

How Einstein confirmed $E_0 = mc^2$

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The equivalence of mass m and rest-energy E_0 is one of the great discoveries of all time. Despite the current wisdom, Einstein did not derive this relation from first principles. Having conceived the idea in the summer of 1905 he spent more than 40 years trying to prove it. We briefly examine all of Einstein's conceptual demonstrations of $E_0 = mc^2$, focusing on their limitations and his awareness of their shortcomings. Although he repeatedly confirmed the efficacy of $E_0 = mc^2$, he never constructed a general proof. Leaving aside that it continues to be affirmed experimentally, a rigorous proof of the mass-energy equivalence is probably beyond the purview of the special theory. © 2011 American Association of Physics Teachers.

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I. INTRODUCTION

Einstein conceived the idea that mass and energy were equivalent in the summer of 1905 while working as a clerk (with only a trial appointment) at the Patent Office in Bern, Switzerland. This paper focuses on his life-long effort to prove that extraordinary insight. We will explore the motivations behind, and the limitations of, his surprisingly many derivations. To this day special relativity is burdened with two competing interpretations of the concepts of momentum, mass, and energy. As we will see, this issue influenced Einstein's later derivations of the mass-energy equivalence.

In the 19th century, a number of European physicists, including the leading theoretician of the time, Hendrik Lorentz, were working to establish that mass was, in whole or in part, electromagnetic.¹ There were primarily two competing theories, one by Lorentz, the other by Max Abraham. Both agreed that depending on the relative direction of its velocity and acceleration, an "electron" (a generic charged particle) moving through the aether could manifest both transverse and longitudinal speed-dependent mass components. There was even experimental evidence that seemed to confirm as much. Electromagnetic theory was then at the center of physics.

Einstein was aware of some of these exciting developments. He boldly entered the discussion, albeit in a novel way, when he added the last section to his groundbreaking June 1905 paper, "On the electrodynamics of moving bodies."² Without utilizing the aether or hypothesizing the physical characteristics of the electron, he employed his new theory, along with Maxwell's equations, and $F = ma$ (alas, an erroneous approach)³ to derive expressions for the transverse $\mu\gamma^2$ and longitudinal $\mu\gamma^3$ mass. Like most German-speaking scientists, he denoted mass by μ . We will simplify the notation and write the Lorentz factor as $\gamma = (1 - v^2/c^2)^{-1/2}$, where v is the particle's speed and c is the vacuum speed of light. Only his longitudinal mass agreed with Lorentz, but that didn't deter Einstein; besides he might not even have known there was a discrepancy at the time.⁴

Young Albert ended his June 1905 paper using his somewhat flawed results⁵ to obtain the correct expression for the relativistic kinetic energy

$$\text{KE} = \gamma mc^2 - mc^2 \quad (1)$$

of "a pointlike particle,"² where (in modern notation) m is mass. Although this derivation left much to be desired, it was a spectacular achievement.

His next discovery, the one Einstein later considered the greatest accomplishment of the special theory, was the equivalence of mass and energy. That revelation—more an epiphany than the result of rigorous analysis—came on the heels of the June paper. It was then that Einstein became aware of the potential significance of the quantity mc^2 . It is argued elsewhere⁵ that his realization that mc^2 was some sort of rest-energy (call it E_0 , though as yet he had no symbol for it) arose from a close reading of his formula, $\text{KE} = \gamma mc^2 - mc^2$.

The assistant examiner at the Patent Office did not derive the equation $E_0 = mc^2$, or anything like it, from first principles in 1905. Nor did he ever explicitly make it known how he came to believe that mass and energy were "identical." He said nothing about it in his June 1905 paper, yet less than three months later he produced his first demonstration of the mass-energy equivalence.

Without a mature relativistic dynamics, Einstein was compelled to work with Maxwell's equations and the Lorentz transformations applied to special situations that were contrived to be amenable to analysis. More often than not, these Gedanken experiments portrayed situations that are unattainable in the laboratory.

After Planck's derivation of relativistic momentum ($p = \gamma mv$) and his confirmation of the formula for relativistic kinetic energy,⁶ Einstein had more tools that he could bring to bear. Nonetheless, he was, time and again, thwarted by having to apply equations which were derived for point masses, to extended objects.

Despite hundreds of books and articles that attribute to Einstein the expression $E = mc^2$, where E is the total energy and m is relativistic (speed-dependent) mass, he never wrote that equation with those meanings for those terms. It is the equivalency between invariant (speed-independent) mass and rest-energy—not relativistic mass and total energy—that was Einstein's great contribution.⁷

Although he affirmed the correctness of the mass-energy equivalence in about 18 different presentations, he was not able to provide a conclusive general proof of this seminal hypothesis. Einstein seems to have been aware of the limitations of these efforts as witnessed by his continued offering of new derivations up to 1946. Even so, establishing ownership of the concept of the equivalence of mass and energy was of great importance to him. We will suggest that the

inability to prove $E_0 = mc^2$ once and for all was not a failing of Einstein's, but an epistemological constraint that he began to recognize as early as 1907.

II. THE SECOND RELATIVITY PAPER

Einstein came to his second paper on relativity (September 1905), "Does the inertia of a body depend upon its energy content?,"⁸ armed with new knowledge. The June 1905 article² contained expressions for the Doppler shift, and the energy of an electromagnetic plane wave as determined by a uniformly moving observer. He also, no doubt, suspected from the analysis leading to Eq. (1) that the energy of an object at rest equals mc^2 . Accordingly, it made sense to construct a Gedanken experiment around something that stayed motionless. We presume Einstein already knew the answer he was looking for before he devised the rather elaborate scheme he used to reveal it.

The treatment outlined here, in which the notation has been simplified, is true to the original derivation. Although it is routinely claimed that this analysis provided the definitive proof of $E = mc^2$, it did not. Moreover, the youthful savant was aware that this inquiry, which he astutely labeled a "special case," did not provide a complete proof of the mass-energy equivalence. Indeed, he later (1907) spoke of that equivalence as "an assumption of such extraordinary generality,"⁹ rather than a proven fact.

The September 1905 study⁸ began with the idea of a body at rest that subsequently emitted two identical electromagnetic plane-wave pulses in opposite directions. Each of these carried off an amount of energy $\Delta E/2$ (Einstein actually wrote it as $L/2$). He then created expressions for conservation of energy before and after the emission, as measured by two observers: one at rest with respect to the radiating body, the other moving with respect to it at speed v .

Because of the Doppler effect, an observer heading toward a source measures the wave blue-shifted (higher energy), whereas an observer heading away from a source measures it red-shifted (lower energy). Einstein had an equation for the energy of electromagnetic plane waves in the June 1905 paper.² From the moving frame, the energy increase for one pulse was greater than the energy decrease for the other. Consequently, the emitted energy ΔE in the rest frame becomes $\Delta E' = \gamma \Delta E$ in the moving frame.

Einstein formulated the difference between the total energy in each frame prior to emission. This difference was the initial kinetic energy KE_i of the object due to the relative motion of the two frames. Similarly, the difference between the total energy in each frame after emission was the final kinetic energy KE_f . Subtracting these two quantities yields

$$KE_i - KE_f = \Delta E(\gamma - 1). \quad (2)$$

Einstein included a constant C in each of the kinetic energy terms "which depends on the choice of the arbitrary additive constants of the energies...." This constant cancels out of Eq. (2).

In this imagined experiment, the emitting body was at rest in the original frame before and after radiating, and hence it did not change its speed in either frame. Because the body's kinetic energy, as determined by the moving observer, decreased [via Eq. (2)] as a result of the emission, its mass must also have decreased. (At that moment young Einstein was a believer, albeit a temporary one, in a speed-dependent mass and had to keep in mind that m was the object's mass

provided it moved slowly.) Knowing that $KE = \frac{1}{2}mv^2$ at low speeds, Einstein expanded the right side of Eq. (2) in powers of v/c requiring $v \ll c$, so that

$$KE_i - KE_f = \frac{1}{2} \frac{\Delta E}{c^2} v^2. \quad (3)$$

For a low-speed observer,

$$\Delta m = \Delta E/c^2. \quad (4)$$

This expression is independent of v and applies, as well, to observers at rest with respect to the body.

Einstein jumped from Eq. (3) directly to the conclusion: "If a body releases the energy ΔE in the form of radiation, its mass decreases by $\Delta E/V^2$." (In his early papers, Einstein used V , as had Lorentz, for the vacuum speed of light.) In other words, if a body at rest, having energy E_0 , loses radiant energy $\Delta E = \Delta E_0$, its mass decreases by Δm where

$$\Delta E_0 = \Delta mc^2. \quad (5)$$

However much Eq. (5) was implied, the patent clerk did not explicitly present it in this paper, nor did he write $E_0 = mc^2$, or $E = mc^2$, or even the equivalent of $\Delta m = \Delta E/c^2$.

Einstein stopped abruptly at Eq. (3) and only went so far as to conclude that a change in energy would be accompanied by a change in mass. He chose his words carefully: "The mass of a body is a measure of its energy content." Mass is a measure of energy content, but he left his readers to wonder if there might be other contributing factors as well.¹⁰ Nowhere did he write that mass and energy are "equivalent;" that would come later. At this time he could imagine that there might be some kind of quiescent matter that possessed residual inert mass even if all of its energy were somehow removed.¹¹ After all, light was an entity with energy and no mass; perhaps there was matter with mass and no energy.

This derivation came very early in the development of relativity, and the formal concept of "rest-energy" had not yet evolved, nor had E_0 been introduced to symbolize it. Unless Einstein was willing to assume that whenever $E_0 = 0$, $m = 0$ —that all of the mass of a material entity was equivalent to energy—he could not take the analysis any farther than he had in September 1905. Unfortunately, he did not overtly share his concerns about this issue with his readers.

Modern authors, knowing all about particle-antiparticle annihilation, tend to jump in on Einstein's behalf, concluding that $E_0 = mc^2$ follows so naturally from Eq. (5) that he must have produced it in 1905.¹² Einstein avoided that temptation, and it would not be until 1912 that he was writing statements like: "According to this conception, we would have to view a body with inertial mass m as an energy store of magnitude mc^2 (rest-energy of the body)."¹³ To attribute $E_0 = mc^2$ (or even worse, $E = mc^2$) (Ref. 12) to Einstein via the September 1905 paper⁸ is to do injustice to the gradual nature of his development of the concepts.

Having established the basis for Eq. (4), Einstein made two intuitive and seemingly innocent leaps that went beyond what he could prove. "Since obviously here it is inessential that the energy withdrawn from the body happens to turn into energy of radiation rather than some other kind of energy, we are led to the more general conclusion: The mass of a body is a measure of its energy content [*Energieinhalt*]...." The concept of rest-energy was still unformulated, and he was not even using the term. Without proof, Einstein guessed that the

loss of other forms of energy besides electromagnetic would result in an equivalent loss of mass. That is no small point, especially because many physicists at the time believed mass was entirely electromagnetic.¹ If it were, why should an object that loses heat, which is more mechanical than electromagnetic, lose mass?

An unspoken assumption was made early in the September 1905 paper that each emitted light pulse was not material, in that it carried energy but had no intrinsic mass. Throughout his work on the special theory, Einstein maintained that light was other than matter. As he put it in 1911, “[t]he comparison of light with other ‘stuff’ is not permissible.”¹⁴ Thus, Einstein avoided a more complicated calculation¹⁵ by conjecturing, without discussion or proof, that light was massless. But of course he could not prove such a thing, one way or the other.

Assuming the previous derivation is structurally flawless—there has been debate about that¹⁶—it is predicated on several assumptions. These seem innocuous today only because they appear to be right, inasmuch as a hundred years of experimentation have not produced any significant contradictions.

Finally, it should be noted that plane waves are a mathematical contrivance.¹⁷ No real extended body can actually radiate electromagnetic plane waves as required by this thought experiment. Naturally enough Einstein said nothing about the physical structure of the hypothetical emitting body that could perform such a feat. This omission likely troubled him because, as we will see, he returned to the issue seven years later with a more elaborate emitter.

What Einstein proved in September 1905 was that if an imaginary material body could somehow emit radiant energy ΔE_0 (which is itself massless) in the form of plane waves, that extraordinary object would diminish in mass by $\Delta E_0/c^2$. He did not prove that every form of energy was equivalent to mass; that, he simply proclaimed. Yet, he was quite aware that his analysis had limitations, and in 1907 he discussed the need for a more general approach.⁹

The fact that positive work done on an object, otherwise moving freely, causes an increase in kinetic energy is known as the work-energy theorem. Although strictly accurate for an idealized point mass, which has no inner structure (that is, no separate parts) and cannot purloin energy internally, the theorem is also applicable to the fictitious translating “rigid body,” because it too cannot appropriate energy internally. If we were to derive a whole new relativistic mechanics, concepts like the kinetic energy of a compound body (whose Newtonian representation is in terms of the motion of and with respect to its center of mass) would have to be reexamined.

The June 1905 derivation of relativistic kinetic energy was an application of the work-energy theorem to a pointlike particle. Einstein argued that his results were also applicable to “ponderable material points.”¹⁸ He then dauntlessly asserted that Eq. (1) “must be valid for ponderable masses.” How “ponderable masses” differ from ponderable material points was not revealed, but a reader could easily get the impression that Einstein was now considering objects such as cannonballs, which was quite a leap.

In the September 1905 paper, the “body” emitting light was observed from the moving frame to be in motion. Moreover it had to be an extended object if it was to radiate plane waves, and that raises issues about the motion of its center of mass. (Einstein usually worked with the essentially equivalent

concept of “center of gravity.”) If mass is speed-dependent—and in 1905 young Einstein still believed it was—the location of the center of mass becomes frame-dependent. Furthermore, because “[t]he mass of a body is a measure of its energy content,”⁸ if the extended body possesses rapidly moving parts, the associated internal energy could significantly contribute to its mass, thereby affecting its translational kinetic energy and perhaps causing the formula for it to differ from that of a point mass. Some of these concerns probably occupied Einstein’s attention because his next paper on mass-energy was entitled “The principle of conservation of motion of the center of gravity and the inertia of energy” (May 1906).¹⁹

III. THE INERTIA OF ENERGY

The patent clerk was not entirely pleased with his September 1905 derivation of the mass-energy equivalence.⁹ Coming at it from his hallmark Gedanken-experiment approach introduced all sorts of limitations, but in the early days before the theory had matured that seemed the best way to go. Still, each thought experiment set up an artificial landscape of assumptions that robbed the conclusions, however brilliantly obtained, of the sought-after generality.

His next effort (May 1906) (Ref. 19) contained two distinctly different analyses. The first constituted Sec. 1 entitled “A special case,” which says a lot. It undertook to create a derivation based on mechanics. Einstein asked his readers to imagine a hollow tube floating in space, wherein a pulse of light transported energy from one end to the other. This energy was absorbed and returned by a massless carrier body, which brought the system back to its original state.

The whole setup was unrealistic: there are no massless absorbing bodies. Although Einstein did not elaborate the point, his treatment relied on the tube being rigid so the two ends moved (recoiled) together, and that too was problematic. This conceptual experiment has been discussed elsewhere,²⁰ and we need not repeat the analysis. Suffice it to say, the ongoing forward motion of the center of mass without an external force, an apparent “contradiction with the principles of mechanics disappears” provided that the energy ΔE_0 imparted to the carrier object at rest “possesses the inertia” $\Delta E_0/c^2$, and by inertia he meant mass.²¹ The technical expert, recently promoted to second class, had affirmed the results of his September 1905 paper, but again he cautiously stopped short of explicitly writing the equivalent of Eq. (5). Although fascinating, this scenario was too unrealistic to stand as an unassailable proof.

His second mass-energy offering in that same May 1906 paper relied on Maxwell’s equations, a formalism over which Einstein displayed complete mastery. In this case, his imagined setup involved a number of moving “discrete [charged] material points.” These had different masses, moved randomly, and were mutually interacting. Einstein dealt with the problem of internal relativistic motions rather glibly by assuming that “all of the [system’s particle] velocities are so small that terms of second order may be neglected....”¹⁹

Intrepidly treating the field energy as a mass distribution, in accord with the “hypothesis on the dependence of the mass on energy,” Einstein included it in the determination of the center of gravity. From that he concluded that “at least in first [nonrelativistic] approximation, the principle of conservation of the motion of the center of gravity”¹⁹ holds, provided we ascribe inertial mass to energy.

The derivation was specialized, approximate, and perforce impotent as a definitive proof. For Einstein, as of May 1906, the mass-energy equivalence was, in his words, a “hypothesis.”¹⁹

A. The complete theoretical system

While still at the Patent Office, the youthful genius addressed the limitations of his special theory in a comment²² responding to a note by his friend Paul Ehrenfest. The two postulates of relativity do not, in and of themselves, represent a complete theoretical system “in which the individual laws are implicitly contained and from which they can be found by deduction alone....” This point is tremendously significant; the special theory is predicated on two postulates and there is no reason to believe that all the conclusions that spring from that theory can be proven true by “deduction alone.”²² He went on: “For the time being we only have the kinematics of parallel translation [no rotation] and an expression for the kinetic energy of a body in parallel translation....” Although derived for point masses, Einstein claimed that Eq. (1) applied to “bodies.” Furthermore, “both the dynamics and the kinematics of a rigid body have at present to be considered as unknown....”²² Without a “complete system,” attempts at establishing the mass-energy relation would by necessity involve using special situations along with appended hypotheses.

B. Two remarkable 1907 papers

Einstein was not content with his earlier treatments of mass-energy, and returned to the subject in his next paper, “On the inertia of energy required by the relativity principle” (May 1907).²³ This discussion began with a recapitulation of the September 1905 analysis. Einstein was aware of the special nature of that exposition and sought a more widely applicable approach. He also knew that “[t]he general answer to the question posed is not yet possible.... Rather, we must limit ourselves to the special cases that we can handle at present without arbitrariness from the standpoint of relativistic electrodynamics.”²³ For the time being he confined himself to thought experiments in electrodynamics, two of which were contained in the May 1907 paper.

In Sec. 1, Einstein asked his readers to “consider a rigid body that is moving in uniform translation” one that is “rigidly electrified.”²³ The body was subsequently acted on by an electric field as it traveled in the direction of the field. Clearly, there must be some sort of additional applied opposing force or the body would accelerate. After an elaborate analysis, Einstein remarked: “We thus get the following strange result.” “[T]hese forces—observed from a coordinate system that is moving relative to the body—perform an amount of work ΔE on the body....” Using electromagnetic theory, he derived an expression for ΔE that depends on the velocity and the forces. He concluded that this additional energy must result in an increased kinetic energy beyond that of the same body without the field, moving at the same velocity. That could only mean there was an increase in the body’s mass, but he left this ultimate surmise to his readers. He stopped at the point of asserting that there was an increased kinetic energy, not even mentioning mass.

His conclusion was reached without elaborating on how the field might do work on a rigid body that could not deform. Besides, he probably already knew that “[a]ccording to

the theory of relativity, a rigid body cannot exist at all.”²⁴ Interestingly, there was no mention of potential energy. Nor did he talk about the necessity for there to be a charged object causing the field in the first place, and hence any change in mass it might have experienced. Apart from that, having a three-dimensional uniform distribution of rigidly fixed charges was unrealistic. So, all in all, this argument, however masterfully developed, was not the sought-after ultimate proof of $E_0=mc^2$.

The second May 1907 attempt again utilized a “rigid, rigidly electrified body in uniform translation....”²³ In this case, there was no external field, and Einstein considered the internal forces between charges that produced an electrostatic energy increase. He derived an expression for the kinetic energy of a charged rigid body and compared it to the kinetic energy of an uncharged body moving at the same speed. He found “that the electrostatically charged body has an inertial mass that exceeds that of the uncharged body by the electrostatic energy divided by the square of the velocity of light. The law of the inertia of energy is thus confirmed by our result in the special case considered,” and it was a special case that could not actually exist.

In the last section of this May 1907 paper,²³ Einstein, seeking “the simplest expression for E ,” the total energy, set $E_0=mc^2$ for the very first time.²⁵ Here, E_0 is the energy “for the stationary mass-point;” the term rest-energy was still not part of his lexicon.

At the request of Johannes Stark (December 1907), Einstein wrote a lengthy review article entitled, “On the relativity principle and the conclusions drawn from it.”²⁶ In Sec. 11, he considered a moving charged system of inertial mass μ . The system was composed of separate parts that were in motion with respect to each other, and was exposed to “the action of electric and magnetic forces.” He handled the daunting problem of an extended entity composed of moving parts by again simply requiring that the motions of those parts could be neglected. The system behaved like a point mass M provided $M=\mu+(E_0/c^2)$, where E_0 was the “energy content” as determined by a comoving observer.²⁷ It is here that Einstein explicitly combined the energy associated with the mass of the system’s component parts μc^2 with its traditional energy content to create the new concept of rest-energy, although he didn’t give the result that name. He went further: “It seems far more natural to consider any inertial mass as a reserve of energy.”²⁶ In so doing, he essentially hypothesized that when $E_0=0$, $m=0$, or as he put it much later, “one can stipulate that E_0 should vanish together with m ” (1935).²⁸ That was the conceptual leap he did not take publicly in September 1905.

Limited to only electrodynamics, and systems with negligible internal motions, this derivation was hardly a universal proof. Nonetheless, it provided one more impetus to accept the relation $E_0=mc^2$.

Section 14 in the same review article²⁶ contained two “examples” of the efficacy of the mass-energy principle. The first involved a massless cavity filled with electromagnetic radiation, and the second considered a massless charged body. Both analyses, which were completely artificial though quite elegant, led to the conclusion that the systems behaved as if each possessed a mass E_0/c^2 .

In the last section of this 1907 article, Einstein went beyond special relativity to determine how gravity might contribute to mass-energy. He had already proposed the *equiva-*

lence principle earlier in the paper, that is, “the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system.”²⁶ The Lorentz transformation for time contained the speed of the moving system, and if this system was accelerating, it would influence its speed and hence its time. It followed that gravity would affect the electromagnetic phenomena unfolding in time. An analysis of the electromagnetic energy supplied to a distribution of matter in a gravitational field showed that if the equivalence principle were correct, the mass-energy principle would apply to gravitational as well as inertial mass. This brilliant discussion confirmed the efficacy of the mass-energy equivalence, but it could not prove either principle.

C. 1909–1912: Three more demonstrations

In July 1909 Einstein submitted his resignation at the Patent Office to take a position as associate professor of theoretical physics at the University of Zürich. He again turned his attention to the mass-energy principle when (September 1909) he prepared a paper “On the development of our views concerning the nature and constitution of radiation.”²⁹ In this study, he essentially repeated the original derivation of September 1905, no doubt because it involved light pulses. This later, more concise, version omitted any discussion of an “additive constant” in the energy expressions. Einstein concluded: “The energy emitted must be reckoned as part of the body’s mass.” The derivation again relied on plane waves and is no less speculative than its 1905 precursor.

The introduction of the equivalence principle in 1907 marked both the beginning of a long effort to formulate a theory of gravity and a shift in emphasis away from the special theory. Einstein’s June 1911 paper, “On the influence of gravitation on the propagation of light,”³⁰ contained another treatment of mass-energy in relation to gravity. This time he envisioned two optical instruments, I_1 on the z -axis at a distance l beneath device I_2 . Both are in a gravitational field in the $-z$ -direction. By the equivalence principle, this situation was identical to a gravity-free system accelerating at g in the $+z$ -direction. A pulse of radiant energy E_2 was emitted by I_2 downward toward I_1 . Because of the acceleration and the resulting Doppler shift, the pulse arrived at I_1 with an energy $E_1 > E_2$. By using the appropriate energy equation from the June 1905 paper,² and introducing several approximations, Einstein showed that the radiant energy increase ($E_1 - E_2$) equals the potential energy decrease experienced by a mass equivalent to E_2/c^2 descending a distance l in a gravitational field. The mass-energy principle was again consistent with the equivalence principle, which was another affirmation, though not a general proof of either principle.

Einstein was a full professor at the German University in Prague in 1911 when he was asked to write, for inclusion in a book, a treatise, “On the special theory of relativity.” Because of the First World War, the book’s publication was delayed and his contribution was ultimately withdrawn. The handwritten manuscript, which began in 1912, still exists. Section 14, “The inertia of energy,”³¹ contains another thought experiment. It imagined a “rectangular parallelepiped-shaped plate” floating freely in space that simultaneously radiates two identical electromagnetic wave trains in opposite directions. Without saying as much, Einstein treated these hypothetical plane-wave pulses as if they were monochromatic. He went on to formulate expressions for their energy densities as measured by a uniformly mov-

ing observer in terms of the energy density determined by a comoving observer. Expressions for the energy of the two wave trains followed. Like the unspecified body in the related September 1905 derivation,⁸ this “plate” could only radiate waves that approximate plane waves, because plane waves are a mathematical idealization.¹⁷ Nor could real pulses be monochromatic regardless of their finite length. Einstein ignored these limitations on the analysis entirely.

He then asserted that if we “consider exclusively the translational motion of the plate as a whole, we surely can view the plate as a material point of a certain mass M .” This assumption was necessary because the equations for relativistic energy and momentum were for point masses. Equating the plate to a material point brings to mind the conundrum of the work-energy theorem, which haunts many of these derivations, and again raises questions. Was the plate supposed to be rigid? If so, it offends relativity.³² If not, before it radiated, were any of its internal parts in rapid motion so that it might not translate as a Newtonian body would (that is, as a point mass at the center of mass)?

Once we are willing to accept the point-mass equivalence of the plate and the notion that it radiates plane waves, its energy before and after emission follows from $E = \gamma Mc^2$. “[T]hus, the inertial mass of the plate increases [or decreases] by $\Delta E'/c^2$ when its rest-energy (the energy for a comoving observer) experiences an increase [or decrease] of $\Delta E'$.”³¹ This outstanding analysis was nevertheless restricted to electromagnetism, somewhat unrealistic, and perforce wanting as the elusive universal proof.

D. Space-time

Minkowski, a mathematics professor at Göttingen, presented his four-dimensional space-time formulation of relativity in 1907–1909.³³ His analysis began with a collection of familiar ideas from electromagnetic theory. There was one equation corresponding to electromagnetic energy density, three expressions for $1/c$ times each of the components of the electromagnetic energy flux, three expressions for c times each component of the electromagnetic momentum density, and six components of what Maxwell referred to as aether stress. The idea was to describe the amount of energy per unit volume, yet when determined by a translating observer, whatever was initially static would now be moving and perforce there would exist energy fluxes and momentum flows.

Minkowski realized that this set of quantities could constitute a single mathematical entity called a *tensor*. The resulting 16-component expression for the energy-momentum density came to be known as the stress-energy (also known as energy-momentum) tensor. Much of the analysis that followed was a reformulation of Maxwell’s equations in which the four-dimensional force density was equal to the negative of the divergence of the electromagnetic energy-momentum tensor.

Laue first recognized the central role that might be played by the energy-momentum tensor in the relativistic dynamics of extended bodies.³⁴ He audaciously generalized this symmetric tensor, claiming that it could embrace all forms of energy other than electromagnetic (for example, mechanical and thermal). By summing the tensor representations of each such contribution, he argued that one could arrive at an all-encompassing energy-momentum tensor $T_{\mu\nu}$, which specified the distribution and flow of energy and momentum in a region of space-time.

To simplify the analysis, Laue limited himself to an arbitrary closed system provided it was “static,” a restriction that later proved unnecessary.³⁵ In Laue’s treatment, there could be no flow of energy, in which case the stress-energy tensor in the rest frame was time independent. In the zero-momentum frame, he showed (written here in modern notation) that “a static closed system in uniform motion behaves like a point mass having the rest-mass $m_0 = E_0/c^2$.”³⁶

Einstein was well aware of Minkowski’s work though he was initially unimpressed. Early on, he said it was “*überflüssige Gelehrsamkeit*,” superfluous learnedness,³⁷ but he was soon moved to change his mind by the accomplishments of “Abraham, Planck, and Laue.”¹³ Einstein’s unpublished 1912 treatise embraced “Minkowski’s four-dimensional theory.”¹³ A vector in this space-time continuum, a 4-vector, is a special mathematical device which under a Lorentz transformation changes its components but not its length. It is therefore invariant from one inertial frame to another. Because all inertial observers must experience the same laws of physics, valid physical quantities should be expressible in terms of 4-vectors.

Einstein (1912) attempted his own proof of the mass-energy equivalence using the stress-energy tensor,¹³ as had Laue. He began with a more general approach considering the action of a 4-vector whose first three spatial components were force densities, and whose last temporal component was proportional to energy density per unit time. Under the influence of this force-density 4-vector, a “system”—he did not explicitly tell us what a system was—would experience a divergence of its energy-momentum tensor $T_{\mu\nu}$. Because force, momentum, and energy were all formulated as densities, integrating this relation over an appropriate volume of the four-dimensional manifold led to expressions corresponding to conservation of momentum and energy in integral form.

Einstein established that the resulting changes in the momentum and energy transformed as a 4-vector. He went on to claim, without proof, that if these changes constituted a 4-vector, the quantities themselves formed a 4-vector. Although his intuition was correct, that unsubstantiated assumption constituted a fatal flaw in the argument. Einstein, aware that he was on thin ice, commented in a footnote: “To be sure this is not rigorous, because additive constants might be present that do not have the character of a vector; but this seems so artificial that we will not dwell on this possibility at all.”³⁸

The $T_{\mu\nu}$ tensor was so constructed that upon integration of its divergence, the components comprising its fourth column corresponded to (what Einstein claimed was) a 4-vector made up of one energy and three momentum components—the functional forms of which were not then explicitly specified. The simplest situation to treat was that of the zero-momentum frame in which the 3-velocity was zero and the energy-momentum 4-vector became, in modern notation, $(0, 0, 0, iE_0/c)$. This representation assumed that the system, whatever it was, had energy ($E = E_0$) when at rest.

Earlier in a section treating “ponderable bodies,” Einstein provided, without explication, a dimensionless variant of the Minkowski velocity 4-vector, which he called G_μ . Other than referring to it as relating to “matter,” he did not explicitly tell us that it was an adaptation of the velocity 4-vector of a point mass. In the zero-momentum frame, it became $(0, 0, 0, i)$. The two 4-vectors in this special reference frame differed

only by a multiplicative constant E_0/c . He therefore concluded that “the latter four-vector $[E_0(G_\mu)/c]$ is identical to the four-vector of the momentum and energy for any other arbitrary system as well. By equating, we obtain the following equations:”¹³ $p_i = \gamma(E_0/c^2)v_i$, where $i = x, y, z$, and $E = \gamma E_0$. “[T]o a first degree of approximation,” $p_i = (E_0/c^2)v_i$ and $E = E_0 + (E_0/c^2)v^2/2$. “From these equations it follows that (E_0/c^2) plays the role of the total mass M of the system in the sense of ordinary mechanics. The energy and mass (of a closed system) differ from each other only by a universal factor, and thus are completely equivalent.”¹³

We might well ask why Einstein limited his (already incomplete) derivation in the last step by apparently expanding the energy and momenta each in a power series, dropping terms higher than v^2/c^2 . It should be remembered that there were already two schools of thought concerning relativistic dynamics. One embraced speed-dependent relativistic mass (which for the sake of clarity we shall denote as m_R), and the other agreed with Einstein that mass m was a speed-independent invariant. The former group took momentum to be $\mathbf{p} = m_R \mathbf{v}$ and total energy to be $E = m_R c^2$. The latter group maintained that momentum is $\mathbf{p} = \gamma m \mathbf{v}$ and the total energy is $E = \gamma m c^2$. Recall that the functional forms of the momenta and energy in the stress-energy tensor were not explicitly specified; Einstein produced the expressions $p_i = \gamma(E_0/c^2)v_i$ and $E = \gamma E_0$. For him these equations meant that invariant mass was $m = E_0/c^2$, but he was no doubt aware that if momentum were taken to be $\mathbf{p} = m_R \mathbf{v}$ he had just as well shown that $m_R = \gamma E_0/c^2$. There was no way to distinguish between these two possibilities, and he simply set $\gamma = 1$ without discussion, limiting the effort “to a first degree of approximation.” He would return to this issue of the uniqueness of the momentum and energy expressions some 20 years later in his penultimate attempt to prove the mass-energy equivalence.

In a response to a question on general relativity, Einstein (December 1913) wrote an exposition in *Physikalische Zeitschrift* that contained a version of his incomplete 4-vector argument.³⁹ A month later, he submitted a paper to the same journal on the general theory that included a similar treatment.⁴⁰ A brief discussion of these matters also appeared in Einstein’s handwritten “Lecture notes for course on relativity” at the University of Berlin, winter semester 1914–1915.⁴¹ He was apparently confident in the overall correctness of his analysis. Sometime after 1 January 1914, he wrote to his friend Michele Besso, “I did prove rigorously that the gravitational as well as the inertial mass of closed systems are determined by the total energy of the system, including the gravitational energy.”⁴²

Felix Klein, a well-known mathematician already retired from Göttingen, corresponded by mail⁴³ with Einstein frequently during 1918. In a June 1 postcard, Klein took exception to Einstein’s assertion that his analysis rigorously led to the energy-momentum 4-vector.⁴⁴ Two days later, Einstein wrote back defending his thesis, but he did not provide a solid proof.⁴⁵ In July, Klein contacted Einstein to tell him that he had found a solution to the energy-momentum 4-vector dilemma that followed the same route that Einstein had outlined in an earlier (June 9) letter.⁴⁶ That year, Klein, guided by his discussions with Einstein, published an extension of Laue’s analysis that applied to closed systems,⁴⁷ static or otherwise.⁴⁸

What Laue, Klein, and Einstein showed was that if the $T_{\mu\nu}$ tensor could be generalized to all interactions, as they as-

serted without any proof, then $E_0=mc^2$ followed and moreover was applicable to “systems” as well as to point masses. One might easily conclude that their work did more to establish the efficacy of the tensor approach than it did to prove the universality of the mass-energy equivalence. Einstein himself (1912) used “[t]he general validity of the conservation laws and of the law of the inertia of energy” to “ascribe a general significance” to his tensor equations “even though they were obtained in a very special case,” that is, via only electromagnetic field considerations.⁴⁹

Einstein produced a more instructive version of his incomplete tensor treatment in the Stafford Lectures delivered at Princeton during a visit in May 1921. These lectures became the basis of a popular book, *The Meaning of Relativity*,⁵⁰ which still left the 4-vector nature of the energy-momentum of a closed system in space-time explicitly unproven.

E. The penultimate derivation—1934

Einstein immigrated to the United States in 1933, settling in Princeton. A year later, he was invited by the American Mathematical Society (AMS) to give the Gibbs Lecture at the Carnegie Institute of Technology. His talk, “An elementary proof of the theorem concerning the equivalence of mass and energy,” was published in the *Bulletin of the AMS* (1935).^{28,51}

The new “proof” was created to utilize only mechanics, avoiding electromagnetic theory altogether. There were good reasons for doing that. Einstein had long believed that Maxwell’s theory was suspect, which was a notion underscored by the success of the photon concept. Moreover, he maintained from the beginning that the mass-energy equivalence was applicable to all interactions. As we have seen, Einstein’s stress-energy-tensor derivation was not wholly satisfactory. He was still wrestling with questions concerning the uniqueness of the expressions for relativistic energy and momentum.

Twenty-five or so years earlier, Lewis and Tolman,⁵² in their derivation of “relativistic mass,” had introduced the idea of using collisions between particles to ground their treatment in mechanics. Ironically, Einstein, who rejected the concept of relativistic mass, seems to have been inspired by their approach, which by 1934 had found widespread acceptance.⁵³ He now took conservation of energy and momentum to be fundamental, along with the two postulates of special relativity (which are the determinants of the Lorentz transformations). Because this explication has been discussed elsewhere,⁵⁴ we need only to outline the development to make the points that are relevant here.

Einstein began with Minkowski’s differential line element $ds^2=dt^2-dx^2-dy^2-dz^2$, the fundamental invariant.²⁸ This quantity immediately led to the velocity 4-vector. Multiplying that by the invariant mass m of a material particle, “[w]e obtain a vector connected with the motion of the latter....” That invariant space-time vector is the well-known energy-momentum 4-vector, but Einstein deliberately avoided calling it that. With his previous derivation no doubt in mind, and with the nature of mass and momentum still controversial, he was being circumspect. In simplified notation using SI units, this 4-vector comprised three momentum terms ($p_i=\gamma mu_i$ where $i=x,y,z$, and the u_i are the 3-velocity components of the particle) and one energy term ($E=\gamma mc^2$). It was “natural to give” γmc^2 the meaning of energy, from which it followed that at zero speed, $E_0=mc^2$, and further-

more that $KE=mc^2(\gamma-1)$. Nonetheless, Einstein asserted that we really have not proven as much, “since in no way is it shown that this impulse [that is, momentum] satisfies the impulse-principle [that is, conservation of momentum] and this energy the energy-principle [that is, conservation of energy]....”⁵⁵

The goal of the derivation was to use the conservation laws on systems of colliding particles connected by Lorentz transformations, and thereby determine if the foregoing kinetic energy and momentum expressions were consistent with such an analysis. Correct representations must reduce to zero at zero speed, but there are many possible formulations that do so. Being consistent with the conservation laws is a more definitive criterion.

Accordingly, Einstein constructed general expressions for the relativistic momentum and energy that could subsequently be particularized by the demands of the conservation laws. He proposed (written here in modern notation) that we start with

$$p_i = mu_i F(u) \quad \text{where } i = x, y, z \quad \text{and} \quad E = E_0 + mG(u). \quad (6)$$

$F(u)$ and $G(u)$ are even continuous functions of the speed u that vanish when $u=0$ and are thus independent of the direction of motion. He assumed that both expressions contained “the same mass-constant m ”—remember that Einstein abjured relativistic mass. Given that $E=E_0+KE$, it follows that $KE=mG(u)$. To determine $F(u)$ and $G(u)$, he considered the case of a perfectly elastic collision between two identical “mass-points.”

Imagine a coordinate system S , wherein two mass-points are traveling toward each other. One can find a zero-momentum frame S' (moving at v with respect to S) wherein the particles—Einstein called them a *particle-pair*—have velocities that are equal in magnitude and opposite in direction. Because momentum must be conserved, the sum of their velocities in S' has to remain unchanged by the elastic collision. No matter what the form of $F(u)$ is, the speeds of the two particles remain equal after the collision. From energy conservation, the same is true for $G(u)$.

By applying the Lorentz transformations, Einstein arrived at two mathematical statements equating speeds before and after the collision. He called these expressions “conservation equations” because except for a multiplicative factor of m , one of them looked exactly like conservation of p_i , and the other looked exactly like conservation of KE. He concluded, “we shall have to regard γmu_i as the impulse [that is, momentum] and $mc^2(\gamma-1)$ as the kinetic energy of the particle.”⁵⁶

“We now turn to the proof of the assertion that the mass is equal to the *rest-energy*.”⁵⁷ Accordingly, Einstein imagined colliding “material points” that are not elastic. He employed “points” because he had just derived $KE=mc^2(\gamma-1)$ for point masses. Nonetheless, in an inelastic collision, some energy is transferred to internal parts of the colliding bodies, and though total energy is conserved, KE is not.

Without defining terms, Einstein used “material point,” “mass-point,” and “particle” interchangeably.⁵⁸ A structureless point mass may arguably be conceivable, but here he conjured up “material points that are not *elastic*.” Yet, a point particle with internal structure is an oxymoron. That is the old conundrum of particle versus a system of particles. Ein-

stein didn't satisfactorily address this problem in his previous thought experiments, and avoided it completely here.

A particle-pair with respect to S' has precollision velocities that are equal in magnitude and opposite in direction. Because momentum must be conserved, their final velocities in S' "must be equal and opposite."²⁸ Einstein then equated the total energy of the particles [$E=E_0+KE=E_0+mc^2(\gamma-1)$] before and after the collision. With $c=1$, the impact produced a change in both the rest-energy and mass of the particles such that $\Delta E_0=\Delta m$.⁵⁹ Because rest-energy, "from the nature of the concept, is determined only to within an additive constant,"⁶⁰ we can stipulate that E_0 should vanish together with m . Then we have simply $E_0=m$.²⁸ Again, Einstein is guessing, although experiments will confirm his intuition.

To generalize his conclusion, Einstein boldly maintained that "[e]very system can be looked upon as a material point as long as we consider no processes other than changes in its translation velocity as a whole."²⁸ But he never proved as much for systems with fast-moving internal components. Furthermore, a self-propelled system (for example, a runner or an automobile) cannot "be looked upon as a material point."²⁸ When such a system accelerates, there is ordinarily no external propelling force that does work on it, as there must be for a material point. Alas, the inelastic material point that Einstein required is a fabrication only approximated in reality.

F. Last attempt—1946

"An elementary derivation of the equivalence of mass and energy"⁶¹ appeared in 1946 and was Einstein's last confirmation of $E_0=mc^2$. The text was short, deceptively simple, and explicitly limited. Remarkably, the treatment "does not presume the formal machinery of the theory [of relativity], but uses only three previously known laws:"⁶¹ conservation of momentum, the equation for radiation pressure, and the expression for stellar aberration of light.

Like his September 1905 thought experiment, it involved "a body" at rest in a frame S , and two pulses of light ("complexes of radiation") each of energy $E/2$. In this imagined experiment, these pulses were absorbed rather than emitted, and Einstein focused on conservation of momentum instead of energy. The pulses, traveling in opposite directions, approached the body along the x -axis in S , and we are presumably to assume they arrived simultaneously, which is no mean feat. This process was viewed by an inertial observer moving in the negative z -direction at speed v . Einstein compared the total momentum of the system before and after absorption, as observed from the moving frame S' . He introduced several approximations which required the speed of the observer to be small compared to c . For example, he took the momentum of the body in the moving frame to be mv "according to classical mechanics," rather than γmv , which would have injected relativity into the analysis.

Einstein established that for momentum to be conserved, the body's mass would have to increase by $\Delta m=E/c^2$. Here, E is a specific energy increase. Inasmuch as the object remained at rest in S , we can let $E=\Delta E_0$ and write $\Delta m=\Delta E_0/c^2$. Then because "energy according to the usual definition leaves an additive constant free, we may so choose the latter that" $E_0=mc^2$. In other words, we may conjecture that when $E_0=0$, $m=0$. Incidentally, even though it is clear that he was considering rest-energy and invariant mass, and

moreover that E was a specific amount of energy and not the total energy, the last expression he wrote in this article was " $E=Mc^2$." In 1946, that equation was already an icon of the "atomic age" and even Einstein would occasionally write it, rather than the more explicit $E_0=mc^2$, so as not to disappoint his readers.³

Einstein's complexes of radiation, which he never described in any detail,⁶² traveled along the x -axis. Classically, they could do that only if they were unrealistically assumed to have planar wavefronts.¹⁷ On arrival, these complexes of radiation were completely absorbed by the finite-sized body. In that respect, they behaved more like narrow pulses of photons than electromagnetic plane waves of infinite extent. Yet, stellar aberration, which he utilized in the analysis, assumes incoming classical plane waves. Einstein didn't address this seeming inconsistency.

This ingenious imaginary experiment, his final affirmation of what he was now calling the "law of equivalence,"⁶¹ was nonetheless approximate, limited to radiant energy, a tad impractical, and no more compelling or universal than his very first proof, but it came after three atomic bombs had inescapably made the point.

IV. CONCLUSIONS

Einstein produced about 18 virtuoso derivations and demonstrations all aimed at establishing the mass-energy principle. We have shown that although each of them gave evidence for the applicability of $E_0=mc^2$ to a particular set of circumstances, no one derivation, or collection of them taken together, succeeded in providing a definitive proof of its complete generality. That should not be surprising because the same situation occurs, for example, with $F=ma$, which is a different kind of relation than $E_0=mc^2$. Even so, 300 years of successful theoretical work have not proven the correctness of $F=ma$. Indeed, relativity showed that this expression, one of the bedrocks of classical mechanics, holds only approximately.

The fact that Einstein continued to create demonstrations of the efficacy of $E_0=mc^2$ up to 1946 tells us that he knew the definitive proof had not been accomplished. That aside, countless experiments from Cockcroft and Walton⁶³ to Rainville *et al.*⁶⁴ have, with increasing accuracy, confirmed that $E_0=mc^2$ is one of the greatest insights of the 20th century.

For several compelling reasons, many physicists have come to accept that mass is invariant,⁶⁵ even though there is no proof—theoretical or experimental—that it is. If mass is speed-independent, then $E_0=mc^2$ and only the internal energy, the rest-energy, of an object determines its mass. However, if mass is relativistic, then $E=m_R c^2$ and the kinetic energy of a system moving as a whole also contributes to its mass. A proof of either $E_0=mc^2$ or $E=m_R c^2$ is therefore inextricably linked to a determination of the fundamental nature of mass and is beyond the purview of the special theory.⁶⁶ That conclusion is supported by the fact that relativity was introduced over a hundred years ago and in all that time no such proof has been forthcoming.

That is ironic because it was Kurt Gödel, a close friend of Einstein's, who showed that it is impossible to prove every true statement contained in a mathematical system using only the conceptual apparatus of that system. It is fascinating to speculate that the youthful dapper logician and the aging

rumpled physicist might have discussed the inability to prove $E_0=mc^2$ on one of their long morning walks to the Institute in Princeton.

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¹For general references, see Russell McCormmach, "H. A. Lorentz and the electromagnetic view of nature," *Isis* **61**(4), 459–497 (1970); Christa Jungnickel and Russell McCormmach, *Intellectual Mastery of Nature* (University of Chicago Press, Chicago, 1986), Vol. 2; Stanley Goldberg, "The Lorentz theory of electrons and Einstein's theory of relativity," *Am. J. Phys.* **37**(10), 982–994 (1969); Michel Janssen and Matthew Mecklenburg, "From classical to relativistic mechanics: Electromagnetic models of the electron," in *Interactions: Mathematics, Physics, and Philosophy*, edited by V. F. Hendricks *et al.* (Springer, Dordrecht, 2007), pp. 65–134; Max Jammer, *Concepts of Mass in Classical and Modern Physics* (Dover, New York, 1997); Edmund Whittaker, *A History of the Theories of Aether and Electricity* (Dover, New York, 1989).

²*The Collected Papers of Albert Einstein, The Swiss Years: Writings, 1900–1909*, translated by Anna Beck (Princeton U. P., Princeton, 1989), Vol. 2, pp. 140–171.

³Eugene Hecht, "Einstein on mass and energy," *Am. J. Phys.* **77**(9), 799–806 (2009).

⁴A. I. Miller, *Albert Einstein's Special Theory of Relativity* (Addison-Wesley, Reading, MA, 1981), p. 87.

⁵Eugene Hecht, "How Einstein discovered $E_0=mc^2$," *Phys. Teach.* (to be published).

⁶Max Planck, "Das Prinzip der Relativität und die Grundgleichungen der Mechanik," *Verh. Dtsch. Phys. Ges.* **8**, 136–141 (1906).

⁷Eugene Hecht, "Einstein never approved of relativistic mass," *Phys. Teach.* **47**(6), 336–341 (2009); Also see W. L. Fadner, "Did Einstein really discover ' $E=mc^2$ '?," *Am. J. Phys.* **56**(2), 114–122 (1988); L. B. Okun, "The Einstein formula: $E=mc^2$. Isn't the Lord laughing?," *Physics-Uspekhi* **51**(5), 513–527 (2008).

⁸Reference 2, pp. 172–174, "Does the inertia of a body depend upon its energy content?" (September 1905).

⁹Reference 2, p. 238, "On the inertia of energy required by the relativity principle" (May 1907).

¹⁰John Stachel, editor and annotator of the German edition, *The Collected Papers of Albert Einstein, Vol. 2, The Swiss Years: Writings, 1900–1909* (Princeton U. P., Princeton, NJ, 1989), p. 428, interpreted Einstein's remark to mean "that the inertia or mass of a body depends on, or is a measure of, its energy content"; This interpretation is in accord with Einstein's 1910 comment: "the mass of any arbitrary body depends on the quantity of energy it contains," *The Collected Papers of Albert Einstein, The Swiss Years: Writings, 1909–1911*, translated by Anna Beck (Princeton U. P., Princeton, NJ, 1993), Vol. 3, p. 142.

¹¹Roberto Torretti, *Relativity and Geometry* (Pergamon, New York, 1983), p. 112. For more discussions of this point, see Ref. 5.

¹²Abraham Pais, *Subtle is the Lord* (Oxford U. P., Oxford, 1982). In a tabular list on p. 522 of this fine biography, the author writes: "1905 September 27. The second paper on special relativity theory is received. It contains the relation $E=mc^2$." It does not.

¹³*The Collected Papers of Albert Einstein, The Swiss Years: Writings, 1912–1914*, translated by Anna Beck (Princeton U. P., Princeton, NJ, 1996), Vol. 4, pp. 3–88; Also see Wallace Kantor, "Inertia of energy," *Am. J. Phys.* **22**, 528–541 (1954).

¹⁴Reference 10, Anna Beck, p. 357, "Discussion following lecture version of 'The theory of relativity'" (January 1911).

¹⁵Mitchell J. Feigenbaum and N. David Mermin, " $E=mc^2$," *Am. J. Phys.* **56**(1), 18–21 (1988). (This paper treats the emitting "body" as a "particle" without specifying if such an entity has internal moving parts.)

¹⁶John Stachel and Roberto Torretti, "Einstein's first derivation of mass-energy equivalence," *Am. J. Phys.* **50**(8), 760–763 (1982); Also see Hans C. Ohanian, "Einstein's $E=mc^2$ mistakes," *Stud. Hist. Philos. Mod. Phys.* **40**, 167–173 (2009) and Ref. 7, Fadner, p. 120.

¹⁷"The concept of a plane wave, like many other physical concepts, is no more than a fiction which can be realized with only a certain degree of

accuracy." Albert Einstein and Leopold Infeld, *The Evolution of Physics* (Simon and Schuster, New York, 1966), p. 105, first published in 1938.

¹⁸Reference 2, p. 170. Einstein took the notion of the "material point" to be fundamental to classical mechanics, and often employed it although he does not seem to have ever defined it. The contemporary equivalent is presumably the "point mass," a concept equally ubiquitous and equally mysterious.

¹⁹Reference 2, pp. 200–206, "The principle of conservation of motion of the center of gravity and the inertia of energy" (May 1906). Incidentally, Einstein employed the terms "internal energy" and "energy content" interchangeably in this treatment; his first use of the designation "rest-energy" was around 1912.

²⁰Reference 3. See also A. P. French, *Special Relativity* (Norton, New York, 1968), p. 17; and Adel Antippa, "Inertia of energy and the liberated photon," *Am. J. Phys.* **44**(9), 841–844 (1976).

²¹In his original notation, Einstein wrote (1906), "if one assumes that any energy E possesses the inertia E/V^2 , then...." (Ref. 19). That statement suggests the famous equation, but it is not identical to it. Einstein is saying that any amount of energy added to a system adds mass, not that every mass corresponds to an amount of energy. This latter assertion requires that mass be equivalent to energy and not just that "mass depends on energy." In 1906, this leap was yet to come.

²²Reference 2, pp. 236–237, "Comment on the note of Paul Ehrenfest: 'The translatory motion of deformable electrons and the area law'" (April 1907).

²³Reference 2, pp. 238–250, "On the inertia of energy required by the relativity principle" (May 1907).

²⁴Reference 14, p. 353, "Discussion following lecture version of 'The theory of relativity'" (January 1911).

²⁵Reference 2, p. 249, "On the inertia of energy required by the relativity principle" (May 1907). Using μ rather than m , V rather than c , and ϵ_0 rather than E_0 , Einstein wrote his famous equation for the first time as $\mu V^2 = \epsilon_0$, and he did it in a footnote. At the end of that paper, he introduced the symbol E_0 to denote energy in the rest frame, and wrote the famous expression again, this time as $\mu = E_0/V^2$.

²⁶Reference 2, pp. 252–311, "On the relativity principle and the conclusions drawn from it" (December 1907).

²⁷H. M. Schwartz, "Einstein's comprehensive 1907 essay on relativity, part II," *Am. J. Phys.* **45**(9), 811–817 (1977).

²⁸A. Einstein, "Elementary derivation of the equivalence of mass and energy," *Bull. Am. Math. Soc.* **41**, 223–230 (1935).

²⁹Reference 2, pp. 379–394, "On the development of our views concerning the nature and constitution of radiation" (September 1909).

³⁰Reference 10, Anna Beck, pp. 379–387, "On the influence of gravitation on the propagation of light" (June 1911).

³¹Reference 13, Anna Beck, pp. 50–54, "On the special theory of relativity" (1912). In this paper, Einstein used the terms "rest-energy" (p. 49) and "rest-mass" (p. 85) both in quotes, seemingly for the first time. The quote marks suggest that he was taking these terms from the works of others (for example, Laue's 1911 paper, Ref. 34).

³²For a discussion, albeit brief, of rigid bodies in relativity, see John Stachel, *Einstein from 'B' to 'Z'* (Birkhäuser, Boston, MA, 2002), pp. 246 and 268. An impulsive force applied at one end of a rigid rod would be instantly conveyed to the other end, thereby violating the special theory. See Ref. 24.

³³Hermann Minkowski, "Das Relativitätsprinzip," first published in 1907, reprinted in *Ann. Phys.* **352**(15), 927–938 (1915); Hermann Minkowski, "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern," *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* **53**–111 (1908), presented December 1907; English translation: "The fundamental equations for electromagnetic processes in moving bodies," M. N. Saha and S. N. Bose, *The Principle of Relativity* (U. P., Calcutta, 1920), pp. 1–69; Hermann Minkowski, "Raum und Zeit," *Jahresber. Dtsch. Math.-Ver.* **18**, 75–88 (1909), presented September 1908; English translation: "Space and time," A. Einstein, H. A. Lorentz, H. Weyl, and H. Minkowski, *The Principle of Relativity* (Dover, New York, 1923), pp. 73–91.

³⁴M. Laue, "Zur Dynamik der Relativitätstheorie," *Ann. Phys.* **340**(8), 524–542 (1911).

³⁵John D. Norton, "Einstein, Nordström and the early demise of Lorentz-covariant, scalar theories of gravitation," *Arch. Hist. Exact Sci.* **45**, 17–94 (1992); Reprinted in *The Genesis of General Relativity, Theories of Gravitation in the Twilight of Classical Physics: Between Mechanics, Field Theory, and Astronomy Vol. 3*, edited by J. Renn and M. Schemmel

- (Springer, Dordrecht, 2007), pp. 413–487.
- ³⁶Reference 34, p. 541.
- ³⁷Reference 12, p. 152.
- ³⁸Einstein's 1912 Manuscript on the Special Theory of Relativity (George Braziller, New York, 1996), pp. 66 and 158.
- ³⁹Reference 13, Anna Beck, pp. 276–281, "Supplementary response to a question by Mr. Reissner" (December 1913).
- ⁴⁰Reference 13, Anna Beck, pp. 282–288, "On the foundations of the generalized theory of relativity and the theory of gravitation" (January 1914).
- ⁴¹*The Collected Papers of Albert Einstein, The Berlin Years: Writings, 1914-1917*, translated by Alfred Engel (Princeton U. P., Princeton, NJ, 1997), Vol. 6, p. 27; See the German version for the complete text: *The Collected Papers of Albert Einstein, The Berlin Years: Writings, 1914-1917*, edited by A. J. Kox, Martin Klein, and Robert Schulmann (Princeton U. P., Princeton, 1996). Vol. 6, p. 27.
- ⁴²*The Collected Papers of Albert Einstein, The Swiss Years: Correspondence, 1902-1914*, translated by Anna Beck (Princeton U. P., Princeton, NJ, 1995), Vol. 5, pp. 373–374.
- ⁴³David Rowe, in *The Symbolic Universe: Geometry and Physics 1890-1930*, edited by Jeremy Gray (Oxford U. P., Oxford, 1999), pp. 189–233.
- ⁴⁴*The Collected Papers of Albert Einstein, The Berlin Years: Correspondence, 1914-1918*, translated by Ann M. Hentschel (Princeton U. P., Princeton, NJ, 1998), Vol. 8, p. 574.
- ⁴⁵Reference 44, p. 576.
- ⁴⁶Reference 44, p. 581.
- ⁴⁷F. Klein, "Über die Integralform der Erhaltungssätze und die Theorie der räumlich-geschlossenen Welt," *Königliche Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse. Nachrichten*, 394–423 (1918).
- ⁴⁸Michel Janssen, "The Trouton experiment, $E=mc^2$, and a slice of Minkowski spacetime," in *Revisiting the Foundations of Relativistic Physics: Festschrift in Honor of John J. Stachel*, edited by Jürgen Renn, Lindy Divarci, and Petra Schröter (Kluwer, Dordrecht, 2003), pp. 27–54.
- ⁴⁹Reference 13, Anna Beck, p. 79, "Manuscript on special relativity" (1912).
- ⁵⁰Albert Einstein, *The Meaning of Relativity* (MJF Books, New York, 1984), p. 44, first published in 1922.
- ⁵¹David Topper and Dwight Vincent, "Einstein's 1934 two-blackboard derivation of energy-mass equivalence," *Am. J. Phys.* **75**(11), 978–983 (2007).
- ⁵²G. N. Lewis, "A revision of the fundamental laws of matter and energy," *Philos. Mag.* **16**(11), 705–717 (1908); G. N. Lewis and R. C. Tolman, "The principle of relativity, and non-Newtonian mechanics," *ibid.* **18**, 510–523 (1909); R. C. Tolman, "Note on the derivation from the principle of relativity of the fifth equation of the Maxwell-Lorentz theory," *ibid.* **21**, 296 (1911); R. C. Tolman, "Non-Newtonian mechanics, the mass of a moving body," *ibid.* **23**, 375–380 (1912); For a different approach, see P. C. Peters, "An alternative derivation of relativistic momentum," *Am. J. Phys.* **54**(9), 804–808 (1986).
- ⁵³W. Pauli, *Theory of Relativity* (Pergamon, New York, 1958), p. 83, first published in 1921; Max Born, *Einstein's Theory of Relativity* (Dover, New York, 1962), first published in 1920.
- ⁵⁴Francisco Flores, "Einstein's 1935 derivation of $E=mc^2$," *Stud. Hist. Philos. Mod. Phys.* **29**(2), 223–243 (1998); Also see Ref. 13, Wallace Kantor, and Ref. 51, David Topper and Dwight Vincent.
- ⁵⁵Reference 28, A. Einstein, p. 224.
- ⁵⁶Reference 28, A. Einstein, pp. 227–228.
- ⁵⁷Reference 28, A. Einstein, p. 228.
- ⁵⁸"The notion of 'material point' is fundamental for mechanics. If now we seek the mechanics of a bodily object which itself cannot be treated as a material point—and strictly speaking every object 'perceptible to our senses' is of this category...." Albert Einstein, *Out of My Later Years* (Philosophical Library, New York, 1950) p. 72. Also see p. 100 where he discusses "mass points with invariable mass" as fundamental to Newtonian mechanics.
- ⁵⁹This expression as written appeared on the blackboard during Einstein's talk (see Ref. 51, David Topper and Dwight Vincent) but not in his paper.
- ⁶⁰Ruth W. Chabay and Bruce A. Sherwood, "Modern mechanics," *Am. J. Phys.* **72**(4), 439–445 (2004) (These authors write, "consider a positron and electron each of mass m initially at rest far from each other. The (relativistic) energy of the two-particle system is $2mc^2$, with no additive constant possible (otherwise the four-momentum would not transform properly).")
- ⁶¹Albert Einstein, "An elementary derivation of the equivalence of mass and energy," *Technion Yearbook* **5**, 16–17 (1946).
- ⁶²Einstein used the phrase "light complex" in the June 1905 paper (Ref. 2). There he was considering electromagnetic plane waves. Here the term seems to correspond to a narrow pulse or localized wave train that could be absorbed completely by a finite object, something that could not happen with plane waves that have unlimited wavefronts.
- ⁶³J. D. Cockcroft and E. T. S. Walton, "Experiments with high velocity positive ions. II.—The disintegration of elements by high velocity protons," *Proc. R. Soc. London, Ser. A* **137**, 229–242 (1932); Also see Saul Dushman, "Mass-energy relation," *G. E. Review*, 6–13 October 1944, reproduced in *Special Relativity Theory, Selected Reprints* (AIP, New York, 1963), pp. 20–25.
- ⁶⁴S. Rainville, J. K. Thompson, E. G. Myers, J. M. Brown, M. S. Dewey, E. G. Kessler, and D. E. Pritchard, "A direct test of $E=mc^2$," *Nature (London)* **438**, 1096–1097 (2005). Their experiment established that $E_0=mc^2$ "holds to a level of at least 0.00004%."
- ⁶⁵G. J. Aubrecht II, "Comment on 'Apparatus to measure relativistic mass increase'," *Am. J. Phys.* **72**(7), 970–971 (2004); L. B. Okun, "Putting to rest mass misconceptions," *Phys. Today* **43**(5), 14, 115, 117 (1990).
- ⁶⁶"[I]t appears to be outside the scope of special relativity to prove that $c^2m\gamma$ is the available (as distinct from conventional) energy of a mass m" J. Ehlers, W. Rindler, and R. Penrose, "Energy conservation as the basis of relativistic mechanics II," *Am. J. Phys.* **35**, 995–997 (1965).