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Radiation and propagation of a surface-wave mode on a curved open waveguide of arbitrary cross section

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A general scheme for calculating the change (both the real part and the imaginary part) of the propagation constant of a surface-wave mode on a general open waveguide structure is presented. By carefully keeping track of lower-order terms in the analysis, it is shown how approximations commonly made in other analyses can result in quite significant errors in the determination of the radiation loss. The resulting formulas require knowledge only of the fields and propagation constant of the corresponding straight waveguide mode, and the value of the radius of curvature of the waveguide axis. As a simple example, the curvature loss of a Goubau line is calculated.

1. INTRODUCTION

One of the practical limitations on the use of open waveguiding structures as a transmission medium is that such structures are much more susceptible to radiation losses than are more conventional closed waveguides [*Kapron et al.*, 1970]. With the recent increase of interest in optical fibers and integrated channel waveguides in communication systems, radiation from curved surface waveguides has received a good deal of attention.

The earliest treatment of a problem of this type seems to have been *Richtmyer's* [1939], which was later followed by more careful treatments of less complicated structures by *Elliott* [1955] and *Miller and Talanov* [1956]. Of the more recent work, that of *Marcatili* [1969, 1970] involves assumptions on the form of the field outside a rectangular dielectric structure whose validity cannot easily be assessed. Ray techniques have also been applied to both two- and three-dimensional structures [*Maurer and Felsen*, 1970; *Gloge*, 1972; *Snyder and Mitchell*, 1974*b*], but their generalization to guides of general cross-section would be rather cumbersome. Particularly simple geometries such as the slab [*Marcuse*, 1971, 1974; *Chang and Barnes*, 1973; *Heiblum and Harris*, 1975] and the circular fiber [*Shevchenko*, 1971; *Lewin*, 1974; *Arnaud*, 1974] can be treated by more or less approximate techniques, but, especially in the case of the fiber, whose finite cross section precludes any simple exact formulation of the boundary problem, the various results are sometimes inconsistent, and the most careful of these analyses [*Lewin*, 1974] has been performed only for these simple geometries.

The method of *Arnaud* [1974] is in principle a generalization to guides of arbitrary cross section of Lewin's analysis of the fiber, since both approaches utilize a spectral expansion of the fields in the direction normal to the plane of the bend. The result for the special case of the fiber treated therein does, in fact, follow from Lewin's result, after the appropriate redefinitions and approximations have been made [*Arnaud*, 1975*a*]. In this paper, the spectral expansion technique is utilized to develop a general formulation applicable to waveguides of arbitrary inhomogeneous cross section, which is, in some respects, the extension of the above approach whose possibility was indicated by *Arnaud* [1974], but which retains much of the simple physical interpretation of the slab case. By carefully keeping track of small-order terms, significant factors may be found in both the attenuation and phase shift which are often incorrectly neglected. Specific application of our theory to several open waveguide structures has been given in a companion paper [*Kuester and Chang*, 1975]. As a simple example, we include here a calculation of the curvature loss of a Goubau line.

2. REVIEW OF THE CURVED SLAB PROBLEM

We consider first a curved homogeneous dielectric slab waveguide of thickness D having an inner radius of curvature R and a refractive index n (with respect to the surroundings) as depicted in Figure

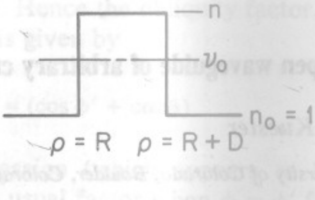

Fig. 1. Straight homogeneous slab.

1. All quantities are assumed independent of z (the direction normal to the bend), and we search for solutions of the form $\exp(i\omega t - ik_0 \nu R\theta)$, where $k_0^2 = \omega^2 \mu_0 \epsilon_0$, which satisfy the usual boundary conditions at the slab and the outgoing radiation condition at sufficiently large radial distances ρ from the slab, where ρ is the radial coordinate in the cylindrical system. Here the (normalized) propagation constant ν, although as yet undetermined, has to approach the value ν_0 corresponding to the straight guide as $R \to \infty$.

We now define a local coordinate system $\tilde{x} = R \ln(\rho/R)$ and $\tilde{y} = R\theta$ so that the governing wave equation becomes [*Chang and Barnes*, 1973]:

$$\{d^2/d\tilde{x}^2 + k_0^2[n_j^2 \exp(2\tilde{x}/R) - \nu^2]\} E_z(\tilde{x}) = 0 \quad (1)$$

where $n_j = 1$ or n for $j = 1$ or 2, is the refractive index corresponding to the medium outside or inside the slab. The slab boundaries have now become $\tilde{x} = 0$ and $\tilde{x} = d = R \ln(1 + D/R) \simeq D$. Thus for all practical purposes we can replace the curved slab of Figure 1 by a straight one of virtually the same thickness but with an inhomogeneous refractive index profile as in Figure 2. For the case of a straight homogeneous slab, the propagation constant for a propagating surface-wave mode satisfies $1 < \nu_0 < n$. Thus inside the slab the solution is a standing wave—a linear combination of $\sin(n^2 - \nu_0^2)^{1/2}\tilde{x}$ and $\cos(n^2 - \nu_0^2)^{1/2}\tilde{x}$. Outside the slab, $n_j^2 - \nu_0^2 < 0$ so that the solution is an evanescent wave of the form $\exp[-(\nu_0^2 - 1)^{1/2}|\tilde{x}|]$.

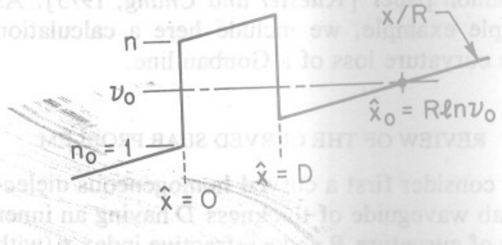

Fig. 2. Straightened slab with exponential profile.

Now in the case of the curved slab, if ν_0 is essentially unchanged, the character of the solutions is unchanged everywhere except the region where $\tilde{x} > \tilde{x}_0 = R \ln \nu_0$ corresponding to $\rho_0 = \nu_0 R$. Beyond this point (which is known as the turning point) the effective refractive index $n_j \exp(\tilde{x}/R) > \nu_0$ and the solution of (1) satisfying the radiation condition is an outgoing unattenuated cylindrical wave. Thus, in spite of the fact that between $\tilde{x} = d$ and $\tilde{x} = \tilde{x}_0$ the fields must be evanescent, the character of the solution at $\tilde{x} = \tilde{x}_0$ must change so that the field is partially transmitted and partially reflected from the turning point, returning toward the slab as an "incoming" evanescent wave. Near the slab surface $\tilde{x} = d$ then, the electric field must have a finite, although small, exponentially growing component [*Chang and Barnes*, 1973]:

$$E_z \simeq E_0[\exp(-k_0\lambda\tilde{x}) + \sigma_0 \exp(k_0\lambda\tilde{x})] \quad (2)$$

where the reflection coefficient is found by the WKB method to be

$$\sigma_0 = -(i/2)\exp(-2\lambda^3 k_0 R/3\nu^2) \quad (3)$$

Here $\lambda = (\nu^2 - 1)^{1/2}$, making $(k_0\lambda)^{-1}$ the penetration depth of the surface wave into the outside medium. It should be noted that the reflection coefficient decreases exponentially with R, so that when $R \to \infty$, $\sigma_0 \to 0$ and (2) reduces to the field outside the homogeneous straight slab. *Chang and Barnes* [1973] have calculated the attenuation coefficient for this structure in a straightforward manner by considering the reflected field in (2) as a perturbation, and calculating the resultant change in impedance at the slab surfaces.

3. SPECTRAL REPRESENTATION OF FIELDS IN FINITE CROSS SECTION WAVEGUIDES

We now attack the problem of a curved section of a dissipationless three-dimensional optical waveguide, shown in Figure 3. We allow the guide to be of arbitrary cross-sectional shape, and possibly inhomogeneous, but the outside medium is required to be homogeneous with refractive index n_0 for $\rho > \rho_{max}$. We construct four coordinate systems for this geometry as shown: two global ones (Cartesian (x,y,z) and cylindrical (ρ,θ,z) as usual) and two local ones (local Cartesian $(\hat{x},\hat{y},\hat{z})$, where $\hat{x} = \rho - R$, $\hat{y} = R\theta$, $\hat{z} = z$, and local cylindrical $(\hat{r},\hat{\phi},\hat{y})$, where $\hat{r}\cos\hat{\phi} = \hat{x}$ and $\hat{r}\sin\hat{\phi} = -\hat{z}$). The radius of curvature R is chosen as the distance between

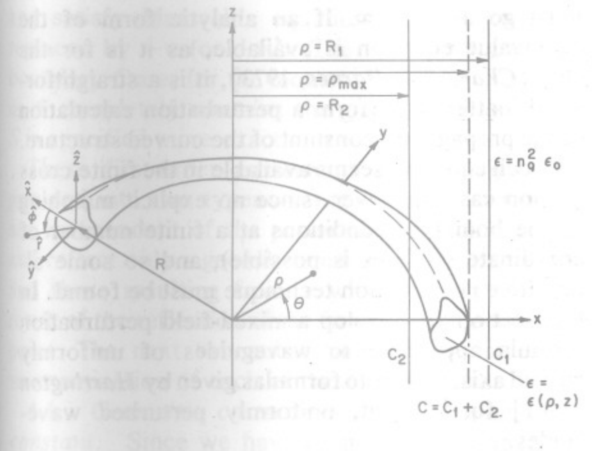

Fig. 3. Section of curved optical waveguide with Cartesian coordinate system (x,y,z), cylindrical system (ρ,θ,z), local Cartesian system $(\hat{x},\hat{y},\hat{z})$, and local cylindrical system $(\hat{r},\hat{\phi},\hat{y})$.

the origins of the local and global systems.

Now any Cartesian field component Φ in the region $\rho > \rho_{\max}$ (the largest value of ρ in the guide cross section, see Figure 3) must satisfy the scalar wave equation

$$(1/\rho)(\partial/\partial\rho)(\rho\, \partial\Phi/\partial\rho) + \partial^2\Phi/\partial z^2 + k_0^2(n_0^2 - \nu^2 R^2/\rho^2)\,\Phi = 0 \qquad (4)$$

where an $\exp(i\omega t - ik_0 \nu R \theta)$ dependence has been assumed as before. Because the medium in this region is uniform in the z direction between $\pm\infty$, we may further reduce (4) using the Fourier transform pair

$$\Phi(\rho,z) = \int_{-\infty}^{\infty} \tilde{\Phi}(\rho,s)\, e^{-iksz}\, ds$$
$$\tilde{\Phi}(\rho,s) = \frac{k}{2\pi}\int_{-\infty}^{\infty} \Phi(\rho,z)\, e^{iksz}\, dz \qquad (5)$$

where $k = k_0 n_0$ is the wave number in the outer medium. The spectrum function $\tilde{\Phi}$ now satisfies the equation:

$$(1/\rho)(\partial/\partial\rho)(\rho\, \partial\tilde{\Phi}/\partial\rho) + k_0^2 [n_0^2(1 - s^2) - \nu^2 R^2/\rho^2]\,\tilde{\Phi} = 0 \qquad (6)$$

The exact solution of (6) which satisfies the radiation condition at $\rho = \infty$ is well known:

$$\begin{aligned}\tilde{\Phi}(\rho,s) &= A_1 H^{(2)}_{\nu k_0 R}[k\rho(1 - s^2)^{1/2}], \quad s^2 < 1\\ &= A_2 (k\rho)^{-\nu k_0 R}, \quad s^2 = 1\\ &= A_3 K_{\nu k_0 R}[k\rho(s^2 - 1)^{1/2}], \quad s^2 > 1\end{aligned} \qquad (7)$$

where A_i are independent of ρ, $H^{(2)}$ is the outgoing Hankel function, and K is the modified Bessel function of the second kind.

For the purpose of calculating the attenuation for large radii of curvature, the exact Bessel functions in (7) can be conveniently replaced by their asymptotic forms for $k_0 R \gg 1$, obtainable by the WKB method [*Chang and Barnes*, 1973; *Chang and Kuester*, 1975], or simply by using the Debye asymptotic forms for the Bessel functions [*Felsen and Marcuvitz*, 1973]. If we examine (7), we see that there are two distinct cases to be considered (the points $s = \pm 1$ may be considered as limiting cases of either). When $s^2 < 1$, and if ν were real, the Debye forms indicate that a turning point occurs at $\rho = \rho_t = \nu R(1 - s^2)^{-1/2}/n_0$. Anticipating that the mode of the curved structure will radiate, however, we realize that ν will actually be complex with a small negative imaginary part accounting for the consequent attenuation.

These considerations allow us to determine approximate forms for the spectrum function by means of the Debye expansions. For convenience we call

$$\begin{aligned}&v = \hat{x}/R, \quad w = [g(v)]^{1/2}, \quad g(v) = n_0^2(1 - s^2)\,e^{2v} - \nu^2\\ &\quad = n_0^2(1 - s^2)\,\rho^2/R^2 - \nu^2 = g(\rho)\\ &\xi = \int_{v_t}^{v} w(v')\,dv' = w - (i/2)\nu\,[\ln(w + i\nu)\\ &\quad - \ln(w - i\nu)] - (1/2)\nu\pi\end{aligned} \qquad (8)$$

where v_t is the turning point (related to $\hat{x}_t$ by (8)) and the branch of w is chosen so that $0 \le \arg w < \pi$, while $-\pi < \mathrm{Im}[\ln(w \pm i\nu)] < \pi$.

Case I: $s^2 < 1$. Except for a small region near the turning point, we have

$$\begin{aligned}\Phi \sim\ & D^i(s)\,w^{-1/2}\{[1 + O(1/k_0 R)]\\ &\cdot \exp[+ik_0 R(\xi - \xi_0)] + \sigma_s[1 + O(1/k_0 R)]\\ &\cdot \exp[-ik_0 R(\xi - \xi_0)]\}, \quad \rho_{\max} < \rho < |\rho_t|\end{aligned} \qquad (9)$$

and

$$\Phi \sim D^+(s)\,w^{-1/2}\exp(-ik_0 R\xi), \quad \rho > |\rho_t|$$

where

$$\xi_0 = \xi(v = 0) = i\nu\left[\eta_0 + \frac{1}{2}\ln\left(\frac{1-\eta_0}{1+\eta_0}\right)\right] \tag{10}$$

$$\eta_0 = [1 - n_0^2(1 - s^2)/\nu^2]^{1/2}$$

This quantity was inserted into (9) in order to make

$$\sigma_s = -(i/2)\exp(-2ik_0R\xi_0) \tag{11}$$

the ratio of the incident field and that reflected from the turning point in the local coordinate system. The functions $D^i(s)$ and $D^+(s)$ represent the amplitude of the spectral component under consideration, inside and outside the turning point, respectively (the two are in fact proportional, but since the fields beyond the turning point are not needed in the following, this point is not pursued).

Case II: $s^2 > 1$. In this case, the turning point is away from the real axis, so that only a single exponential in the Debye expansion is required, giving

$$\Phi \sim D^i(s)w^{-1/2}[1 + O(1/k_0R)]\exp[+ik_0R(\xi - \xi_0)] \tag{12}$$

for all $\rho > \rho_{max}$.

In summary, then, the fields of a curved guide outside of the guide can be represented by a spectral expansion in the z direction, and the spectrum function $\tilde{\Phi}(\rho,s)$ can be represented for large k_0R by its Debye approximation. For those components with $s^2 > 1$, formula (12) applies, i.e., locally decaying field spectrum components of this type remain evanescent all the way until $\rho = \infty$. (These are the so-called "stable waves" described by *Miller and Talanov* [1956]. See also *Lewin* [1974].) On the other hand, those components with $s^2 < 1$, which appear evanescent (locally to the guide) must in fact possess a small exponentially growing wave below the turning point as in (9), and beyond the turning point become an outwardly propagating rather than evanescent field. It is this, as in the slab case, which accounts for the attenuation in the curved waveguide.

4. PERTURBATION FORMULA FOR THE PROPAGATION CONSTANT

Having now a few general ideas about how the fields in a curved waveguide must behave, we ask how to determine the change in ν, knowing the value ν_0 and the fields in the corresponding straight structure, into which the curved guide quantities must go as $R \to \infty$. If an analytic form of the eigenvalue equation is available, as it is for the slab [*Chang and Barnes*, 1973], it is a straightforward matter to perform a perturbation calculation of the propagation constant of the curved structure. No such equation seems available in the finite cross section case, however (since no explicit matching of the boundary conditions at a finite number of coordinate surfaces is possible), and so some alternative perturbation technique must be found. In this section we develop a mixed-field perturbation formula applicable to waveguides of uniformly curved axis, similar to formulas given by *Harrington* [1961] for straight, uniformly perturbed waveguides.

By expanding Maxwell's equations, written in global cylindrical coordinates (ρ,θ,z in Figure 3), in inverse powers of R using local Cartesian coordinates ($\hat{x},\hat{y},\hat{z}$ in Figure 3) centered in the guide cross section, and utilizing the fact that $\bar{E}_0$, $\bar{H}_0$, ν_0 (the unperturbed fields and propagation constant of the straight guide) will satisfy these equations to the zeroth order in R^{-1}, one may, with the help of an analog of the two-dimensional divergence theorem, obtain the first-order perturbation to each of the real and imaginary parts of ν_0 as [*Chang and Kuester*, 1975]:

$$k_0(\nu - \nu_0) \simeq -i\,c/P + \Delta/P \tag{13}$$

where

$$c = \oint_C \frac{\rho}{R}[\bar{E}_0^- \times \bar{H}_p^+ - \bar{E}_p^+ \times \bar{H}_0^-] \cdot \bar{a}_n\, dl \tag{14}$$

$$P = 2\int_S \bar{a}_\theta \cdot [\bar{E}_0^+ \times \bar{H}_0^+]\, dS \tag{15}$$

$$\Delta = \frac{1}{R}\int_S \hat{x}\,[\omega\mu_0 \bar{H}_0^+ \cdot \bar{H}_0^+ + \omega\epsilon \bar{E}_0^+ \cdot \bar{E}_0^+]\, dS \tag{16}$$

The fields $\bar{E}_0$ and $\bar{H}_0$ are expressed in the local coordinate system (in which the integrations are performed as well), and $\bar{E}_p$ and $\bar{H}_p$ are those portions of $\bar{E}_0$ and $\bar{H}_0$ "reflected" from the turning point, in the manner of the σ_s term in (9). The transpose fields (superscript minus) are related to the original (plus) fields by the relations

$$\bar{a}_\theta \times \bar{E}^+ = \bar{a}_\theta \times \bar{E}^-, \qquad \bar{a}_\theta \times \bar{H}^+ = -\bar{a}_\theta \times \bar{H}^-,$$

$$\bar{a}_\theta \cdot \bar{E}^+ = -\bar{a}_\theta \cdot \bar{E}^-, \qquad \bar{a}_\theta \cdot \bar{H}^+ = \bar{a}_\theta \cdot \bar{H}^-$$

and satisfy Maxwell's equations if ν (or ν_0) is replaced by $-\nu$ (or $-\nu_0$). The surface S and its boundary C are, for the moment, arbitrary, so long as S completely contains the guide cross section. S is specified more exactly in section 4.2.

These first-order corrections can be logically grouped into two types—those involving Δ which are independent of the perturbation fields $\bar{E}_p, \bar{H}_p$ and reflect only a geometric influence of the bend, and those resulting from c which are directly a result of the reflection of the fields from the turning point. The next two subsections are devoted to an examination of each of these corrections.

4.1. *Geometric corrections to the propagation constant.* Since we have assumed for simplicity that the waveguide is lossless, we can show that the phase relationship of the various field components is such that Δ/P is purely real [*Chang and Kuester*, 1975], and so this term is a correction to the phase constant only which contributes no attenuation. The form of (16) suggests the geometrical interpretation of Δ as a shift of the phase velocity reference point to the center of gravity of the "energy" $\omega(\mu_0 \bar{H}_0^+ \cdot \bar{H}_0^+ + \epsilon \bar{E}_0^+ \cdot \bar{E}_0^+)$ from the (arbitrarily chosen) coordinate origin within the guide. It can in fact be shown [*Chang and Kuester*, 1975] that this $(1/R)$ correction is zero for a symmetrical structure, provided that the mode itself possesses certain symmetry properties. This result is well known for certain closed waveguides [*Lewin*, 1975] and open slab waveguides [*Chang and Barnes*, 1973] but not in the general case. Note that the inherently unsymmetrical nature of single-reactive-surface waveguides precludes this correction from vanishing, which indeed has been shown to be the case [*Miller and Talanov*, 1956].

For those asymmetrical waveguide modes where the $(1/R)$ correction is not equal to zero, one can in principle compute Δ from (16) for any guide configuration once the fields of the mode on the straight guide are known. Because of the dependence $\exp(-ik_0\nu R\theta)$, any finite angle θ of bend will cause accumulation of an appreciable excess phase shift in comparison with the straight guide. When the $(1/R)$ correction does vanish, at most a $(1/R^2)$ correction will occur; however, the total resulting excess phase shift will be of order $(1/R)$ and can thus reasonably be neglected. This phase shift may contribute substantially to both single-mode and multimode pulse distortion if its frequency dependence is strong enough; more importantly, however, this phase shift can also affect the amount of radiation loss as detailed in section 5.

4.2. *Radiation corrections to the propagation constant.* The remaining correction to the propagation constant is given by $-ic/P$. In contrast to the geometric correction term, it depends critically upon the perturbation fields $\bar{E}_p$ and $\bar{H}_p$, and will turn out to be an attenuation term. In this connection, it is interesting to note the similarity of this term, as given by (14) and (15), to the power balance relation given by *Snyder and Mitchell* [1974*a*], as well as to the mode coupling coefficients for surface waves on parallel open waveguides [*Arnaud*, 1974; *Chang and Kuester*, 1975]. *Arnaud* [1974], in fact, obtains a similar expression for the attenuative part of ν, which he interprets as coupling to a whispering gallery mode propagating along an artificially introduced perfectly conducting cylinder. The cylinder is then allowed to approach infinity, circumventing in the process the mathematical difficulty that such modes in the absence of the cylinder cannot be normalized.

The authors prefer to think of (14) as representing codirectional coupling to a second, image guide whose fields are $\bar{E}_p$ and $\bar{H}_p$. An investigation of crosstalk carried out by *Arnaud* [1975*b*] suggests that such fields could be produced by the image of the actual guide in a semi-infinite lossy medium, or by an actual second guide separated from the original one by a lossy layer. Comparison of the coupling coefficients obtained for several cases in the work of *Arnaud* [1975*b*] with the attenuation due to curvature [*Kuester and Chang*, 1975] allows the distance from the guide to its image to be calculated (see section 5).

In order to obtain from (14) a useful expression for c, we first note that the only field components which are Cartesian in all of the coordinate systems of Figure 3 are E_z and H_z. We will find it useful, then, to write (14), as far as possible, solely in terms of these. Furthermore, to use the spectrum function to the greatest possible advantage, we choose the surface S so that its boundary C consists of the two infinite lines at $\rho = R_1$ and $\rho = R_2$, between $z = -\infty$ and $z = +\infty$. R_1 and R_2 must be away from the waveguide, on the outer and inner side of the bend, respectively, but are otherwise as yet arbitrary. The resulting expression consists of two contributions (one for each part of the boundary C). Utilizing the spectrum representation of the fields and the convolution theorem, these contributions can be transformed into integrals over the spectral variable s.

At this point it is appropriate to introduce an approximation which appears at several points in the analysis and greatly simplifies matters. We assume R_1 and R_2 may be taken far enough away from the guide so that essentially all of the "power flow" is included in (19) (strictly speaking, P is merely proportional to the power), but not so far that R_1 is near or past the WKB turning point. This requirement may be stated simply in the form

$$k_0\lambda_0 R \gg 1, \quad \lambda_0 = \lambda(\nu_0, 0)$$

where

$$\lambda(\nu,s) = [\nu^2 + n_0^2(s^2-1)]^{1/2} \tag{17}$$

As a result, all terms in c involving $\exp[-2k_0\lambda(\mu_0,s)\hat{x}]$ may be considered as smaller than any inverse power of R, and the only significant remaining terms are those in which the exponential dependence has been cancelled, i.e., terms involving the product of a locally growing and a locally evanescent wave. To first order then, we have a contribution only from C_1 [*Chang and Kuester*, 1975]:

$$c = \frac{4\pi}{n_0}\int_{-1}^{1}\frac{\lambda(\nu_0,s)\sigma_s}{1-s^2}\left[\frac{\tilde{E}(s)\tilde{E}(-s)}{i\omega\mu_0} + \frac{\tilde{H}(s)\tilde{H}(-s)}{i\omega\epsilon_0 n_0^2}\right]ds \tag{18}$$

where $\tilde{E}$ and $\tilde{H}$ are the spectrum functions of $E_{0\hat{z}}$ and $H_{0\hat{z}}$ (after the $\hat{x}$ dependence $\exp[-k_0\lambda(\nu_0,s)\hat{x}]$ has been removed):

$$\left.\begin{matrix}\tilde{E}(s)\\ \tilde{H}(s)\end{matrix}\right\} = \frac{k_0 n_0}{2\pi}\exp[k_0\lambda(\nu_0,s)\hat{x}]\cdot\int_{-\infty}^{\infty}\left\{\begin{matrix}E_{0\hat{z}}(\hat{x},\hat{z})\\ H_{0\hat{z}}(\hat{x},\hat{z})\end{matrix}\right\}\cdot\exp(iks\hat{z})\,d\hat{z} \tag{19}$$

Expression (19) is independent of $\hat{x}$ by virtue of the fact that $E_{0\hat{z}}$ and $H_{0\hat{z}}$ satisfy the scalar wave equation outside the guide. The integral (18) is only from -1 to $+1$ since we have, according to (12), $\sigma_s = 0$ outside this range. It should be further recognized that while $\tilde{E}$ and $\tilde{H}$ have no explicit dependence on R, there is an indirect variation which arises when the choice of the origin of the $\hat{x}$ axis is varied, and hence this choice will affect the perturbed value of ν.

5. STEEPEST-DESCENT EVALUATION OF THE ATTENUATION

Consistent with the foregoing approximations involving the magnitude of R, the appropriate method to evaluate (18) is the method of steepest descents [*Felsen and Marcuvitz*, 1973]. We have, by substitution of (10) and (11) into (18), the following expression:

$$c = \frac{2\pi\omega}{k_0^2 n_0^3}\int_{-1}^{1}[\epsilon_0 n_0^2\tilde{E}(s)\tilde{E}(-s) + \mu_0\tilde{H}(s)\tilde{H}(-s)]\cdot\exp[k_0 Rq(s)][\lambda(\nu_0,s)/(1-s^2)]\,ds \tag{20}$$

where

$$q(s) = 2\left[\lambda(\nu,s) + \frac{\nu}{2}\ln\frac{\nu-\lambda(\nu,s)}{\nu+\lambda(\nu,s)}\right]$$

Assuming that the exponential behavior of $\tilde{E}(s)$ and $\tilde{H}(s)$ is negligible in comparison with $k_0Rq(s)$, it is easily verified that the steepest-descent path is essentially the real axis between $s = \pm 1$, and that the exponent goes to $-\infty$ as $s^2 \to 1$. Choosing k_0R as the large parameter, we have by the usual techniques that the saddle point (which satisfies $q'(s) = 0$) is located at the point $s = 0$. (Two spurious complex zeroes of $q'(s)$ also occur; these, however, are branch points of $q(s)$ and will in fact provide the limitation on R which insures the validity of the steepest-descent approximation.) Thus the first-order steepest-descent asymptotic approximation is:

$$c \simeq (2\pi\omega/k_0^2 n_0^3)(\pi\lambda_0/k_0R)^{1/2}[\epsilon_0 n_0^2\tilde{E}^2(0) + \mu_0\tilde{H}^2(0)]\exp(-2\tau) \tag{21}$$

where we abbreviate $\lambda = \lambda(\nu,0) = [\nu^2 - n_0^2]^{1/2}$, and

$$\tau = -k_0R\{\lambda + (\nu/2)\ln[(\nu-\lambda)/(\nu+\lambda)]\} \tag{22}$$

In the special case when $|\lambda/\nu| \ll 1$ (quite typical of both fibers and integrated optics), we have approximately

$$\tau \simeq (1/3)k_0R\lambda^3/\nu^2 \simeq (2/3)k_0\lambda(\rho_{t0} - R) \tag{23}$$

where ρ_{t0} is the WKB turning point for $s = 0$. This way of expressing τ, in addition to the well-known dependence of $\exp(-k_0\lambda W)$ of the parallel-guide coupling coefficient [*Arnaud*, 1974] where W is the separation between guides, allows us to roughly identify the distance between the first guide and the fictitious second guide as $W = (4/3)(\rho_{t0} - R)$. Finally, for (21) to be valid, we must require the distance from the saddle point to the nearest singularities of $q(s)$ (see above) to be large. The condition is easily shown to be

$$|\tau| \gg 1 \tag{24}$$

which, for the case $|\lambda/\nu| \ll 1$, is obviously more stringent than the condition $k_0 \lambda_0 R \gg 1$ assumed in section 4.2. Thus (24) is the single criterion of applicability for any finite cross section guide.

It would ordinarily be sufficient to evaluate τ by replacing ν everywhere by ν_0 for a symmetric mode when the difference $(\nu - \nu_0)$ is of order $(1/R^2)$ as we explained in section 4.1. However, for the more general case, the $(1/R)$ correction of section 4.1 does not vanish and we must take it into account. Thus, if we call $\lambda_0 = \lambda(\nu = \nu_0)$ and $\tau_0 = \tau(\nu = \nu_0)$, we have by a Taylor series expansion:

$$\tau \simeq \tau_0 - \frac{1}{2} k_0 (\nu - \nu_0) R \ln \left(\frac{\nu_0 - \lambda_0}{\nu_0 + \lambda_0} \right) \tag{25}$$

or, again for $|\lambda_0/\nu_0| \ll 1$,

$$\tau \simeq \tau_0 + k_0 (\nu - \nu_0) R \lambda_0 / \nu_0 + (1/3) k_0 (\nu - \nu_0) \cdot R \lambda_0^3 / \nu_0^3 + \ldots \tag{26}$$

The appropriate procedure, then, for calculating $(\nu - \nu_0)$ is to first calculate the real $(1/R)$ correction as in section 4.1 and then, using this quantity in (25) or (26), proceed to calculate the attenuation from (21). The only fields required in (21) are actually "averaged" in the $\hat{z}$ direction by the integration of (19); thus it is to be expected that less error will be incurred as a result of using inexact fields at this point than would be if the "naked" inexact fields were used to calculate the loss. This may explain why *Marcatili's* [1969] field approximations for the rectangular dielectric waveguide are incapable of predicting the $R^{-1/2}$ dependence of the attenuation which by (21) should occur in all finite cross section structures. Finally, for the case of a symmetrical mode, $\nu \simeq \nu_0$ so that the expression for τ reduces to

$$\tau \simeq \tau_0 = (1/3)\, k_0 R \lambda_0^3 / \nu_0^2 \tag{27}$$

6. A SIMPLE EXAMPLE: THE GOUBAU LINE

As a simple example, let us consider the lowest-order (axially symmetric) TM mode on a Goubau line [*Collin*, 1960], since no formula for the bending loss of such lines seems to have appeared in the literature. For a Goubau line consisting of an inner conductor of circular cross section with radius a, and an annular dielectric coating of outer radius b and refractive index n, situated in free space, the fields of this mode are well known. It is straightforward, if somewhat tedious, to calculate the P integral (15) [*Chang and Kuester*, 1975]:

$$P = -(2\pi \nu_0 A^2 / Z_0) D(\nu_0) \tag{28}$$

where A is an arbitrary constant,

$$D(\nu_0) = b^2 (1 - n^2) \left[\frac{n^2}{\kappa_0^2} F_1^2(\nu_0) + \frac{1}{\lambda_0^2} F_0^2(\nu_0) + \frac{n^2}{\lambda_0^2 \kappa_0^2} F_0(\nu_0) F_2(\nu_0) \right] - \frac{4n^2}{\pi^2 k_0^2 \kappa_0^4 Y_0^2(k_0 \kappa_0 a)}$$

$$\kappa_0 = (n^2 - \nu_0^2)^{1/2}$$

$$Z_0 = (\mu_0 / \epsilon_0)^{1/2}$$

$$F_j(\nu_0) = J_j(k_0 \kappa_0 b) - J_0(k_0 \kappa_0 a) \frac{Y_j(k_0 \kappa_0 b)}{Y_0(k_0 \kappa_0 a)}$$

(Though not obvious, $D(\nu_0) > 0$ when A is real, as may be shown using the eigenvalue equation.) Similarly, since the fields outside the line have the same form as for the optical fiber TM modes, we may utilize this spectrum function [*Kuester and Chang*, 1975] to obtain

$$c = -\frac{\pi}{k_0 Z_0} \left(\frac{\pi}{k_0 R} \right)^{1/2} e^{-2\tau_0} \frac{A^2 F_0^2(\nu_0)}{2 \lambda_0^{7/2} K_0^2(k_0 \lambda_0 b)} \cdot (\nu_0^2 \sin^2 \phi_0 + \cos^2 \phi_0) \tag{29}$$

where τ_0 is given by (22) with $\nu = \nu_0$ and $\lambda = \lambda_0$, and ϕ_0 is a mode polarization angle with respect to the plane of the bend. Note that the radiative attenuation

$$c/P = (\pi / k_0 R)^{1/2} e^{-2\tau_0} [1 + \lambda_0^2 \sin^2 \phi_0] F_0^2(\nu_0) \div 4 k_0 \nu_0 \lambda_0^{7/2} K_0^2(k_0 \lambda_0 b) D(\nu_0) \tag{30}$$

is polarization dependent; only weakly so if $\nu_0 \simeq 1$, but more and more strongly as the surface wave becomes slower.

7. CONCLUDING REMARKS

To summarize the technique, then: given a straight surface waveguide of arbitrary cross section whose fields $\bar{E}_0$ and $\bar{H}_0$ and propagation constant ν_0 are known, we may calculate the shift in the real part of the propagation constant due to a uniform curvature of the waveguide axis of radius R to be Δ/P, where Δ and P are given in (15) and (16). This done, we may calculate the radiative attenuation as $-ic/P$, where c is given by (21), (19), and (25) or (26), using the previously determined value for the real part of the correction to ν_0 in (25) or (26).

The application of this method to two-dimensional slab structures is straightforward, with the following simplifications taking place. The integrals (15) and (16) become line integrals instead of surface integrals, and the line integral (14) actually degenerates into an evaluation of the integrand at the single point $\rho = R_1$ (see section 4.2). In this case the steepest-descent evaluation of c is unnecessary, as is the use of spectrum functions for the waveguide fields, and the factor $R^{-1/2}$ no longer appears in c [*Kuester and Chang*, 1975]. The application of this procedure for the asymmetric slab is given by *Kuester and Chang* [1975]. Discrepancies are found with *Marcuse's* [1971] result for the asymmetric slab which are traceable to a neglect of the $(\nu - \nu_0)$ corrections appearing in (29) and (30). That these discrepancies can be of quite significant magnitude in this case has been previously demonstrated [*Chang and Barnes*, 1973].

It is interesting to consider the transition from a finite cross section guide to a slab as the former is elongated in the z direction. If we follow a mode whose variation in the z direction has only a single maximum, this mode pattern becomes broader and broader as the guide is elongated, and in the sense that field variations in this direction become slower and slower, it approximates a mode on the corresponding infinite slab (whose fields are independent of z). As this happens, however, the Fourier transforms $\tilde{E}(s)$ and $\tilde{H}(s)$ of the fields become narrower and narrower, eventually becoming delta functions in the slab limit. But before this occurs, the exponential dependence of $\tilde{E}(s)$ and $\tilde{H}(s)$ will have become so strong that the steepest-descent evaluation (21) for c will have ceased to be valid. It is easy to see, in fact, that taking this additional exponential dependence into account in the steepest-descent evaluation of (20) can explain the transition from the additional $R^{-1/2}$ dependence of c in the finite cross section case to the purely exponential dependence on R in the slab case. An additional condition on the validity of (21) thus obtains, namely, that the characteristic dimension of the mode in the z direction (hence for low-order modes, of the guide itself) must be small compared to R.

Another limitation to the present method appears because in the derivation of formulas (13)-(16), it has been assumed that the fields and propagation constant of the mode on the curved guide can be developed as asymptotic expansions in $(k_0 R)^{-1}$ with the corresponding straight waveguide quantities as leading terms. When this is not the case (e.g., for so-called "edge-guided" or "whispering gallery" modes [*Heiblum and Harris*, 1975], which, due to a curvature-induced or -shifted caustic are not well approximated by any one straight guide mode), it would be necessary to know $\bar{E}_p$ and $\bar{H}_p$ independently in order to apply the present approach.

In addition, it is not necessary to restrict the medium external to the guide to be homogeneous. By choosing λ instead of s as the independent transform variable, the fields above and below the guide can be expressed as Fourier transforms with respect to the x (i.e., the ρ) variable. Once this is done, the introduction of planar boundaries parallel to the plane of the bend presents no substantial problems, and the asymptotic estimation of the attenuation may be carried out in a similar fashion. In this way, we may treat, for instance, dielectric channels embedded in a substrate, accounting for the effect of the air-substrate interface. This problem is currently under investigation.

The application of this method to the circular fiber is also given by *Kuester and Chang* [1975]. The results agree with those of *Arnaud* [1974] and *Lewin* [1974] once the differences in definitions and the appropriate approximations are made.

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