. By defining a characteristic distance $\delta=\sqrt{\frac{2\mu}{\rho\omega}}$, the condition can be further simplified to:

$$l >> \frac{d^2}{\delta^2} a_o \quad (3.97)$$

We will find in §3.3.3 that the effective distance, δ, corresponds to the distance that the velocity decays away from the plate by a factor of e (=2.718...) in the z-direction. The curve in Fig. 3.18 shows the dependence of δ on the frequency in air at 1 *atm.* at 20°C.

(3) Two flow models

Under the condition that δ>>d (i.e., $\omega << \mu / \rho d^2$), $\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} >> \frac{\partial u}{\partial t}$. In this case, Eq. (3.93) can be further simplified to:

$$\frac{\partial^2 u}{\partial z^2} = 0 \quad (3.98)$$

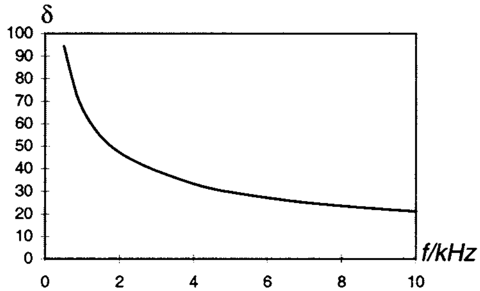


Fig. 3.18. The effective distance, δ, as a function of frequency

In the following sections, two different damping models will be considered: a Couette-flow model governed by Eq. (3.98) when δ is much larger than d and a Stokes-flow model governed by Eq. (3.93) for more general conditions [8].

§3.3.2. Couette-flow model

Suppose that a large plate over a static substrate oscillates laterally as shown in Fig. 3.19. If the oscillating frequency is so low so that δ>>d, the flow pattern of the air around the plate is called Couette-flow. We will consider the damping effect caused by the viscous fluid between the plate and the substrate by the Couette-flow model with the boundary conditions:

$$u(0) = u_o \cos \omega t, \; u(d) = 0 \quad (3.99)$$

According to Eqs. (3.98) and (3.99), the velocity distribution of the fluid is:

$$u(y) = u(0)(1 - \frac{z}{d}) \tag{3.100}$$

where $u(0)$ is the velocity of the moving plate. The shearing force applied to the plate to oppose its motion is:

$$F = -\mu \frac{u(0)}{d} A \tag{3.101}$$

where A is the area of the plate. According to the Couette-flow model, the velocity gradient on the open (top) side of the plate is zero. Therefore, there is no damping force on the top side of the plate and the Q factor of the lateral vibration system is determined only by the damping force described in Eq. (3.101). (As a matter of fact, if d on the top side is large, the condition for Couette-flow, $\delta >> d$, is no longer valid, but we will just assume that the damping force on the top side of the plate is negligible.)

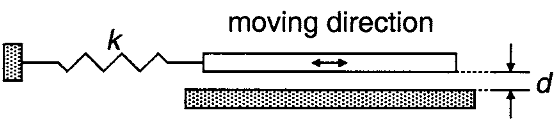


Fig. 3.19. Laterally oscillating plate over a substrate

The energy dissipated by the damping force in one cycle is:

$$\Delta E_{Cd} = \int_0^T A\mu \frac{u(0)}{d} u(0) dt$$

As $u(0) = u_o \cos \omega t$, we have:

$$\Delta E_{Cd} = \frac{\pi}{\omega} {u_o}^2 \frac{\mu}{d} A$$

According to the second definition of the Q factor in Chapter 2:

$$Q_{Cd} = \frac{\pi m {u_o}^2}{\Delta E_{Cd}} = \frac{m\omega d}{\mu A} \tag{3.102}$$

If the specific mass of the material is ρ and the thickness of the plate is h, Eq. (3.102) can be written as:

$$Q_{Cd} = \frac{\rho h \omega d}{\mu}$$

Note that Q_{Cd} is not dependent on the area of the plate, A.

§3.3.3. Stokes-flow model

In the Couette-flow model, the velocity profile in the fluid between the plate and the substrate is linear. The model becomes invalid when the distance between the plate and the substrate is large enough. For a very large distance, the velocity profile of the fluid is governed by the differential equation Eq. (3.93). If the effective distance, δ, is not much larger than d, then Eq. (3.99) applies.

By solving Eq. (3.93) with the boundary conditions given in Eq. (3.99), the velocity profile of the fluid is:

$$u = u_o \frac{-e^{-\tilde{d}+\tilde{z}}\cos(\omega t + \tilde{z} - \tilde{d} - \theta) + e^{\tilde{d}-\tilde{z}}\cos(\omega t - \tilde{z} + \tilde{d} - \theta)}{\sqrt{e^{2\tilde{d}} + e^{-2\tilde{d}} - 2\cos(2\tilde{d})}} \tag{3.103}$$

where $\tilde{d} \equiv \frac{d}{\delta}$, $\tilde{z} \equiv \frac{z}{\delta}$ and θ is a phase lag angle against the oscillation of the plate ($u(0) = u_o \cos\omega t$). The expression for θ is:

$$\theta = \arctan\frac{(e^{\tilde{d}} + e^{-\tilde{d}})\sin\tilde{d}}{(e^{\tilde{d}} - e^{-\tilde{d}})\cos\tilde{d}} \tag{3.104}$$

The force applied on the plate (on one side) is:

$$\begin{aligned} F_{Sd} &= A\mu \left.\frac{\partial u}{\partial z}\right|_{z=0} \\ &= \frac{A\mu u_o}{\delta\sqrt{e^{2\tilde{d}} + e^{-2\tilde{d}} - 2\cos 2\tilde{d}}}\Big(-e^{-\tilde{d}}\cos(\omega t - \tilde{d} - \theta) \\ &\quad + e^{-\tilde{d}}\sin(\omega t - \tilde{d} - \theta) - e^{\tilde{d}}\cos(\omega t + \tilde{d} - \theta) + e^{\tilde{d}}\sin(\omega t + \tilde{d} - \theta)\Big) \end{aligned} \tag{3.105}$$

With the damping force on the plate shown by Eq. (3.105), the energy dissipation in one cycle of oscillation is found to be:

$$\Delta E_{Sd} = \int_0^T F_{Sd} u_o dt = \frac{\pi A\mu u_o^2}{\omega\delta}\frac{\sinh(2\tilde{d}) + \sin(2\tilde{d})}{\cosh(2\tilde{d}) - \cos(2\tilde{d})} \tag{3.106}$$

and the Q factor is:

$$Q_{Sd} = \frac{m\omega\delta}{A\mu}\cdot\frac{\cosh(2\tilde{d}) - \cos(2\tilde{d})}{\sinh(2\tilde{d}) + \sin(2\tilde{d})} \tag{3.107}$$

For the extreme condition of $d \ll \delta$, we have $\theta = \frac{\pi}{4}$, $F_{Sd} = F_{Cd} = -A\frac{\mu}{d}u(0)$, $\Delta E_{Sd} = \Delta E_{Cd}$ and $Q_{Sd} = Q_{Cd}$, i.e., the results of Stokes-flow model coincide with those of Couette-flow model.

For another extreme condition of $d \gg \delta$, from Eq. (3.103), we have:

$$u = u_o e^{-\tilde{z}} \cos(\omega t + \tilde{d} - \theta)$$

This shows that the fluid around the plate oscillates with the same frequency as the plate but the oscillation amplitude in the fluid decays exponentially away from the plate. δ is the distance over which the amplitude decreases by a factor of e (=2.718).

Under this condition, the energy dissipation in one cycle is:

$$\Delta E_{S\infty} = \frac{\pi}{\omega} u_o^2 \frac{\mu}{\delta} A$$

and

$$Q_{S\infty} = \frac{m\omega\delta}{\mu A} = \frac{\rho h\omega\delta}{\mu}$$

If $Q_{S\infty}$ is compared with Q_{Cd} in Eq. (3.102), we can conclude that the damping force now is:

$$F_{S\infty} = \frac{\mu A u(0)}{\delta} \tag{3.108}$$

As the condition of $d \gg \delta$ means that the effect of the neighboring substrate is negligible for the oscillating plate, the plate can be considered as an isolated object in the fluid. Now let us compare the result here with the drag force on an isolated object given in §3.1.3.

According to §3.1.3, the drag force exerted by the viscous fluid on a circular dish moving in its plane direction is:

$$F_d = \frac{32}{3}\mu r v = \frac{32}{3\pi}\frac{\mu A v}{r} \tag{3.109}$$

where v is the velocity of the plate, equivalent to the $u(0)$ in Eq. (3.108), and r is the radius of the dish, i.e., its characteristic dimension. When Eq. (3.109) is compared with Eq. (3.108), we can make the following conclusions:

a) if ω is small ($\delta \gg r$), the damping force should be estimated by Eq. (3.109);

b) if ω is large ($\delta << r$), the damping force should be estimated by Eq. (3.108).

§3.3.4. Air damping of a comb resonator

Silicon micro resonators are typical micro mechanical structures and have very useful applications in sensors and actuators. Many types of silicon micro-resonators have been so far developed. Among them, the lateral driving comb micro resonators, formed by surface micromachining technology, have wide applications.

Fig. 3.20 shows schematically the basic structure of the lateral driving comb resonator. The shaded areas are fixed fingers (or, fixed electrodes) and act as anchors for the movable parts. The movable parts include the flexures (narrow beams), supporting plate (with etching holes) and fingers (movable electrodes). The moving parts of the structure can be driven into lateral oscillation by applying alternating voltage (often with a dc bias, see Chapter 5) between the movable and the fixed electrodes. Quite often, the frequency of the driving force coincides with the resonant frequency of the structure so that the structure is driven into a resonant state. Thus, the structure is often referred to as a comb resonator.

One of the most important characteristics of a resonator is its mechanical quality factor, Q. For a comb resonator operating in an atmospheric environment, air damping is the dominant factor that determines the Q factor of the resonator.

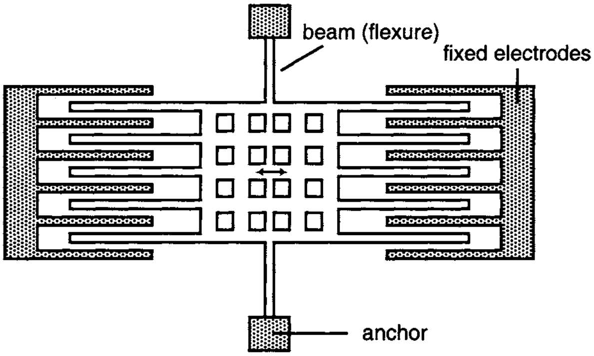


Fig. 3.20. A schematic drawing of a comb resonator

The air damping force for a comb resonator consists of many damping force components related to the geometries of the structure and different damping mechanisms. These damping force components are:

(a) The slide-film damping force on the bottom

As the distance, d_p, between the moving parts and the substrate beneath the movable structure is much smaller than δ, the damping force is of Couette-flow type and can be expressed as:

$$F_1 = \mu \frac{A_p}{d_p} \dot{x} = c_1 \dot{x}$$

This is usually the most important force component of all the damping force components

(b) The slide-film damping on top

Suppose the structure is placed far away from any external objects above it. The damping force above the moving parts of the structure is of Stokes-flow type. The damping force component is:

$$F_2 = \mu \frac{A_p}{\delta} \dot{x} = c_2 \dot{x}$$

where δ is the effective distance defined in §3.3.1 as $\delta = \sqrt{\frac{2\mu}{\rho\omega}}$ and ρ is the density of air. From Fig. 3.18, for a resonant frequency of 1 kHz, we find δ = 67 μm. A_P is the effective plate area for the damping calculation which includes the areas of the plates, fingers and beams. The etch hole region is also included in A_P as the dimension of the etch hole is usually small when compared with δ.

(c) Slide-film damping of the sidewalls

The damping force is:

$$F_3 = \mu \frac{A_s}{d_s} \dot{x} = c_3 \dot{x}$$

where A_s is the area of the sidewalls that are in parallel with the moving direction and d_s is the distance between the sidewalls and their neighboring structures. Here we have assumed that: $d_s \ll \delta$.

(d) Air drag force

The air drag force on the moving plate is difficult to estimate. Referring to the air drag force on a circular dish moving along its plane direction (see Eq. (3.13) in §3.1.3.), the force can be approximated by:

$$F_4 \approx \frac{32}{3}\mu l\dot{x} = c_4\dot{x}$$

where l is the characteristic dimension of the moving structure that can be taken as half the width of the plate.

Therefore, the total damping force on the entire structure is:

$$F = F_1 + F_2 + F_3 + F_4 = (c_1 + c_2 + c_3 + c_4)\dot{x} \equiv c\dot{x}$$

and the quality factor is:

$$\frac{1}{Q} = 2\zeta = \frac{c}{m\omega_o}$$

or

$$\frac{1}{Q} = \frac{\mu}{m\omega_o}\left((\frac{A_p}{d_p} + \frac{A_p}{\delta} + \frac{A_s}{d_s} + 10.7l)\right) \tag{3.110}$$

Due to the finite dimension of the structure and the fringe effect at the edges and corners, Eq. (3.110) is only a semi-quantitative approximation to the exact value. However, it does provide useful information in designing a comb resonator.

The resonant frequency ω_o in Eq. (3.110) can be written as:

$$\omega_o = \sqrt{\frac{Ehb^3}{mL_{eff}{}^3}}$$

where L_{eff} is the effective length of the beams, b the width and h the thickness of the beam flexures. As $m \cong A_p h\rho$, we have:

$$\frac{1}{Q} = \frac{\mu}{h}\sqrt{\frac{L_{eff}{}^3}{E\rho A_p b^3}}\left((\frac{A_p}{d_p} + \frac{A_p}{\delta} + \frac{A_s}{d_s} + 10.7l)\right) \tag{3.111}$$

For most situations, the Couette-flow slide-film damping term, the first term in Eq. (3.111), is the dominant factor of air damping. If only the Couette-flow slide-film damping is considered in estimating the quality factor of the structure shown in Fig. 3.20, we have:

$$Q \approx \frac{hd_p}{\mu}\sqrt{\frac{E\rho}{A_p}(\frac{b}{L_{eff}})^3} \tag{3.112}$$

This means that, for a high quality factor, the structure should be thick and far away from the substrate. Also, the flexures should have as large a flexure rigidity as possible in their moving direction.

§3.4. Damping in rare air

§3.4.1. Free molecule model for rare air damping

In a wide band of pressure range around and above atmospheric pressure, the viscosity of gas is independent of pressure. Therefore, the damping effect of gas is independent of pressure in this range. This phenomenon can be explained by the kinetic theory of gas. The viscosity coefficient of a gas found by the simple kinetic model is:

$$\mu = \frac{1}{3}\rho\lambda\bar{v}$$

where ρ is the density, λ the mean free path and $\bar{v}$ the average velocity of the gas molecules. As $\bar{v}$ and the product of ρ and λ are independent of pressure, the viscosity coefficient, μ, is not a function of pressure.

However, experiments show that the air damping force on a microstructure reduces significantly in rare air where the air pressure is below several hundreds Pa. It is believed that the gas molecules are so far apart in low pressure that the interaction between gas molecules can be neglected. Therefore, a model called the free molecule model is used for rare air [9].

Now let us consider the air damping force acting on a plate oscillating in its normal direction (x-direction). If the interaction between gas molecules can be neglected, the damping force on the plate resonator is caused by momentum transfer during its collisions with individual gas molecules. If the speed of the plate is $\dot{x}$, the pressure caused by the collisions on the front side of the plate is:

$$P_f = 2mn\int_{-\dot{x}}^{\infty}(v_x + \dot{x})^2 f(v_x)dv_x \quad (3.113)$$

where m is the mass of a molecule, n the molecule density, v_x the x-component of velocity of the colliding molecule moving against the plate and $f(v_x)$ the normalized distribution function of the molecules. For the distribution function, the Maxwellian function is usually used:

$$f(v_x) = \sqrt{\frac{m}{2\pi kT}}e^{-\frac{mv_x^2}{2kT}} \quad (3.114)$$

Similarly, the pressure caused by the collisions on the back side of the plate is:

$$P_b = 2mn\int_{\dot{x}}^{\infty}(v_x - \dot{x})^2 f(v_x)dv_x \tag{3.115}$$

According to Eqs. (3.113), (3.114) and (3.115), the pure damping force caused by collision is:

$$F_r = A(P_f - P_b)$$

where A is the area of the plate. If the velocity of the plate, $\dot{x}$, is much smaller than that of the majority of the gas molecules, we have:

$$F_r \cong 8mnA\int_0^{\infty} v_x \cdot \dot{x}F(v_x)dv_x = 8mnA\sqrt{\frac{kT}{2\pi m}}\dot{x} \tag{3.116}$$

For the gas in the standard condition (P_o=1 *atm.*, T_o=273 K), the molecule density of gas is:

$$n_o = \frac{N_A}{V_o}$$

where $N_A=6.023\times10^{23}$ and $V_o=2.24\times10^{-2}m^3$. The molecule density at pressure P and temperature T is:

$$n = n_o\frac{PT_o}{P_oT} = \frac{N_AP}{RT}$$

where R=8.31 kg·m^2/sec^2/°K and is referred to as the universal molar gas constant. Therefore:

$$F_r = 4\sqrt{\frac{2}{\pi}}\sqrt{\frac{M}{RT}}PA\dot{x} \tag{3.117}$$

where M is Molar mass of the gas. The coefficient of damping force in rare air by the free molecule model is:

$$c_r = 4\sqrt{\frac{2}{\pi}}\sqrt{\frac{M}{RT}}PA \tag{3.118}$$

Eq. (3.118) shows that the damping effect in rare air decreases in a linear way with decreasing pressure. Therefore, if the pressure is in the range where the rare air damping is the dominant damping factor, the Q factor of the system is:

$$Q = \frac{M_P\omega_o}{c_r}$$

where M_P is the mass of the plate and ω_o the natural frequency of the system. As $M_p=Ah\rho$, we have:

$$Q = \frac{h\rho\omega_o}{4} \cdot \sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{RT}{M}} \frac{1}{P} \tag{3.119}$$

where h is the thickness of the plate and ρ is the specific mass of the plate.

Eq. (3.119) has been compared with experimental data. The results agree to within an order of magnitude. Though the air damping is indeed inversely proportional to pressure P, the value of the Q factor is overestimated quantitatively, i.e., the damping force in rare air is underestimated by the free molecular model.

Disregarding the quantitative difference, Eq. (3.117) is now used to estimate the pressure where the transition from viscous flow model to free molecule model occurs. Suppose the plate is a square of 1mm by 1mm and it is 20μm away from the neighboring substrate, the squeeze-film damping force in the normal direction is:

$$F_s = 0.42 \frac{\mu L B^3}{h^3} \dot{x} = 9.5 \times 10^{-8} \cdot \dot{x} \tag{3.120}$$

The damping force by Eq. (3.117) is:

$$F_r = 4\sqrt{\frac{2}{\pi}} \sqrt{\frac{M}{RT}} PA\dot{x} = 1.148 \times 10^{-10} P\dot{x} \tag{3.121}$$

By equating F_s and F_r in Eqs. (3.120) and (3.121), we find the transition pressure of P_t=828 Pa. This result means that, for the specific structure described, the transition from squeeze-film air damping to rare air damping occurs at a pressure of about 828 Pa. Obviously, the transition pressure is dependent on the geometries of the microstructure.

§3.4.2. Damping in a vacuum

According to the free molecule model, the damping force in rare air is proportional to the air pressure. Therefore, the air damping force goes down, or, the Q factor of the system goes up, with a decrease in the air pressure. However, experimental results show that the Q factor levels off when the vacuum is high enough, i.e., when the effects of internal friction and support losses become the dominant mechanisms of energy dissipation [6]. As the internal friction and the support losses are very hard to predict theoretically, they are evaluated by the quality factor in a high vacuum condition. For micro structures made of silicon, the quality factors at high vacuum, Q_o, range from 10^4~10^5. Once Q_o is found through experimental measurements the coefficient of damping force caused by internal friction and support losses can be found using:

$$c_o = \frac{M_p \omega_o}{Q_o} \tag{3.122}$$

where M_p is the mass of the oscillating plate and ω_o is the natural frequency of the system. Therefore, the differential equation for vibration in rare air can be modified to:

$$m\ddot{x} + (c_r + c_o)\dot{x} + kx = F \tag{3.123}$$

where c_r and c_o can be found from Eqs. (3.118) and (3.122), respectively.

Assume Q_o of the oscillating plate is found to be 5×10^4, the plate is made of silicon with a thickness of 200 μm and the natural frequency of the structure is 1 kHz. The critical pressure, P_o, where the Q factor starts to level off with decreasing pressure can be found by equating Eq. (3.118) and Eq. (3.122), i.e.,

$$4\sqrt{\frac{2}{\pi}}\sqrt{\frac{M}{RT}}P_o A = \frac{A\rho h \omega_o}{Q_o}$$

The critical pressure is thus found to be about P_o=5 Pa, or, 4×10^{-2} *Torr*.

As a summary, the dependence of the Q factor on the air pressure from atmospheric pressure to high vacuum is schematically shown by the curve in Fig. 3.21. Let us start at pressures higher than atmospheric pressure. The Q factor at high pressure is independent of pressure as shown by sector A of the curve in Fig. 3.21. The Q factor in this pressure range is determined by the geometries and the moving directions of the structure: the damping could be squeeze-film damping, slide-film damping, drag force damping or a combination of these mechanisms.

When the pressure is pumped down to a certain extent (10^2~10^3 Pa), the Q factor starts to rise when the mechanism of rare air damping starts to play an important role. The Q factor is inversely proportional to the air pressure in the pressure range when the rare air damping plays a major role in air damping, as shown by sector B of the curve in Fig. 3.21. The transition pressure, P_t, for sector A to sector B is usually in the range of several hundred Pa. The exact pressure is dependent on the geometries and vibration mode of the micro structure.

At high vacuum, when the air damping is very small, the effects of internal friction and energy losses via the structure supports have to be considered. Generally, the Q factor in very low pressure must be determined by rare air damping, as well as the internal friction and the support losses of the structure. Obviously, the Q factor will be mainly determined by the internal friction and the support loss when the vacuum is high enough and the Q factor becomes a constant as shown by sector C of the curve in Fig. 3.21. The Q factors of silicon microstructure in high vacuum are usually in the order of 10^5. The exact value is dependent on the geometry design of the micro structure.

The pressure, P_o in the figure, where the Q factor begins to level off with decreasing pressure is usually in the range of several Pa, the exact value is also dependent on the geometries of the micro structure.

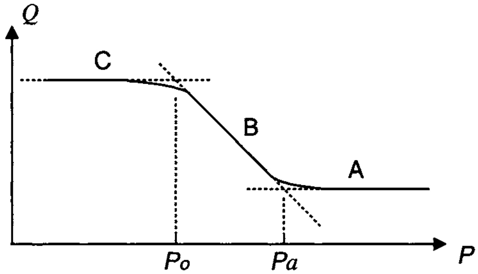


Fig. 3.21. The dependence of Q factor on air pressure

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