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UNIFORM ASYMPTOTIC SOLUTION OF EIGENVALUE PROBLEMS FOR CONVEX PLANE DOMAINS*

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Abstract. The ray method of Keller and Rubinow [1] determines asymptotically certain of the large eigenvalues and corresponding eigenfunctions of the Laplace operator in convex domains, but the expression for each eigenfunction is singular at a certain caustic of the rays. We shall obtain uniform asymptotic expansions of these eigenfunctions in the two-dimensional case, and improved expressions for the eigenvalues, by using the uniform expansion of Kravtsov [2], [3] and Ludwig [4]. For caustics near the boundary the results agree with these of Buldyrev [5]. We also treat the case of caustics near the boundary by using the uniform expansion of Lewis, Bleistein and Ludwig [6] and the boundary layer method used by Matkowsky [7], both of which enable us to treat the impedance boundary condition and to obtain further terms more easily.

1. Introduction. By ray methods, Keller and Rubinow [1] found asymptotic expressions for certain of the large eigenvalues and corresponding eigenfunctions of the Laplace operator, with appropriate boundary conditions, in convex domains. Their expression for each eigenfunction becomes infinite, and is therefore not valid, at the caustic of the rays, which is a curve in the two-dimensional case. We shall obtain uniform asymptotic expansions for the eigenfunctions which are valid at the caustic, and also obtain more accurate expressions for the eigenfunctions. To do so we shall first use the uniform asymptotic expansion of solutions of the reduced wave equation due to Kravtsov [2], [3], and Ludwig [4]. This will enable us to treat the case in which the caustic is at any distance from the boundary. When the caustic is far enough from the boundary, our results for the eigenvalues reduce to those of Keller and Rubinow [1], as we should expect. When the caustic is near the boundary, they reduce to those of Buldyrev [5] who used the parabolic equation method to obtain a uniform expansion of the eigenfunctions in this case.

We shall also treat the case when the caustic is near the boundary by two other methods—the uniform expansion method used for creeping waves by Lewis, Bleistein and Ludwig [6], and the boundary layer method used by Matkowsky [7]. Both these methods enable us to treat the impedance boundary condition and to obtain further terms in the expansion relatively easily. The parabolic equation method could also be used for these purposes.

The results of Keller and Rubinow [1] for the eigenvalues of circular domains are compared with the exact values in their paper. For elliptic domains the comparison is given by Isaacson, Logemann, Rosen and Keller [8]. In both cases the agreement is very good. The present improvement should make it even better.

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2. The Kravtsov–Ludwig expansion. We seek the eigenfunctions $u(\mathbf{x})$ and eigenvalues k of the equation

$$[\Delta + k^2 n^2(\mathbf{x})]u = 0, \quad \mathbf{x} \text{ in } D, \tag{2.1}$$

with either of the boundary conditions

$$u = 0, \qquad \mathbf{x} \text{ on } B, \tag{2.2a}$$

or

$$\partial_N u = 0, \quad \mathbf{x} \text{ on } B. \tag{2.2b}$$

Here D is a convex plane domain bounded by the smooth closed curve B, ∂_N is the inward normal derivative and the refractive index $n(\mathbf{x})$ is a given smooth positive function.

We shall find a particular family of eigenfunctions by assuming, with Keller and Rubinow [1], that there is a one-parameter family of closed convex curves $C(\alpha)$ inside B such that for each α, every ray tangent to $C(\alpha)$ goes into another ray tangent to $C(\alpha)$ upon reflection from B. The rays are those determined by the refractive index $n(\mathbf{x})$ according to the laws of optics. Each $C(\alpha)$ is a caustic of the family of rays tangent to it. With any caustic curve $C(\alpha)$, a uniform asymptotic expansion of a solution u of (2.1) is given by Kravtsov [2], [3] and Ludwig [4] in the form

$$u(\mathbf{x}) \sim e^{ik\theta(\mathbf{x})}\{\mathrm{Ai}\,[-k^{2/3}\rho(\mathbf{x})]g(x) + ik^{-1/3}\,\mathrm{Ai}'\,[-k^{2/3}\rho(\mathbf{x})]h(\mathbf{x})\}. \tag{2.3}$$

Here Ai denotes the Airy function and g and h are asymptotic series in k^{-1} with leading terms g_0 and h_0 respectively.

The functions θ, ρ, g_0 and h_0 are given in terms of the geometrical optics phases $\phi^{\pm}(\mathbf{x})$ and amplitudes $z^{\pm}(\mathbf{x})$ on the two rays through $\mathbf{x}$ which are tangent to $C(\alpha)$, by the equations

$$\begin{aligned} \theta &= (\phi^+ + \phi^-)/2, & \rho &= [\tfrac{3}{4}(\phi^+ - \phi^-)]^{2/3}, \\ g_0 &= \rho^{1/4}(z^+ + z^-)/2, & h_0 &= \rho^{-1/4}(z^+ - z^-)/2. \end{aligned} \tag{2.4}$$

The phases can be expressed simply as $\phi^{\pm} = \sigma^{\pm} \pm t^{\pm}$ in terms of the optical lengths $\sigma^{\pm}$ of the caustic C from some fixed point on it to the points of tangency of the two rays through $\mathbf{x}$, and the optical lengths $t^{\pm}$ of these two rays from C to $\mathbf{x}$. (The optical length of a curve is the integral of $n\,ds$ along it, where s is arc length.) The amplitudes $z^{\pm}$ are given by

$$z^{\pm} = a(\sigma^{\pm})|j^{\pm}(t^{\pm}, \sigma^{\pm})|^{-1/2}. \tag{2.5}$$

Here $a(\sigma)$ is an arbitrary function and j^+ and j^- are the Jacobians of the transformations from $\mathbf{x}$ to the ray coordinates t^+, σ^+ and t^-, σ^- respectively (see Fig. 1).

When the expressions just given for $\phi^{\pm}$ and $z^{\pm}$ are used in (2.4), they yield

$$\begin{aligned} \theta &= (\sigma^+ + t^+ + \sigma^- - t^-)/2, \\ \rho &= [\tfrac{3}{4}(t^+ + t^- + \phi^+ - \phi^-)]^{2/3}, \\ g_0 &= \frac{\rho^{1/4}}{2}\{a(\sigma^+)|j^+(t^+, \sigma^+)|^{-1/2} + a(\sigma^-)|j^-(t^-, \sigma^-)|^{-1/2}\}, \\ h_0 &= \frac{\rho^{-1/4}}{2}\{a(\sigma^+)|j^+(t^+, \sigma^+)|^{-1/2} - a(\sigma^-)|j^-(t^-, \sigma^-)|^{-1/2}\}. \end{aligned} \tag{2.6}$$

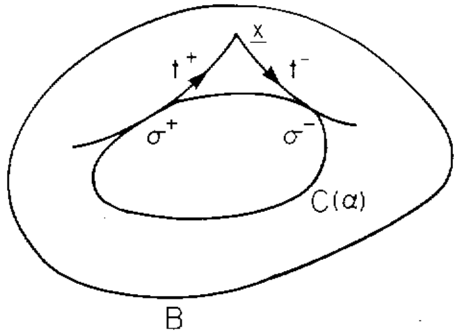

FIG. 1. *B is the boundary of a convex plane domain and $C(\alpha)$ is a caustic curve. Through each point $\mathbf{x}$ between B and $C(\alpha)$, there are two rays which are tangent to $C(\alpha)$. The two points of tangency at the caustic are denoted by their optical distances σ^+ and σ^- along the caustic from some fixed point on it. The lengths t^+ and t^- are the optical distances of $\mathbf{x}$ from σ^+ and σ^- respectively.*

When σ^+ and σ^- increase by $L(\alpha)$, the optical length of $C(\alpha)$, (2.6) shows that θ increases by L and ρ is unchanged. The Jacobians $j^{\pm}$ and the optical lengths $t^{\pm}$ are also unchanged. In order that g_0 and h_0 remain unchanged we require

(2.7) $$a(\sigma + L) = a(\sigma).$$

Now the requirement that u, given by (2.3), remain single-valued when σ^+ and σ^- increase by $L(\alpha)$ yields

(2.8) $$kL(\alpha) = 2\pi m, \qquad m = 1, 2, \cdots.$$

To make u satisfy the boundary condition (2.2a) we observe that $\rho(\mathbf{x})$ is constant for $\mathbf{x}$ on B [1] and we write the constant value $\rho = (3\alpha/4)^{2/3}$, since it depends upon the caustic $C(\alpha)$. By (2.6) this defines the parameter α to be $\alpha = t^+ + t^- + \sigma^+ - \sigma^-$ evaluated on B. Then we make the first term in (2.3) vanish on B by setting

(2.9) $$\mathrm{Ai}\,[-(3k\alpha/4)^{2/3}] = 0.$$

We make the second term in (2.3) vanish by setting $h(\mathbf{x}) = 0$ on B. In view of (2.4) and (2.5), this yields the relation

(2.10) $$a(\sigma^+)|j^+(t^+, \sigma^+)|^{-1/2} = a(\sigma^-)|j^-(t^-, \sigma^-)|^{-1/2}.$$

Here $t^{\pm}$ are evaluated on B so they are functions of σ^+ and σ^- respectively. From (2.9) we obtain

(2.11a) $$k(t^+ + t^- + \sigma^+ - \sigma^-) = \tfrac{4}{3}(q_n)^{3/2}, \qquad n = 1, 2, \cdots.$$

Here $-q_n$ is the nth root of the Airy function.

Now for each pair of integers m and n, (2.8) and (2.11a) are equations for the eigenvalue k and the parameter α. Then the corresponding eigenfunction u is given asymptotically by (2.3)–(2.5) with $a(\sigma)$ a solution of (2.10) satisfying (2.7). For the boundary condition (2.2b) we replace (2.9) by $\mathrm{Ai}' = 0$ and then instead of (2.11a) we obtain

(2.11b) $$k(t^+ + t^- + \sigma^+ - \sigma^-) = \tfrac{4}{3}(q_n')^{3/2}, \qquad n = 1, 2, \cdots.$$

Here $-q_n'$ is the nth root of the derivative of the Airy function.

Our main result is the pair of equations (2.8) and (2.11) for k and α, and (2.3) for u. When n is large, $q_n \sim [(3\pi/2)(n + 1/2)]^{2/3}$ and $q_n' \sim [(3/2)(n + 1/4)]^{2/3}$. Then (2.11a) and (2.11b) become

$$k(t^+ + t^- + \sigma^+ - \sigma^-) = 2\pi(n + 1/2) \tag{2.12a}$$

and

$$k(t^+ + t^- + \sigma^+ - \sigma^-) = 2\pi(n + 1/4). \tag{2.12b}$$

The equations (2.8) and (2.12b) are exactly the same as (136) and (137) of Keller and Rubinow [1] which were derived for the case $n(\mathbf{x}) \equiv 1$. Their method also yields (2.12a) for the boundary condition (2.2a). Thus our results improve upon theirs by having the exact root of the Airy function or its derivative instead of its asymptotic form.

When the caustic $C(\alpha)$ is near the boundary B, which is the case when α is small, we can simplify and solve the equations (2.8) and (2.11). To do so we introduce arc length s along B and distance y from B measured positively along the inward normal to B. Then it can be shown that for small α the normal distance $y(s)$ from B to $C(\alpha)$ is asymptotically given by $(3\alpha/4)^{2/3}(2n^2K)^{-1/3}$. Here $n(s, 0)$ is evaluated on B while $K(s)$ is the relative curvature of B and a ray tangent to B, and is given by

$$K(s) = \kappa(s) - n_y(s, 0)/n(s, 0). \tag{2.13}$$

The curvature of B is $\kappa(s)$. By using (2.11a) for α we can write the normal distance from B to C as

$$y(s) \sim q_n k^{-2/3}(2n^2K)^{-1/3}. \tag{2.14}$$

Now we compute the optical length $L(\alpha)$ of $C(\alpha)$ by transforming from integration with respect to arc length s' along C to arc length s along B, noting that $ds' \sim (1 - \kappa y)\,ds$, and using (2.14). Then we have

$$\begin{aligned} L(\alpha) &= \oint n[s, y(s)]\,ds' = \oint \{n(s,0) + y(s)n_y(s,0) + \cdots\}\{1 - \kappa(s)y(s) + \cdots\}\,ds \\ &= \oint n(s,0)\,ds - k^{-2/3}2^{-1/3}q_n \oint [n(s,0)K^2(s)]^{1/3}\,ds + \cdots . \end{aligned} \tag{2.15}$$

Upon using (2.15) in (2.8) and solving for k, we obtain

$$k = \frac{2\pi m}{\oint n(s,0)\,ds} + (\pi m)^{1/3} q_n \frac{\oint [n(s,0)K^2(s)]^{1/3}}{\{\oint n(s,0)\,ds\}^{4/3}} + O(m^{-1/3}). \tag{2.16a}$$

For the boundary condition (2.2b), we obtain a result we shall call (2.16b) by replacing q_n by q_n' in (2.16a). The result (2.16b) agrees with (143) of Keller and Rubinow [1] when q_n' is replaced by its asymptotic form and $n(\mathbf{x}) \equiv 1$, provided that a computational error in their result is corrected,[1] as was noted by Matkowsky

[1] Their equations (139) through (143) are in error; (139) and (140) should be

$$\rho(\tau) = \frac{(6\alpha)^{2/3}a^{1/3}(\tau)}{2}, \tag{139}$$

$$k\alpha = \frac{\pi}{2}(n + \tfrac{1}{4}). \tag{140}$$

Then their (141) through (143) will be changed accordingly and (143) will agree with our (2.16b) for n large.

[7]. Both results (2.16a) and (2.16b) with $n(\mathbf{x}) \equiv 1$ agree with those of Buldyrev [5] who derived them by the parabolic equation method.

When B is a circle, the caustics $C(\alpha)$ are the concentric circles and when B is an ellipse they are the confocal ellipses. Poritsky [9] has shown that these are the only cases in which a continuous nonconstant one-parameter family of caustics exists, and therefore they are the only cases to which the preceding analysis applies as described. In order to apply it to other curves B, different from ellipses or circles, we note that the only property of a caustic used in that analysis is the constancy of $\rho(\mathbf{x})$ on B. Therefore, to apply it to any smooth convex curve B we need merely replace the caustic curves $C(\alpha)$ by curves which are nearly caustics. Such curves may be defined by requiring that on each of them $\rho(\mathbf{x})$ differ from a constant by a small amount, say a term of order k^{-2}. Then the above analysis applies equally well. As k tends to infinity these nearly caustic curves must approach actual caustics, such as B itself or one of the discrete set of caustics inside B. The case in which they approach B itself has been treated in the last part of this section.

3. The Lewis–Bleistein–Ludwig expansion. We shall now solve the eigenvalue problem again by a slightly different method which simplifies the calculation of additional terms in the asymptotic expansion of the eigenvalue and also enables us to treat the case of an impedance boundary condition. However, this method is restricted to the case in which the caustic is near the boundary. It is based upon the expansion used by Lewis, Bleistein and Ludwig [6] to treat creeping waves on the exterior of a convex surface. They used the representation (2.3) with

$$\begin{aligned} \theta &= \theta_0(\mathbf{x}) + k^{-2/3}\theta_1(\mathbf{x}), \qquad & \rho &= \rho_0(\mathbf{x}) + k^{-2/3}\rho_1(\mathbf{x}), \\ g &\sim \sum_{m=0}^{\infty} k^{-m/3} g_m(\mathbf{x}), & h &\sim \sum_{m=0}^{\infty} k^{-m/3} h_m(\mathbf{x}). \end{aligned} \tag{3.1}$$

They found that θ_0, ρ_0, g_0 and h_0 are given by (2.4) with $\sigma^{\pm}$ denoting optical length along B, and that θ_1 and ρ_1 satisfy their equations (3.7) and (3.8).

We shall use their expansion to solve (2.1) with the impedance boundary condition

$$\partial_N u - ik^{2/3}\zeta(\mathbf{x})u = 0, \quad \mathbf{x} \text{ on } B. \tag{3.2}$$

When (2.3) is used in (3.2), the result can be simplified by requiring that $\rho_0 = 0$ on B. From the leading term in (3.2) it can be shown that on B, ρ_1 satisfies the equation

$$i\zeta \operatorname{Ai}(-\rho_1) = (2n^2K)^{1/3} \operatorname{Ai}'(-\rho_1), \quad \mathbf{x} \text{ on } B \tag{3.3}$$

(see [6, (3.18)]). From this and the next equation obtained from (3.2) and the equations referred to above, the quantities $\theta_0, \rho_0, \theta_1, \rho_1, g_0$ and h_0 are found in [6]. We note from their results that θ_0 and θ_1 are multiple-valued while ρ_0, ρ_1, g_0 and h_0 are single-valued. In order that the expansion (2.3) of u be single-valued, the different values of the exponent $k\theta_0 + k^{1/3}\theta_1$ must differ by $2\pi m$, where m is an integer. This condition yields

$$k \oint n(s, 0)\, ds - \left(\frac{k}{2}\right)^{1/3} \oint [n(s, 0)K^2(s)]^{1/3} \rho_1(s)\, ds + O(k^{-1/3}) = 2\pi m. \tag{3.4}$$

Upon solving (3.4) for k, we obtain (2.16a) with q_n replaced by the nth root $\rho_1(s)$ of (3.3) in the integrand of the numerator of the second term. This result reduces to (2.16a) when $\zeta = \infty$ and to (2.16b) when $\zeta = 0$.

To obtain another term in the expansion of the eigenvalue k, we must calculate g_1 and h_1. From (2.1) we obtain for g_1 and h_1 the equations

$$\begin{aligned}(3.5)\quad &2\nabla g_1 \cdot \nabla\theta_0 + g_1\Delta\theta_0 + (\nabla\rho_0)^2 h_1 + \rho_0\Delta\rho_0 h_1 + 2\rho_0\nabla\rho_0\cdot\nabla h_1 \\ &\quad = -i\{g_0[(\nabla\theta_1)^2 + \rho_0(\nabla\rho_1)^2 + 2\rho_1\nabla\rho_0\cdot\nabla\rho_1] + 2h_0\rho_0\nabla\rho_1\cdot\nabla\theta_1\},\end{aligned}$$

$$\begin{aligned}(3.6)\quad &2\nabla h_1 \cdot \nabla\theta_0 + h_1\Delta\theta_0 + 2\nabla\rho_0\cdot\nabla g_1 + g_1\Delta\rho_0 \\ &\quad = -i\{h_0[\rho_0(\nabla\rho_1)^2 + 2\rho_1\nabla\theta_0\cdot\nabla\rho_1 + (\nabla\theta_1)^2] + 2g_0\nabla\theta_1\cdot\nabla\rho_1\}.\end{aligned}$$

From (3.2) we get the boundary condition

$$(3.7)\quad h_1\left[i\,\mathrm{Ai}\,\rho_1\frac{\partial\rho_0}{\partial z} + \mathrm{Ai}'\right] = -ig_1\,\mathrm{Ai}\frac{\partial\theta_1}{\partial z} - \mathrm{Ai}\frac{\partial g_0}{\partial z} + \mathrm{Ai}'\left[h_0\frac{\partial\theta_1}{\partial z} + g_0\frac{\partial\rho_1}{\partial z}\right].$$

Although we have two equations (3.5) and (3.6) for g_1 and h_1, we have only one boundary condition (3.7). Just as in [6], this is related to the fact that the boundary B is characteristic, the equations reducing to ordinary differential equations along it.

We now use (3.7) to eliminate h_1 from (3.5) on B and we simplify the resulting equation by using expressions in [6]. In this way we obtain

$$(3.8)\quad \frac{dg_1}{ds} + \frac{1}{2n}g_1\Delta\theta_0 + \frac{1}{2}g_1\frac{\partial\rho_1}{\partial s}\mathrm{Ai}^2(-\rho_1)[\rho_1\,\mathrm{Ai}^2(-\rho_1) + \mathrm{Ai}'^2(-\rho_1)]^{-1} = H(s).$$

Here $H(s)$ is defined by

$$\begin{aligned}H(s) = -\frac{i}{2n}\Big\{&(2n^2K)^{1/3}\,\mathrm{Ai}(-\rho_1)\Big[-\mathrm{Ai}(-\rho_1)\frac{\partial g_0}{\partial z} \\ (3.9)\qquad &\qquad + \mathrm{Ai}'(-\rho_1)\left(h_0\frac{\partial\theta_1}{\partial z} + g_0\frac{\partial\rho_1}{\partial z}\right)\Big] \\ &\cdot[\rho_1\,\mathrm{Ai}^2(-\rho_1) + \mathrm{Ai}'^2(-\rho_1)]^{-1} + g_0(\nabla\theta_1)^2 - 2(2n^2K)^{1/3}\rho_1\frac{\partial\rho_1}{\partial z}\Big\}.\end{aligned}$$

The solution of (3.8) yields for g_1 on B the expression

$$\begin{aligned}(3.10)\quad g_1(s) = \left(\frac{n}{K}\right)^{-1/6}\beta^{-1/2}(\rho_1)\Big\{&g_1(0)\left[\left(\frac{n}{K}\right)^{1/6}\beta^{1/2}(\rho_1)\right]_{s=0} \\ &+ \int_0^s\left(\frac{n}{K}\right)^{1/6}\beta^{1/2}(\rho_1)H(s')\,ds'\Big\}.\end{aligned}$$

In (3.8) we have used $\Delta\theta_0 = (n_sK - nK_s)/3K$, and in (3.10), $\beta(\rho_1)$ is defined by

$$(3.11)\qquad \beta(\rho_1) = \rho_1\,\mathrm{Ai}^2(-\rho_1) + \mathrm{Ai}'^2(-\rho_1).$$

The solution (3.10) shows that g_1 is not single-valued. When we use this expression for g_1 in (2.3) and require u to be single-valued, we obtain (3.4) with an additional term proportional to $k^{-1/3}$ and to the difference between two values of g_1. The solution of that equation for k is

$$k = \frac{2\pi m}{\oint n(s,0)\,ds} + (\pi m)^{1/3}\frac{\oint [n(s,0)K^2(s,0)]^{1/3}\rho_1(s)\,ds}{[\oint n(s,0)\,ds]^{4/3}} + \left[\frac{\pi m}{\oint n(s,0)}\right]^{-1/3}$$

$$\cdot\left\{\left(\frac{n}{K}\right)^{-1/6}\beta^{-1/2}(\rho_1\oint\left(\frac{n}{K}\right)^{1/6}\beta^{1/2}(\rho_1)H(s)\,ds - \frac{\oint [n(s,0)K^2(s,0)]^{1/3}\rho_1(s)\,ds}{6[\oint n(s,0)\,ds]^2}\right\} + O(m^{-2/3}). \tag{3.12}$$

This is our result for k when the caustic is near B. The first two terms agree with (2.16a) and (2.16b) for $\zeta = 0$ and $\zeta = \infty$.

4. Boundary layer analysis. The boundary layer method is another way to solve (2.1) and (3.2) for the eigenfunctions which are nearly zero outside a narrow strip near B, and it was used for this purpose by Matkowsky [7]. To use it, following him, we introduce as coordinates the arc length s along B and the stretched coordinate $\tau = k^{2/3}z$, where z is normal distance from B. We also write u in terms of the three new functions θ_0, θ_1 and v in the form

$$u = e^{ik\theta_0(s) + ik^{1/3}\theta_1(s)}v(s,\tau,k). \tag{4.1}$$

We then assume that v has the asymptotic expansion

$$v(s,\tau,k) \sim \sum_{j=0}^{\infty} v_j(s,\tau)k^{-j/3}. \tag{4.2}$$

Inserting (4.1) and (4.2) into (2.1) and (3.2), and equating the coefficient of each power of k to zero yields

$$(\theta_0')^2 = n^2(s,0), \tag{4.3}$$

$$[\partial_\tau^2 - 2n^2(s,0)K(s)\tau - 2\theta_0'\theta_1']v_j = F_j(\theta_0, \theta_1, v_0, \cdots, v_{j-1}), \quad j = 0, 1, \cdots, \tag{4.4}$$

$$\partial_\tau v_j(s,0) - i\zeta(s)v_j(s,0) = 0, \qquad j = 0, 1, \cdots. \tag{4.5}$$

The prime denotes differentiation with respect to s and the functions F_j can be found from (2.1). In particular, $F_0 = 0$ and

$$F_1 = -2i\theta_0'v_0' - in'(s,0)v_0. \tag{4.6}$$

From (4.3) we see that $\theta_0(s)$ is just the optical length along B. Next we consider (4.4) with $j = 0$ and seek a solution which satisfies (4.5) and which vanishes at $\tau = \infty$ since we want solutions which are small away from B. The general solution satisfying these conditions is

$$v_0(s,\tau) = c(s)\,\mathrm{Ai}\,[(2n^2K)^{1/3}\tau - \rho_1]. \tag{4.7}$$

Here $n = n(s,0)$, $c(s)$ is arbitrary and $\rho_1(s)$ is a solution of (3.3). In order that (4.7) satisfy (4.4) we must have $\theta_1' = -(nK^2/2)^{1/3}\rho_1$. Thus without loss of generality,

$$\theta_1(s) = -\int_0^s [n(s',0)K^2(s',0)/2]^{1/3}\rho_1(s')\,ds'. \tag{4.8}$$

Since $c(s)$ is single-valued, as we shall show, while θ_0 and θ_1 are not, the condition that u given by (4.1) be single-valued is just (3.4).

To find v_1 we set $j = 1$ in (4.4) and (4.5) to obtain an inhomogeneous form of the problem solved by v_0. In order that this problem have a solution vanishing at $\tau = \infty$, the right side F_1 given by (4.6) must be orthogonal to v_0 which gives us

$$\int_0^\infty (2nv_0' + n'v_0)v_0 \, d\tau = 0. \tag{4.9}$$

When we insert (4.7) for v_0 into (4.9) we obtain a first order linear ordinary differential equation for the coefficient $c(s)$. The coefficients in this equation involve integrals of Airy functions which can be simplified by using the identities

$$2\int_0^\infty y \operatorname{Ai}(y-q)\operatorname{Ai}'(y-q)\,dy = -\int_0^\infty \operatorname{Ai}^2(y-q)\,dy, \tag{4.10}$$

$$2\int_0^\infty \operatorname{Ai}(y-q)\operatorname{Ai}'(y-q)\,dy = -\operatorname{Ai}^2(-q). \tag{4.11}$$

The solution of the simplified equation is, aside from a multiplicative constant,

$$c(s) = \{\beta[\rho_1(s)]\}^{-1/2}[K(s)/n(s,0)]^{1/6}. \tag{4.12}$$

Here $\beta(t)$ is defined by (3.11) and we see that $c(s)$ is a single-valued function of s.

With $c(s)$ given by (4.12), v_0 is completely determined and v_1 can be found because the solvability condition (4.9) is satisfied. The solution of (4.4) with $j = 1$ which satisfies (4.5) and vanishes at $\tau = \infty$ is

$$\begin{aligned} v_1 = c(s)\Bigg\{&\left[b(s) - \frac{i}{4}\left(\frac{2}{nK^2}\right)^{1/3}\left(\frac{(n^2K)_s}{3n^2K}(x^2 + 2\rho_1 x) - 2\rho_1' x\right)\right]\operatorname{Ai}(x) \\ &+ \frac{i}{2}\left(\frac{2}{nK^2}\right)^{1/3}\frac{\beta'(\rho_1)}{\beta(\rho_1)}\rho_1 \operatorname{Ai}'(x)\Bigg\}. \end{aligned} \tag{4.13}$$

Here $b(s)$ is arbitrary and x is defined by $x = [2n^2(s,0)K(s)]^{1/3}\tau - \rho_1(s)$. By considering the solvability condition for v_2 we can determine $b(s)$, but we shall not do so.

Upon using the value of θ_0, (4.8) for θ_1, (4.7) for v_0, (4.12) for c and (4.13) for v_1 in (4.1), we obtain

$$\begin{aligned} u(s,\tau) = {}& \{\beta[\rho_1(s)]\}^{-1/2}[K(s)/n(s,0)]^{1/6} \\ & \cdot \exp i\left[k\int_0^s n(s',0)\,ds' - k^{1/3}\int_0^s \{n(s',0)K^2(s')/2\}^{1/3}\rho_1(s')\,ds'\right] \\ & \cdot \Bigg[\Bigg\{1 + k^{-1/3}b(s) + i[4kn(s',0)K^2(s)]^{-1/3} \\ & \cdot \left[\frac{(n^2K)_s}{6n^2\mathrm{K}}(x^2 + 2\rho_1(s)x) - \rho_1'(s)x\right]\Bigg\}\operatorname{Ai}(x) \\ & - i[4kn(s,0)K^2]^{-1/3}\frac{\beta'(\rho_1)}{\beta(\rho_1)}\rho_1'(s)\operatorname{Ai}'(x)\Bigg] + O(k^{-2/3}). \end{aligned} \tag{4.14}$$

If we determined $b(s)$ and imposed the single-valuedness condition on (4.14), we would obtain (3.12). The leading term in (4.14) in the case $n(x) = 1$ and $\zeta = x$ was found by the present method by Matkowsky [7].

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