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INDENTATION SIZE EFFECTS IN CRYSTALLINE MATERIALS: A LAW FOR STRAIN GRADIENT PLASTICITY

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ABSTRACT

We show that the indentation size effect for crystalline materials can be accurately modeled using the concept of geometrically necessary dislocations. The model leads to the following characteristic form for the depth dependence of the hardness:

$$\frac{H}{H_0} = \sqrt{1+\frac{h^*}{h}},$$

where H is the hardness for a given depth of indentation, h, H_0 is the hardness in the limit of infinite depth and h^* is a characteristic length that depends on the shape of the indenter, the shear modulus and H_0. Indentation experiments on annealed (111) copper single crystals and cold worked polycrystalline copper show that this relation is well-obeyed. We also show that this relation describes the indentation size effect observed for single crystals of silver.

We use this model to derive the following law for strain gradient plasticity:

$$\left(\frac{\sigma}{\sigma_0}\right)^2 = 1+\hat{l}\chi,$$

where σ is the effective flow stress in the presence of a gradient, σ_0 is the flow stress in the absence of a gradient, χ is the effective strain gradient and $\hat{l}$ is a characteristic material length scale, which is, in turn, related to the flow stress of the material in the absence of a strain gradient,

$$\hat{l} \approx b\left(\frac{\mu}{\sigma_0}\right)^2.$$

For materials characterized by the power law

$$\sigma_0 = \sigma_{\text{ref}}\varepsilon^{1/n},$$

the above law can be recast in a form with a strain-independent material length scale l,

$$\left(\frac{\sigma}{\sigma_{\text{ref}}}\right)^2 = \varepsilon^{2/n}+l\chi \quad l = b\left(\frac{\mu}{\sigma_{\text{ref}}}\right)^2 = \hat{l}\left(\frac{\sigma_0}{\sigma_{\text{ref}}}\right)^2.$$

This law resembles the phenomenological law developed by Fleck and Hutchinson, with their phenomenological length scale interpreted in terms of measurable material parameters.

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1. INTRODUCTION

As discussed by Fleck and Hutchinson (1993, 1997), conventional theories of plasticity do not include material length scales. For such theories the flow stress at any particular point in a solid is uniquely related to the strain at that point and is unrelated to any strain gradient that might be present. Yet a number of experiments have shown that the flow properties of crystalline solids can depend not only on the strain but also on the strain gradient. This is expected to be particularly important for the modeling of crack tip fields where the strain gradients are large.

Fleck *et al.* (1994) have pointed out that the well-known indentation size effect for metals, wherein the hardness is observed to increase with decreasing indentation size, especially in the sub-micrometer depth regime (Stelmashenko *et al.*, 1993; De Guzman *et al.*, 1993; Ma and Clarke, 1995), can be understood by noting that large strain gradients inherent in small indentations lead to geometrically necessary dislocations that cause enhanced hardening. This same physical description was given earlier by Stelmashenko *et al.* (1993) and De Guzman *et al.* (1993) to account for the depth dependence of the hardness, but the connection to strain gradient plasticity theory was not made in these previous studies. Ma and Clarke (1995) used the same physical description and recognized its connection to strain gradient plasticity. In these descriptions, statistically stored dislocations, which are created by homogeneous strain, and geometrically necessary dislocations, which are related to the curvature of the crystal lattice or to strain gradients, both contribute to the flow stress. This nomenclature and these concepts follow from the early work of Ashby (1970). Similarly Fleck *et al.* (1994) have shown that fine copper wires deformed in torsion exhibit strengths that increase with decreasing wire diameter, whereas the tensile properties of these same wires are almost independent of wire diameter. They interpreted these results in terms of the geometrically necessary dislocations associated with the strain gradients in torsion.

To account for these strain gradient effects, Fleck and Hutchinson (1993) have developed a phenomenological theory of plasticity, using a single, constant, material length scale, l, within the general framework of couple stress theory. The couple stress theory used by Fleck and Hutchinson (1993) bears some resemblance to the early work of Kröner (1963) who studied the connection between lattice curvature associated with dislocations and couple stresses and developed a non-local continuum theory based on that connection. Xia and Hutchinson (1996) and Huang *et al.* (1995, 1997) have used the phenomenological theory to describe the stresses and strains at crack tips in elastic–plastic solids, where the effects of strain gradients are large and must be taken into account. A more general formulation of the phenomenological theory, which involves up to three independent material length scales, has been developed by Fleck and Hutchinson (1997).

In the present paper we use the simple model of geometrically necessary dislocations to describe the depth dependence of hardness of crystalline materials, following the derivations of Stelmashenko *et al.* (1993), De Guzman *et al.* (1993). We show that this model leads to a characteristic relation for the depth dependence of hardness that is in excellent agreement with nanoindentation experiments. In particular, we present new experimental data for the depth dependence of the hardness of annealed (111)

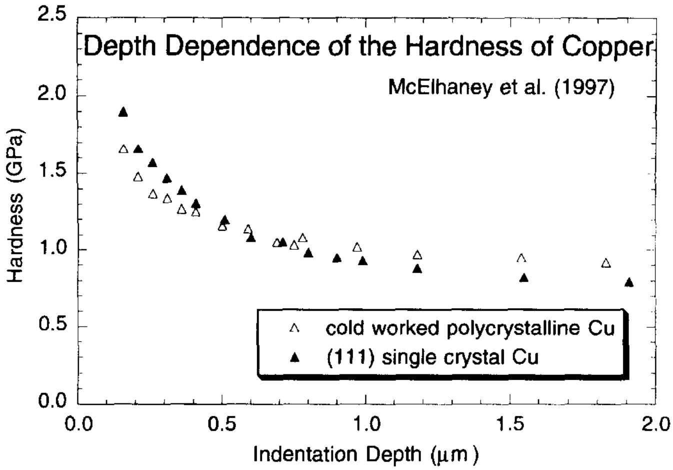

Fig. 1. Depth dependence of the hardness of (111) single crystal copper and cold worked polycrystalline copper, taken from the work of McElhaney *et al.* (1997). Experiments were conducted with a Berkovich diamond indenter ($\tan\theta = 0.358$) and the effects of pile-up and sink-in were taken into account in the determination of contact areas.

single crystals of copper and cold worked polycrystalline copper to support the model. We also show that the nanoindentation experiments of Ma and Clarke (1995) on single crystals of Ag provide support for the model. We focus much of our attention on the implications of this physical model for the theory of strain gradient plasticity. We show that, for materials characterized by a power law, a constant material length scale l can indeed be defined, as assumed in the present phenomenological treatments (Fleck and Hutchinson, 1993; Xia and Hutchinson, 1996; Huang *et al.*, 1997). This length scale is related to a microstructural characteristic length $\hat{l}$ which scales with L^2/b where L is the spacing between dislocation obstacles and b is the Burgers vector. We briefly discuss the implications of these results on strain gradient plasticity theory.

2. MODEL

The depth dependence of the hardness of copper is shown in Fig. 1 for both (111) single crystal copper and for a cold worked sample of polycrystalline copper. These measurements, which were made by nanoindentation, were taken from the work of McElhaney *et al.* (1997). Special care was taken in this study to account for the effects of pile-up and sink-in which occur during indentation. Thus the depth dependence of the hardness shown in the figure arises from real material behavior and is not associated with errors in the contact area. We wish to describe these results using the simple model of geometrically necessary dislocations developed first by Stelmashenko *et al.* (1993) and De Guzman *et al.* (1993). The present development closely follows the approach of De Guzman *et al.* (1993) and is presented in detail here for the convenience of the reader. Consider the indentation by a rigid cone, as shown sche-

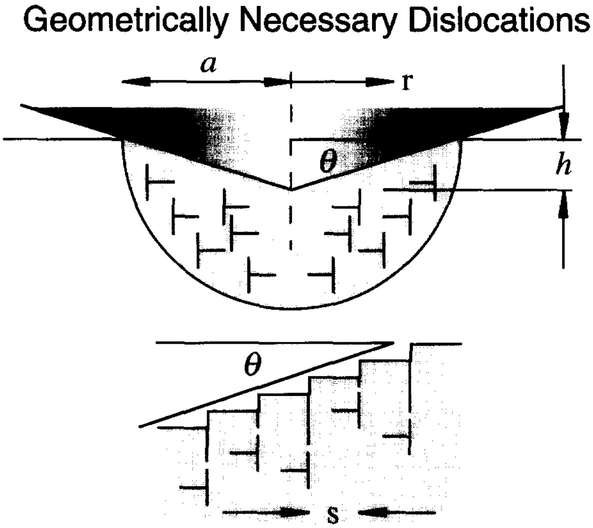

Fig. 2. Geometrically necessary dislocations created by a rigid conical indentation. The dislocation structure is idealized as circular dislocation loops.

matically in Fig. 2. For simplicity we assume that the indentation is accommodated by circular loops of geometrically necessary dislocations with Burgers vectors normal to the plane of the surface. As the indenter is forced into the surface of a single crystal, geometrically necessary dislocations are required to account for the permanent shape change at the surface. Of course other dislocations, called statistically stored dislocations by Ashby (1970), not shown in the figure, would also be created and they would contribute to the deformation resistance. We take the angle between the surface of the conical indenter and the plane of the surface to be θ, the contact radius to be a and the depth of indentation to be h. If we think of the individual dislocation loops as being spaced equally along the surface of the indentation then it is easy to show that

$$\tan\theta = \frac{h}{a} = \frac{\mathrm{b}}{s}, \quad s = \frac{\mathrm{b}a}{h}, \tag{1}$$

where s is the spacing between individual slip steps on the indentation surface, as shown in the figure. If λ is the total length of the injected loops, then between r and $r+\mathrm{d}r$ we have

$$\mathrm{d}\lambda = 2\pi r\frac{\mathrm{d}r}{s} = 2\pi r\frac{h}{\mathrm{b}a}\mathrm{d}r, \tag{2}$$

which after integration gives

$$\lambda = \int_0^a \frac{h}{\mathrm{b}a} 2\pi r\,\mathrm{d}r = \frac{\pi h a}{\mathrm{b}}. \tag{3}$$

We assume that all of the injected loops remain within the hemispherical volume V defined by the contact radius

$$V = \tfrac{2}{3}\pi a^3, \tag{4}$$

so that the density of geometrically necessary dislocations becomes

$$\rho_G = \frac{\lambda}{V} = \frac{3h}{2\mathrm{b}a^2} = \frac{3}{2\mathrm{b}h}\tan^2\theta. \tag{5}$$

To estimate the deformation resistance we use the Taylor relation to find the shear strength as follows:

$$\tau = \alpha\mu\mathrm{b}\sqrt{\rho_T} = \alpha\mu\mathrm{b}\sqrt{\rho_G+\rho_s}, \tag{6}$$

where ρ_T is the total dislocation density in the indentation, ρ_s is the density of statistically stored dislocations, μ is the shear modulus, b is the Burgers vector and α is a constant to be taken as 0.5 in all of the analyses below. We note that ρ_s is not expected to depend on the depth of indentation. Rather it depends on the average strain in the indentation, which is related to the shape of the indenter ($\tan\theta$). We assume that the von Mises flow rule applies and that Tabor's factor of 3 can be used to convert the equivalent flow stress to hardness:

$$\sigma = \sqrt{3}\tau, \quad H = 3\sigma. \tag{7}$$

With these relations we may now write the hardness using both eqns (5) and (6) as

$$\frac{H}{H_0} = \sqrt{1+\frac{h^*}{h}}, \tag{8}$$

where

$$H_0 = 3\sqrt{3}\alpha\mu\mathrm{b}\sqrt{\rho_s}, \tag{9}$$

is the hardness that would arise from the statistically stored dislocations alone, in the absence of any geometrically necessary dislocations, and

$$h^* = \frac{81}{2}\mathrm{b}\alpha^2\tan^2\theta\left(\frac{\mu}{H_0}\right)^2 \tag{10}$$

is a length that characterizes the depth dependence of the hardness. We note that h^* is not a constant for a given material and indenter geometry. Rather it depends on the statistically stored dislocation density through H_0.

3. COMPARISON WITH INDENTATION EXPERIMENTS

The characteristic form for the depth dependence of the hardness shown in eqn (8) suggests that the square of the hardness should be plotted against the reciprocal of the depth of indentation. When the data of Fig. 1 are plotted in this way a good straight line is found, the intercept of which is H_0 and the slope of which is h^*. Then

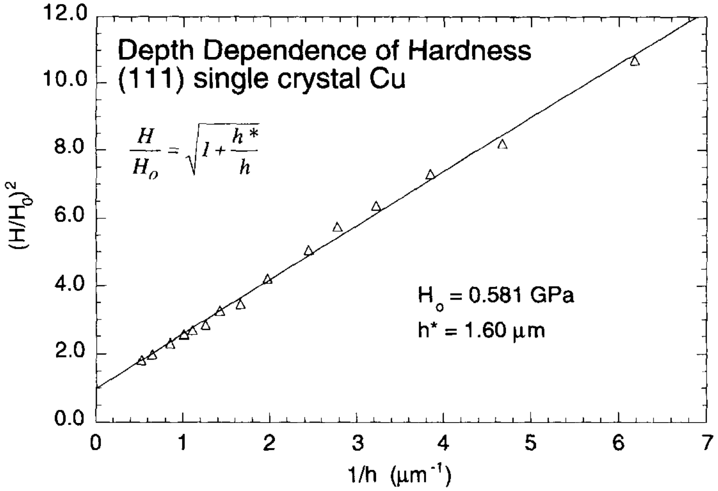

Fig. 3. Depth dependence of the hardness of (111) single crystal copper, taken from Fig. 1, plotted according to eqn (8).

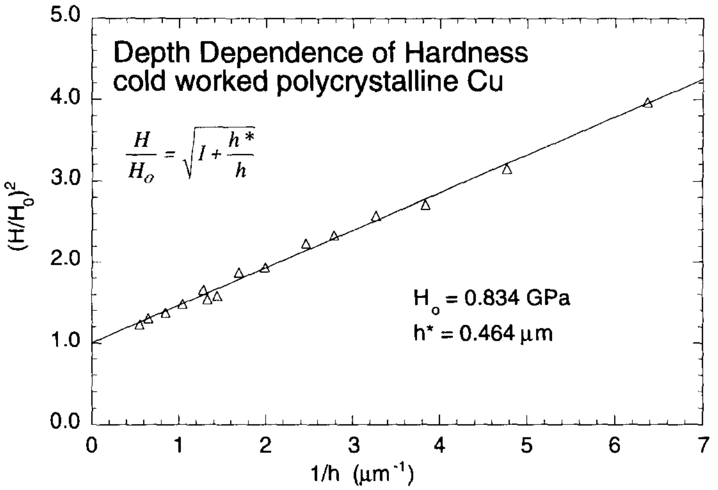

Fig. 4. Depth dependence of the hardness of cold worked polycrystalline copper, taken from Fig. 1, plotted according to eqn (8).

the data can be displayed as a plot of $(H/H_0)^2$ vs $1/h$, as shown for the (111) single crystal copper in Fig. 3 and for the cold worked polycrystalline copper in Fig. 4. Hardness data for indentation depths less than about 0.1 μm have been excluded from Fig. 1 and Figs 3–4 both because the shape of the indenter is not self similar at small indentation depths, as assumed in the model, and because uncertainties in the contact area arise at small depths of indentation. We observe excellent agreement with the predictions of the model.

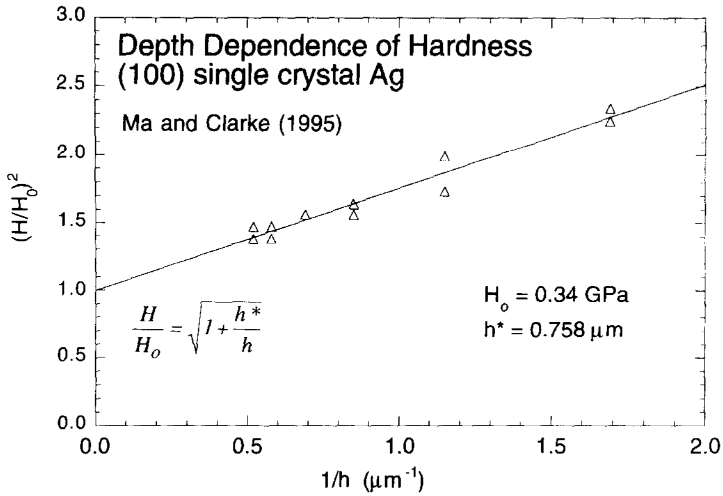

Fig. 5. Depth dependence of the hardness of (100) single crystal silver, taken from the work of Ma and Clarke (1995), plotted according to eqn (8).

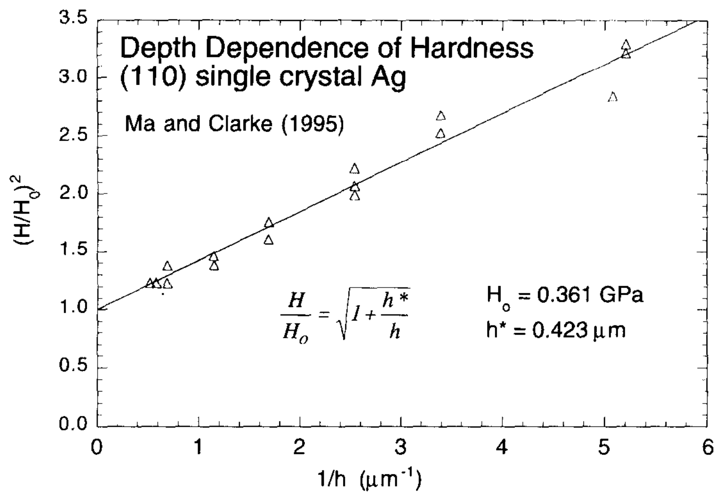

Fig. 6. Depth dependence of the hardness of (110) single crystal silver, taken from the work of Ma and Clarke (1995), plotted according to eqn (8).

To illustrate the utility of this model for describing the depth dependence of hardness of metals we plot the hardness data of Ma and Clarke (1995) for (100) and (110) single crystals of Ag in Figs 5 and 6, again using the form of eqn (8) as a guide. As in the case of the copper data, we have excluded hardness data for indentation depths less than 0.1 μm. Again we find good agreement with the predictions of the model.

As discussed above, the quantities H_0 and h^*, as described by the model, are not

Table 1. *Comparison of predicted [eqn (10)] and measured values of the characteristic length,* h*

Material	H_0 (GPa)	h^* (μm)	μ (GPa)	b (nm)	α	h (predicted) (μm)
(111) single crystal Cu (annealed)	0.581	1.60	42	0.256	0.5	1.73
Polycrystalline Cu (cold worked)	0.834	0.464	42	0.256	0.5	0.840
(100) single crystal Ag**	0.340	0.757	26.4	0.286	0.5	2.23
(110) single crystal Ag**	0.361	0.432	26.4	0.286	0.5	1.98

** Ma and Clarke (1995).

independent. Rather they are related through eqn (10). Thus a check of self consistency can be made by computing the expected value of h^* from the measured values of H_0 using the model and the known shape of the indenter. All of the experiments described here were conducted using a three-sided Berkovich indenter for which the nominal contact area varies as

$$A_c = 24.5h^2 = \pi a^2. \tag{11}$$

Using this relation together with eqn (1) gives

$$\tan\theta = \frac{h}{a} = \sqrt{\frac{\pi}{24.5}} = 0.358. \tag{12}$$

Table 1 shows the predicted values of h^* for the four sets of indentation data discussed above using eqn (10). The data needed to make these calculations are given in the table. We note that the predicted values of h^* are in reasonably good agreement with the measured values. It should be noted that the results of these calculations are quite sensitive to the chosen value of α. As mentioned above, we have used $\alpha = 0.5$ for all of these analyses. Slightly different but still reasonable values of α would be required to make the predicted and measured values of h^* coincide exactly.

The model developed here suggests that the hardness of a material should not depend strongly on the depth of indentation if the material is intrinsically hard. Specifically, large values of H_0 would cause h^* to be very small [eqn (10)] and this would cause the hardness to depend less strongly on depth at a given depth of indentation. This is shown in Fig. 7 for fused quartz, where the hardness is essentially independent of depth of indentation. If the hardness at a depth of 200 nm is taken to be about 9 GPa, then the hardness at a depth of 600 nm would be predicted by eqn (10) to be only slightly lower, 8.95 GPa, provided we use a shear modulus of $\mu = 30$ GPa, a "Burgers vector" of $b = 0.25$ nm and $\tan\theta = 0.358$ in the analysis. Naturally, the present model is based on crystalline materials and the dislocation model does not strictly apply. Nevertheless, the relative independence of hardness on depth of indentation of quartz illustrates the behavior of intrinsically hard materials and is consistent with the predictions of the model.

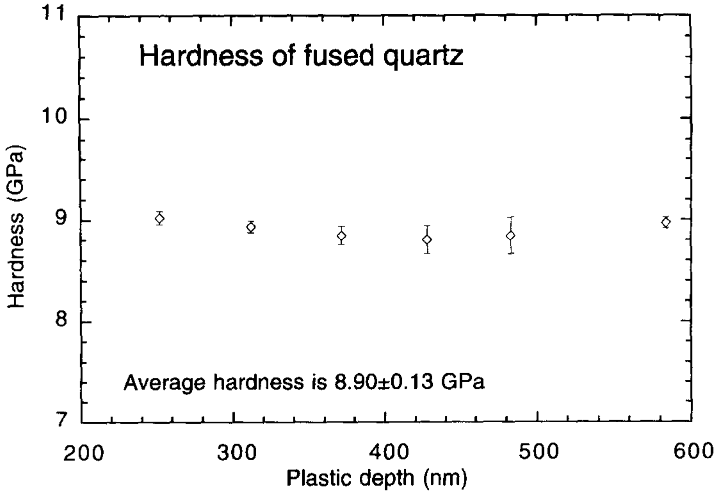

Fig. 7. Depth dependence of the hardness of fused quartz. Essentially no variation in hardness is observed, as expected from eqn (8) for a material with high intrinsic hardness and small h^*.

4. A LAW FOR STRAIN GRADIENT PLASTICITY

As discussed by Fleck *et al.* (1994) and Ma and Clarke (1995), the depth dependence of the hardness, which is expected to arise from the presence of geometrically necessary dislocations, can be associated with the strain gradient. Here we use the simple model of geometrically necessary dislocations to derive a law for strain gradient plasticity. For the indentation problem a measure of the strain gradient is

$$\chi \equiv \frac{\tan\theta}{a}. \tag{13}$$

Using this relation, together with the von Mises flow rule, Tabor's correction, eqns (1) and (10) and $\alpha = 0.5$, the expression for the depth dependence of the hardness [eqn (8)] may be written as

$$\left(\frac{\sigma}{\sigma_0}\right)^2 = 1 + \frac{9}{8}\mathrm{b}\chi\left(\frac{\mu}{\sigma_0}\right)^2, \tag{14}$$

or, more simply,

$$\left(\frac{\sigma}{\sigma_0}\right)^2 \approx 1 + \mathrm{b}\left(\frac{\mu}{\sigma_0}\right)^2\chi \tag{15}$$

where σ is the effective flow stress in the presence of a strain gradient and σ_0 is the flow stress in the absence of a gradient. We recognize the quantity

$$\hat{l} = \mathrm{b}\left(\frac{\mu}{\sigma_0}\right)^2 \tag{16}$$

as a length scale with which any strain gradient must be compared to determine the effect of the strain gradient on the flow stress. We note $\hat{l}$ is not constant for a given material but is required to change as the flow stress in the absence of a strain gradient changes. Specifically, in ordinary strain hardening the flow stress rises with increasing homogeneous strain, thus causing the characteristic length scale $\hat{l}$ to decrease according to eqn (16).

It may be useful to provide a microscopic interpretation of the length scale $\hat{l}$. For the case of pure FCC metals, which are strengthened primarily by dislocation interactions, $\hat{l}$ has a clear microstructural meaning. Using eqn (9) to obtain

$$\sigma_0 = \sqrt{3}\alpha\mu \mathrm{b}\sqrt{\rho_s}, \tag{17}$$

and noting that the density of statistically stored dislocations can be expressed approximately as

$$\rho_s \approx \frac{1}{L_s^2}, \tag{18}$$

where L_s is the mean spacing between statistically stored dislocations, the material length scale becomes

$$\hat{l} \approx \mathrm{b}\left(\frac{\mu}{\sigma_0}\right)^2 = \frac{4}{3}\frac{1}{\mathrm{b}\rho_s} = \frac{4}{3}\frac{L_s^2}{\mathrm{b}}. \tag{19}$$

Thus $\hat{l}$ depends on the mean spacing between dislocations and the Burgers vector, both natural physical dimensions for this problem. For the case of materials strengthened by dispersoids or precipitates, the flow stress in the absence of gradients might be expressed in terms of the mean spacing between particles, Lp, according to

$$\sigma_0 \approx \sqrt{3}\frac{\mu \mathrm{b}}{L_p}. \tag{20}$$

For this case,

$$\hat{l} \approx \mathrm{b}\left(\frac{\mu}{\sigma_0}\right)^2 \approx \frac{1}{3}\frac{L_p^2}{\mathrm{b}}. \tag{21}$$

Again we find that $\hat{l}$ is related to the microstructural dimension L_p and b, as expected physically. For these reasons, we shall call $\hat{l}$ the microstructural length scale for strain gradient plasticity.

The comparison of the indentation size effect for annealed and cold worked copper provides another test of the model. The microstructural length scale $\hat{l}$ can be computed with eqn (16) using the measured value of H_0 and Tabor's factor of three. The resulting values are $\hat{l}_{\mathrm{annealed}} = 12.0\ \mu\mathrm{m}$ and $\hat{l}_{\mathrm{cold\ worked}} = 5.84\ \mu\mathrm{m}$. We note that $\hat{l}$ for the cold worked sample is a factor of two smaller than the value for annealed copper, indicating that larger strain gradients are necessary to produce the same strengthening effects in cold worked Cu as for annealed Cu. This is, of course, evident in the depth dependence of the hardness for these two materials, Fig. 1, where the depth dependence for a particular strain gradient, as defined by the shape of the indenter, is smaller for the cold worked sample.

Strictly speaking, an indent with constant slope, as assumed in the previous section, is not entirely consistent with the approximation of a constant strain gradient. Alternatively, one may use the assumption of a constant strain gradient as the starting point and repeat the analysis given above. In this case, the dislocation spacing becomes nonuniform,

$$s(r) = \frac{a\mathrm{b}}{(a-r)}\frac{1}{\tan\theta}. \tag{22}$$

This leads to a higher density of geometrically necessary dislocations at the center of the indent,

$$\rho_{G\max} = \frac{1}{2\mathrm{b}h}\tan^2\theta. \tag{23}$$

This is twice the average density

$$\overline{\rho_G} = \frac{1}{4\mathrm{b}h}\tan^2\theta. \tag{24}$$

If ρ_G is replaced with $\rho_{G\max}$ in eqn (6), an analysis similar to eqns (7)–(10) leads to a similar expression for h^*

$$h^* = \frac{27}{2}\mathrm{b}\alpha^2\tan^2\theta\left(\frac{\mu}{H_0}\right)^2. \tag{25}$$

This is still within a reasonable range of the experimental data presented in Table 1.

5. STRAIN GRADIENT PLASTICITY

The indentation experiments described above suggest a strain gradient plasticity law of the form

$$\sigma^2 = \sigma_0^2 + \mu^2\mathrm{b}\chi, \tag{26}$$

where χ measures the strain gradient. In order to compare this form of strain gradient plasticity with that developed by Fleck and Hutchinson (1993, 1997), we consider the usual power hardening law in the absence of a strain gradient

$$\sigma_0 = \sigma_{\mathrm{ref}}\varepsilon^{1/n}, \tag{27}$$

where n is a hardening exponent and σ_{ref} is a reference stress taken to be a measure of the yield stress. Inserting eqn (27) into eqn (26) yields a strain gradient plasticity law with a strain-independent material length scale

$$\left(\frac{\sigma}{\sigma_{\mathrm{ref}}}\right)^2 = \varepsilon^{2/n} + \mathrm{b}\left(\frac{\mu}{\sigma_{\mathrm{ref}}}\right)^2\chi. \tag{28}$$

This can be compared to the corresponding law in the Fleck–Hutchinson (1997) strain gradient plasticity framework, which, in the case of a single material length scale, becomes

$$\left(\frac{\sigma}{\sigma_{\text{ref}}}\right)^2 = (\varepsilon^{\beta} + (l\chi)^{\beta})^{2/n\beta}, \tag{29}$$

where l is a phenomenological length scale, β is an exponent usually taken to be 2 (Fleck and Hutchinson, 1993), but can be generalized to an arbitrary number larger than 1 without destroying the basic theoretical framework (Fleck and Hutchinson, 1997). The effective strain ε and the effective strain gradient χ are related to their tensor components by

$$\varepsilon = \sqrt{\frac{2}{3}\varepsilon_{ij}\varepsilon_{ij}}, \tag{30}$$

and

$$\chi = \sqrt{\frac{2}{3}\chi_{ij}\chi_{ij}}, \tag{31}$$

where χ_{ij} is the so-called curvature tensor (Fleck and Hutchinson, 1993) which is related to the third order strain gradient tensor by

$$\chi_{ij} = e_{ikl}\varepsilon_{jk,l} \quad (e_{ikl} = \text{permutation tensor}). \tag{32}$$

The effective strain gradient χ corresponds to the invariant of the curvature tensor χ_{ij}. A numerical factor on the order of one needs to be introduced to relate the simple definition of strain gradient in eqn (13) and the more rigorous definition of a tensor invariant. This factor is ignored here.

The strain gradient law of eqn (28) exactly matches the Fleck–Hutchinson phenomenological law (29) only under the following conditions

$$n = 2 \quad \beta = 1 \quad l \approx \mathrm{b}\left(\frac{\mu}{\sigma_{\text{ref}}}\right)^2 = \hat{l}\left(\frac{\sigma_0}{\sigma_{\text{ref}}}\right)^2. \tag{33}$$

The condition $n = 2$ is not unreasonable for some materials, particularly some annealed crystalline solids. Indeed, this value is assumed by Stoken (1997) in analyzing Ni filament bending experiments. The phenomenological length scale l of Fleck and Hutchinson is now related to measurable physical parameters, and to the microstructural length scale $\hat{l}$ previously discussed. The condition $\beta = 1$ is incompatible with the Fleck–Hutchinson formulation. Serious numerical problems are observed and reported as the value of β is allowed to approach 1 (Fleck and Hutchinson, 1997). Apparently, further developments of the strain gradient plasticity theory are required. Begley and Hutchinson (1997) have initiated such work using the theory of Fleck and Hutchinson (1997) with two constitutive length parameters to model indentation experiments. They concluded that the length parameter associated with the so-called stretch gradient is especially important for describing indentation experiments.

However, the physical meaning of the stretch gradient has not yet been fully investigated.

6. CONCLUDING REMARKS

We have shown that the model of geometrically necessary dislocations provides an excellent description of the depth dependence of hardness of Cu and Ag in the micrometer depth regime and that the form of the relation predicted by the model leads to a new law for strain gradient plasticity:

$$\left(\frac{\sigma}{\sigma_0}\right)^2 = 1+\hat{l}\chi,$$

where $\hat{l}$ is a characteristic length scale. This length scale depends on the strain rate of the material or, equivalently, the flow stress of the material in the absence of a strain gradient:

$$\hat{l} \approx \mathrm{b}\left(\frac{\mu}{\sigma_0}\right)^2.$$

The model of geometrically necessary dislocations shows that $\hat{l}$ scales naturally with L^2/b, where L is the spacing between dislocation obstacles and b is the Burgers vector.

For power hardening materials, the above law can be recast in a form with a strain-independent length scale l,

$$\left(\frac{\sigma}{\sigma_{\mathrm{ref}}}\right)^2 = \varepsilon^{2/n}+l\chi \quad l = \mathrm{b}\left(\frac{\mu}{\sigma_{\mathrm{ref}}}\right)^2 = \hat{l}\left(\frac{\sigma_0}{\sigma_{\mathrm{ref}}}\right)^2.$$

Although this form resembles the Fleck–Hutchinson framework of strain gradient plasticity theory, exact match with that theory is not possible due to some intrinsic incompatibility problems. Nevertheless, the phenomenological length scale l is now clearly linked to meaningful physical parameters and the microstructural length scale $\hat{l}$.

Clearly the strengthening effects that arise from gradients in the strain become especially important when the strain gradients are large. Thus, these effects are expected to be significant when the material in question is plastically deformed in very small volumes, such as at the tips of cracks or in sub-micrometer indentations. However, other effects that do not appear to be associated with strain gradients can also be large when plasticity is constrained to occur in small volumes, and they should be distinguished from strain gradient effects. For example, the biaxial plastic deformation of a thin film on a substrate caused by thermal expansion mismatch between the film and the substrate typically requires much higher stresses than would be required for bulk deformation of the same material [Venkatraman and Bravman (1992)]. This is thought to arise from fine structure effects (small grain size, small film thickness) and not from plastic strain gradients, as the biaxial strains imposed on such thin films are essentially homogeneous. No significant strain gradients are thought to be present when a thin film is plastically deformed in this way. One could envision the high strengths of such thin films to arise from the blockage of dislocations at the film/substrate interface, leading to the formation of dislocation pile ups in the film. This would lead to a plastic strain gradient in the film and also to a gradient of biaxial stress and elastic strain. But Doerner and Brennan (1988) and Venkatraman *et al.* (1994) have tried, without success, to find evidence for such stress gradients in the thin aluminum films that show the film strength effect. They used the grazing incidence X-ray scattering (GIXS) technique to measure the in-plane elastic strain as a function of distance from the free surface. They found essentially no elastic strain gradients in these films, except very near the free surface. Over most of the film thickness, the strains seem to be constant and independent of depth. This suggests that plastic strain gradients are not large and that the strengths of these films should not be associated with strain gradient effects.

The thin film strength problem does have some characteristics in common with the identation/strain gradient problem. In both cases one has "extra" dislocation storage as a consequence of the constraints on the plastic deformation. In indentation one must have strain gradients and geometrically necessary dislocations. These geometrically necessary dislocations are in excess of the ones that would form by statistical processes. In like manner "extra" dislocations are created at the film/substrate interface in the thin film problem because of the constraint of the substrate. This extra storage process occurs even if all of the dislocations run right to the film/substrate interface. In this sense the two problems have something in common.

For another example, the tensile strengths of finely structured microlaminate films are much higher than the strengths of monolithic films [Was and Foecke (1996)], even though these materials are not subjected to large plastic strain gradients. Again this strengthening is thought to be associated with the small dimensions of the individual layers and not with the presence of strain gradients.

Thus, accounting for the high strengths of thin films and multilayers does not necessarily require the use of strain gradient plasticity. Even when these materials are deformed homogeneously under biaxial loading, it is still not possible to use the constitutive properties of bulk materials to make predictions about the mechanical behavior. The constitutive properties of the thin film materials themselves must be used for such predictions.

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