Solid-State Electronics Vol. 28, No. 11, pp. 1143–1151, 1985
Printed in Great Britain.

0038-1101/85 $3.00 + .00

CHARGE COLLECTION IN A SCHOTTKY DIODE AS A MIXED BOUNDARY-VALUE PROBLEM

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(*Received* 21 *November* 1984; *in revised form* 19 *February* 1985)

Abstract—An analysis is given of the determination of bulk diffusion lengths in semiconductors from the induced current profiles that are obtained by scanning an electron beam with normal incidence on a Schottky diode. The discussion assumes that the carrier recombination velocity at the free semiconductor surface is $v_s = 0$. In this case the mixed boundary conditions of the diffusion problem for excess minority carriers can be converted into normal ones by using polar coordinates, and an explicit expression for the induced current profile can be given. This expression is compared to that already known for the opposite case $v_s = \infty$, to establish the influence of the surface recombination velocity on a number of profile properties, such as symmetry, asymptotic decay, or low-order moments of the derivative. It is shown that by evaluating the variance of the profile derivative at two beam energies the diffusion length can be determined independently of the knowledge of the value of v_s.

1. INTRODUCTION

The electron-beam induced-current (EBIC) technique of the scanning electron microscope has been widely used to determine the minority-carrier diffusion length in semiconductors[1, 2]. In this kind of measurements, a number of different beam-sample configurations have been employed; one of these[3, 4] is such that the electron beam is incident normal to the plane of the collecting barrier formed by a Schottky diode (Fig. 1) or a shallow p–n junction. The value of the diffusion length L is obtained by recording the decay of the induced current with the beam to Schottky-diode distance and analyzing the data on the basis of analytical expressions provided by the theory.

Ioannou and Dimitriadis[4] gave a closed-form expression for the current profile in a semi-infinite sample with a Schottky diode on half of the surface plane, assuming infinite recombination velocity at the remaining semiconductor surface. Thus the diffusion problem for the excess minority carrier density p could be formulated and solved explicitly with homogeneous Dirichlet boundary conditions ($p = 0$) over the whole surface.

The analysis for a finite value v_s of the surface recombination velocity is much more difficult, since the boundary-value problem becomes of the mixed type, in that the condition $p = 0$ only holds at the Schottky contact, while over the remainder of the surface the condition is that p is proportional to its normal gradient through v_s. Von Roos[5] has studied a related problem, but adapting his results to the present case appears rather difficult. He also analyzed the configuration consisting of a circular Schottky diode on a sample having finite thickness[6], but obtained a system of dual integral equations which could not be solved in closed form. However, since the diode radius is usually large in comparison to L, the straight-edge approximation of [4] is adequate while being easier to deal with.

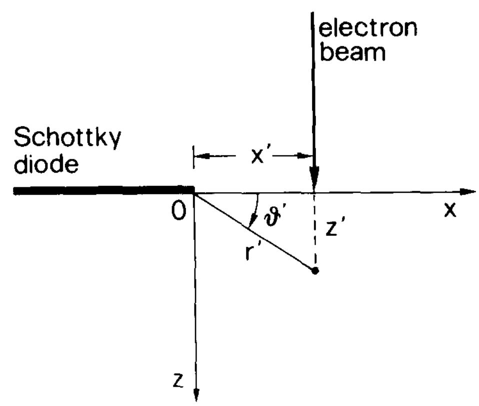

Fig. 1. Schematic representation of the diffusion length measurements by the Schottky-barrier EBIC technique.

The present paper analyzes charge collection in this simplified configuration assuming $v_s = 0$; in this case an explicit expression for the current profile can be obtained by the eigenfunction expansion method. It will be shown that the knowledge of the solution in the two limiting cases $v_s = 0, \infty$ elucidates the influence of the surface recombination velocity on the current profile and also yields a method of determining L that is free from that influence.

2. THE MIXED BOUNDARY-VALUE PROBLEM FOR $v_s = 0$

The configuration to be analyzed is illustrated in Fig. 1. The semiconductor surface is coincident with the x–y plane, and the Schottky diode covers the half-plane $x < 0$; the surface half-plane $x > 0$ is characterized by $v_s = 0$.

Since the configuration has translational invariance along the y axis, the charge collection problem needs to be discussed in two dimensions (x, z) only. Under the usual simplifying assumptions[4,6], the density of excess minority carriers $p(x, z)$ injected by a unit point source at (x', z') obeys the diffusion equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{L^2} p = -\frac{1}{D} \delta(x - x')\, \delta(z - z'), \tag{1}$$

where D is the minority carrier diffusion coefficient and δ is the Dirac delta function. The boundary conditions are

$$p(x, 0) = 0, \qquad x < 0; \tag{2a}$$

$$\left.\frac{\partial p}{\partial z}\right|_{z=0} = 0, \qquad x > 0. \tag{2b}$$

The difficulty in solving this boundary-value problem originates from the fact that the boundary condition changes from homogeneous Dirichlet to homogeneous Neumann (normal gradient of p equal to zero) over the coordinate surface $z = 0$. Any attempt to express the solution in terms of a Green's function which satisfies either (2a) or (2b) along the entire x axis would require the introduction of unknown boundary values and lead to an integral equation. Although integral equations resulting from boundary-value problems can often be solved by complex variable methods relying on the Wiener–Hopf technique ([7, p. 978]; [5]), in the present case a direct solution of (1) with (2) is possible. As suggested by Naylor[8] for a class of similar problems, it is only necessary to convert the mixed boundary conditions (2) into normal ones by introducing polar coordinates.

2.1. *Direct solution in polar coordinates*

Using polar coordinates r, θ, with $x = r\cos\theta$, $z = r\sin\theta$, (1) and the conditions (2) become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \lambda^2 p = -\frac{1}{Dr}\delta(r - r')\,\delta(\theta - \theta'), \tag{3a}$$

$$p(r, \pi) = 0, \tag{3b}$$

$$\left.\frac{\partial p}{\partial \theta}\right|_{\theta=0} = 0, \tag{3c}$$

where $\lambda = 1/L$. The original boundary conditions (2a), (2b) are now given on the distinct coordinate lines $\theta = 0, \pi$ and the boundary-value problem becomes of the normal type. Using a standard procedure ([7], p. 825), the function $p(r, \theta)$ is expanded in terms of the eigenfunctions of the angular part of the Laplace operator that satisfy (3b) and (3c)

$$p(r, \theta) = \sum_{n=0}^{\infty} a_n(r) \cos(n + \tfrac{1}{2})\theta. \tag{4}$$

Inserting this expansion into (3a) and making use of the delta function representation ([7], p. 719) with $q = n + \frac{1}{2}$

$$\delta(\theta - \theta') = (2/\pi) \sum_{n=0}^{\infty} \cos(q\theta)\cos(q\theta'), \tag{5}$$

we obtain an ordinary differential equation for the Fourier coefficients $a_n(r)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_n}{\partial r}\right) - (q^2/r^2 + \lambda^2) a_n = -\frac{2}{\pi D r}\cos(q\theta')\,\delta(r - r'). \tag{6}$$

The solution of this equation can be expressed in terms the modified Bessel functions of half-odd-integral order I_q, K_q of the argument λr. Application to this case of the standard method illustrated in [9] yields

$$a_n(r) = \frac{2}{\pi D}\cos(q\theta') I_q(\lambda r_<) K_q(\lambda r_>), \tag{7}$$

where $r_< (r_>)$ is the smaller (larger) of r and r'. Hence from (4) and (7) the solution can be written as

$$p(r, \theta) = \frac{2}{\pi D} \sum_{n=0}^{\infty} \cos(q\theta)\cos(q\theta') \times I_q(\lambda r_<) K_q(\lambda r_>). \tag{8}$$

The particle current collected by the semi-infinite Schottky diode is given by

$$I = D\int_{-\infty}^{0} \left.\frac{\partial p}{\partial z}\right|_{z=0} \mathrm{d}x = -D\int_0^{\infty} \frac{1}{r}\left.\frac{\partial p}{\partial \theta}\right|_{\theta=\pi} \mathrm{d}r. \tag{9}$$

Inserting (8) into (9) we obtain the final result

$$I(r', \theta') = \frac{2}{\pi}\int_0^{\infty} \frac{\mathrm{d}r}{r} \sum_{n=0}^{\infty} (-1)^n q \cos(q\theta') \times I_q(\lambda r_<) K_q(\lambda r_>). \tag{10}$$

It is not difficult to check that this expression is a particular case of eqn (19) of ref. [8]. Although (10) is the exact solution of the original mixed boundary-value problem, it has two disadvantages. First, $I(r', \theta')$ is expressed in polar coordinates, which are not natural for the configuration of Fig. 1. Second, expression (10) contains both an integral and a series; this can make a numerical evaluation rather troublesome.

Nevertheless, (10) has the useful property of yielding a simple expression for I when the generation point is on the positive x axis. It is shown in the Appendix A that

$$I(r',0) = I(x',0) = \operatorname{erfc}\left[(\lambda x')^{1/2}\right], \qquad x' \geqq 0. \tag{11}$$

The next paragraph illustrates how this knowledge can be used to express $I(x', z')$ through a single integral.

2.2. *The collected current in rectangular coordinates*

It is proved in the Appendix B that the function $I(x', z')$ satisfies the homogeneous version of (1):

$$\frac{\partial^2 I}{\partial x'^2} + \frac{\partial^2 I}{\partial z'^2} - \lambda^2 I = 0, \tag{12}$$

with the boundary conditions (of the mixed type)†

$$I(x',0) = 1, \qquad x' \leqq 0; \tag{13a}$$

$$\left.\frac{\partial I}{\partial z'}\right|_{z'=0} = 0, \qquad x' > 0. \tag{13b}$$

The condition (13a) means that complete charge collection occurs when the point source is at the collecting plane. Although (13b) specifies the normal derivative of I for $x' > 0$, the value of the function itself is now known from (11). Hence I can also be found as the solution of (12) with the boundary conditions of the normal type

$$I(x',0) = 1, \qquad x' \leqq 0; \tag{14a}$$

$$I(x',0) = \operatorname{erfc}\left[(\lambda x')^{1/2}\right], \qquad x' \geqq 0. \tag{14b}$$

The identity of the solutions of (12) with the boundary conditions (13) or (14) follows from the uniqueness of the solution.

Since the new boundary conditions are of the Dirichlet type along the entire x axis, we may express I through the Green's function of (12) that is zero at $z = 0$. The Green's function with this property can be found by the method of images ([7], p. 812), and is given by

$$G = \frac{1}{2\pi}\left[K_0\left\{\lambda\left[(x'-x)^2 + (z'-z)^2\right]^{1/2}\right\} - K_0\left\{\lambda\left[(x'-x)^2 + (z'+z)^2\right]^{1/2}\right\}\right], \tag{15}$$

where K_0 is the modified Bessel function of the second kind of order zero. The Green's theorem gives now (see (B1))

$$I(x', z') = \int_{-\infty}^{+\infty} I(x,0)\left.\frac{\partial G}{\partial z}\right|_{z=0} \mathrm{d}x. \tag{16}$$

Substituting (15) into (16) and performing the derivative we obtain

$$I(x', z') = \frac{\lambda z'}{\pi}\int_{-\infty}^{+\infty} I(x,0) \times \frac{K_1\left\{\lambda\left[(x'-x)^2 + z'^2\right]^{1/2}\right\}}{\left[(x'-x)^2 + z'^2\right]^{1/2}}\,\mathrm{d}x, \tag{17}$$

where K_1 is the modified Bessel function of the second kind of order one. Use of (14) in (17) finally yields

$$I(x', z') = \frac{\lambda z'}{\pi}\int_{-\infty}^{0} \frac{K_1\left\{\lambda\left[(x'-x)^2 + z'^2\right]^{1/2}\right\}}{\left[(x'-x)^2 + z'^2\right]^{1/2}}\,\mathrm{d}x + \frac{\lambda z'}{\pi}\int_{0}^{\infty} \operatorname{erfc}\left[(\lambda x)^{1/2}\right] \times \frac{K_1\left\{\lambda\left[(x'-x)^2 + z'^2\right]^{1/2}\right\}}{\left[(x'-x)^2 + z'^2\right]^{1/2}}\,\mathrm{d}x. \tag{18}$$

The equivalence between (10) and (18) is a consequence of the uniqueness of the solution of the boundary-value problem; however, a direct proof of this property seems to be difficult. In (18), it is easy to recognize that the first term on the right-hand side corresponds to the current that would be collected in the case $v_s = \infty$ [4]. The second term gives the current increase due to carrier reflection at the semiconductor surface with $v_s = 0$. Both for $v_s = 0$ and $v_s = \infty$, and actually for any value of v_s, we have that

$$\lim_{x' \to -\infty} I(x', z') = \frac{\lambda z'}{\pi}\int_{-\infty}^{+\infty} \frac{K_1\left[\lambda(x^2 + z'^2)^{1/2}\right]}{(x^2 + z'^2)^{1/2}}\,\mathrm{d}x = \exp(-\lambda z'), \tag{19}$$

the last equality following from eqn (6.596.3) of Ref. [11]. The well-known result of (19) reflects the fact that when the generation takes place well inside the Schottky diode the collected current is hardly influenced by the properties of the free semiconductor surface.

3. COMPARISON OF THE CURRENT PROFILES WHEN $v_s = \infty$ AND $v_s = 0$

Figure 2 shows some computed current profiles according to (18); the required integrations have been performed numerically using the approxima-

† The fact that $I(x', z')$ satisfies (12) with (13) expresses a reciprocity property of charge collection; a more detailed discussion on this point can be found in [10].

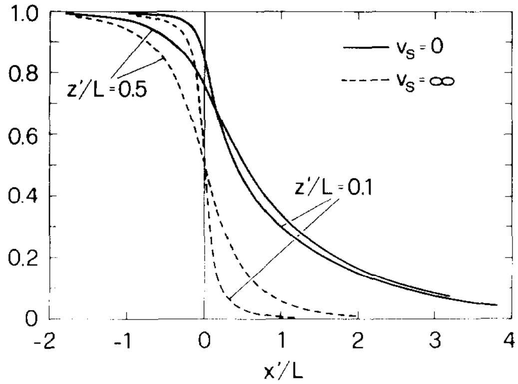

Fig. 2. Normalized EBIC profiles in a Schottky diode for different generation depths and $v_s = \infty, 0$, as calculated from eqn (18).

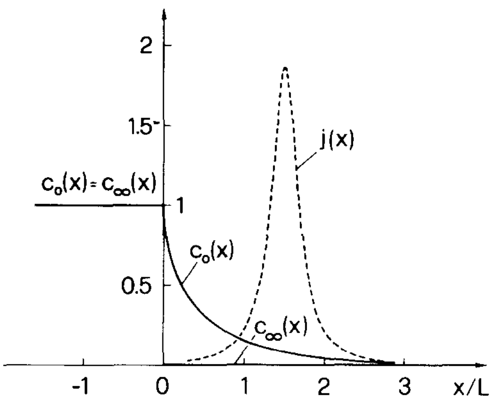

Fig. 3. Plot of the functions $c_\infty(x)$, $c_0(x)$ (eqn (26), (27)), and $j(x)$ (eqn (25) with (30b)) for $z'/L = 0.2$. The function $j(x)$ has been shifted for illustration purposes.

tions for the special functions given in [12]. The profiles have been normalized to $I(-\infty, z')$ using the result of (19), and are plotted versus the reduced distance x'/L.

For $v_s = \infty$ the profiles have a symmetry about $x' = 0$, in that the function $i(x') - \frac{1}{2}$ is an odd function of x'; for $v_s = 0$ no symmetry is observed. In addition, the profiles with $v_s = \infty$ decrease more rapidly with x' than the corresponding ones with $v_s = 0$. Figure 2 also shows that a higher generation depth produces a smoother profile. These properties will be studied in a more quantitative way in the following sections.

3.1. *Asymptotic expansions*

It is of interest to study the asymptotic behaviour of $I(x', z')$ of (18) for large values of x', since it is known that the decrease of I at large scan distances bears information about the value of the diffusion length L [3,4]. It has been shown[4] that for $v_s = \infty$, $z' \ll x'$, the current at $x' \gg L$ decays as

$$I(x') \cong z'(\tfrac{1}{2}L/\pi)^{1/2} x'^{-3/2} \exp(-x'/L). \quad (20)$$

This expansion also applies to the first term I_1 on the right-hand side of (18), since I_1 is just the current for $v_s = \infty$. The second term I_2 can be written in dimensionless quantities $X = \lambda x$, $X' = \lambda x'$, $Z' = \lambda z'$, after changing the integration variable to $Y = X - X'$, as

$$I_2 \cong \frac{Z'}{\pi} \int_{-X'}^{\infty} \operatorname{erfc}\left[(X' + Y)^{1/2}\right] \times \frac{K_1\left[(Y^2 + Z'^2)^{1/2}\right]}{(Y^2 + Z'^2)^{1/2}} \, \mathrm{d}Y. \quad (21)$$

If $Z' \ll 1$, i.e. for generation depths z' small in comparison to L, the function

$$\frac{Z'}{\pi} \frac{K_1\left[(Y^2 + Z'^2)^{1/2}\right]}{(Y^2 + Z'^2)^{1/2}} \quad (22)$$

is sharply peaked at $Y = 0$, having a maximum value $\cong 1/(\pi Z')$ and a full width at half-maximum $\cong 2Z'$, as can be seen using the approximation $K_1(x) \cong 1/x$ [12] for $x \ll 1$. The erfc function of (21) for large values of X' undergoes only minor changes in the neighborhood of $Y = 0$ where the function (22) has relevant values (for $Z' \ll 1$ this function is $\cong j(x)$ of Fig. 3), and can be treated approximately as a constant to give

$$I_2 \cong \operatorname{erfc}(X'^{1/2}) \frac{Z'}{\pi} \int_{-X'}^{\infty} \frac{K_1\left[(Y^2 + Z'^2)^{1/2}\right]}{(Y^2 + Z'^2)^{1/2}} \, \mathrm{d}Y. \quad (23)$$

Under the above assumptions, the lower integration limit in (23) can be replaced by $-\infty$ without significant error. The resulting integral is known from (19); since $\exp(-Z') \cong 1$ for $Z' \ll 1$, (23) becomes

$$I_2 \cong \operatorname{erfc}(X'^{1/2}) \cong (\pi X')^{-1/2} \exp(-X') = (L/\pi)^{1/2} x'^{-1/2} \exp(-x'/L), \quad (24)$$

where use has been made of the asymptotic expansion of the complementary error function[12]. At a given value of x', the ratio I_1/I_2 is $\cong z'/x' \ll 1$, therefore I_2 of (24) represents the asymptotic trend of the current for $v_s = 0$. As can be expected, for $v_s = 0$ the collected current is both larger and decreases more slowly than in the case $v_s = \infty$.

3.2. *The profile as a convolution*

It is useful to express in a unique more compact form the current collected in the two limiting cases $v_s = 0, \infty$. In fact, both (18) and the corresponding eqn (A5) of Ref. [4] (i.e. the first term in (18)) have the form of a convolution; the convoluted functions are

$$J(x) = \frac{\lambda z'}{\pi} \frac{K_1\left[\lambda(x^2 + z'^2)^{1/2}\right]}{(x^2 + z'^2)^{1/2}}, \quad (25)$$

which represents the current density of minority carriers from the point source to the surface when $v_s = \infty$, and

$$c_\infty(x) = \begin{cases} 1 & x \leqq 0 \\ 0 & x > 0 \end{cases} \quad (v_s = \infty), \tag{26}$$

or

$$c_0(x) = \begin{cases} 1 & x \leqq 0 \\ \operatorname{erfc}(\lambda x)^{1/2} & x \geqq 0 \end{cases} \quad (v_s = 0). \tag{27}$$

The function $c(x)$ can be interpreted as the probability that a minority carrier reaching the surface will be collected by the junction. This probability is equal to one for $x \leqq 0$ (at the Schottky contact), and is zero for $x > 0$ when $v_s = \infty$, since in this case all carriers reaching the surface at $x > 0$ are lost by recombination. For $v_s = 0$ the virtual flux of carriers to the surface at $x > 0$ contributes to the collected current, but this contribution decreases with the distance x from the Schottky diode. A plot of the functions $c_\infty(x)$ and $c_0(x)$ is shown in Fig. 3. Thus, omitting for simplicity the dependence on z', the collected current can be written as

$$I(x') = \int_{-\infty}^{+\infty} c(x) J(x'-x)\, dx = \int_{-\infty}^{+\infty} c(x'-x) J(x)\, dx, \tag{28}$$

where $c(x)$ is to be specified according to the value of v_s. It is convenient to normalize $I(x')$ to its value at $x' = -\infty$, as done in Fig. 2; by comparing (19) with (25) we see that

$$I(-\infty) = \int_{-\infty}^{+\infty} J(x)\, dx. \tag{29}$$

In terms of the normalized functions

$$i(x) = I(x)/I(-\infty), \tag{30a}$$

$$j(x) = J(x)/I(-\infty), \tag{30b}$$

(28) becomes

$$i(x') = \int_{-\infty}^{+\infty} c(x) j(x'-x)\, dx. \tag{31}$$

In the following we will need the absolute value of the derivative of $i(x')$, which by (31) can be written as

$$|i'(x')| = -\frac{di(x')}{dx'} = \int_{-\infty}^{+\infty} -c'(x) j(x'-x)\, dx, \tag{32}$$

since $i(x')$ is a decreasing function of x'. According

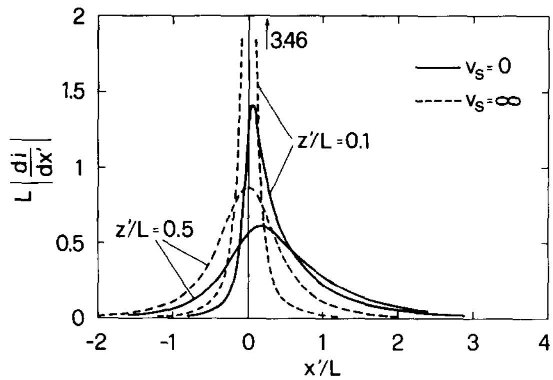

Fig. 4. Derivative of the profiles of Fig. 2.

to the definitions (26) and (27), we have

$$-c'_\infty(x) = \delta(x) \tag{33}$$

and

$$-c'_0(x) = \begin{cases} 0 & x \leqq 0 \\ (\lambda/\pi)^{1/2} x^{-1/2} \exp(-\lambda x) & x > 0, \end{cases} \tag{34}$$

the second equality being a consequence of the definition of the complementary error function ([12], p. 297).

The derivative of the profiles of Fig. 2 are shown in Fig. 4. It is seen that a number of properties of $i(x')$, as symmetry or smoothness, are reflected with better evidence in $i'(x')$. In addition, since $i'(x')$ approaches rapidly zero for increasing $|x'|$, these properties can be described conveniently by examining integral quantities related to $i'(x')$, as the moments about the origin.

3.3. *The moments of the profile derivative*

According to the usual language of probability, the moment of order n about the origin of a distribution function $f(x)$ is defined as

$$\alpha_n[f] = \int_{-\infty}^{+\infty} x^n f(x)\, dx \tag{35}$$

The moment of order zero, i.e. the area under the curve defined by $f(x)$, is equal to one for a function representing a distribution. It is easy to check that both $j(x)$ as defined in (30b) and $-c'(x)$ of (33) and (34) have this property. The same holds consequently for $|i'(x)|$, as can be seen by integrating (32) over x' in the interval $(-\infty, +\infty)$. We will need the moments of $|i'|$ of order $n = 1, 2$, which can be expressed in terms of the corresponding moments of $-c'(x)$ and $j(x)$. Taking into account that

$$\alpha_1[j] = 0 \tag{36}$$

because $j(x)$ is an even function of x, it follows

Table 1. Values of the moments about the origin of order $n = 1, 2$ and variance of $|i'|$ for $v_s = \infty, 0$

Moments of $\|i'(x')\|$	α_1	α_2	$\sigma^2 = \alpha_2 - \alpha_1^2$
$v_s = \infty$	0	Lz'	Lz'
$v_s = 0$	$\frac{1}{2}L$	$Lz' + \frac{3}{4}L^2$	$Lz' + \frac{1}{2}L^2$

from the definitions that

$$\alpha_1[|i'|] = \alpha_1[-c'] \quad (37)$$

and

$$\alpha_2[|i'|] = \alpha_2[-c'] + \alpha_2[j]. \quad (38)$$

The required moments are evaluated in the Appendix C and turn out to have very simple expressions. The resulting moments of $|i'|$, as well as the variance σ^2 (the second moment about the mean), are summarized in Table 1 for $v_s = \infty, 0$.

The moments of $|i'|$ can be calculated starting from the values of $i(x)$ without actually performing the derivative of the profile. This is useful in practice, since the differentiation of an experimental profile with some noise can result in large fluctuations of i', unless some precautions are taken. In fact, let $h(x)$ be the function (Fig. 5)

$$h(x) = \begin{cases} i(x) & x \geq 0 \\ i(x) - 1 & x < 0. \end{cases} \quad (39)$$

This function has the same sign of x and is in general discontinuous at $x = 0$. However, $h(x)$ approaches rapidly zero for $|x| \to \infty$ and its moments are finite, whereas the moments of $i(x)$ do not exist. Using the definition (35) and performing some integrations by parts, the following relations between the moments of $|i'|$ and h are found[†]:

$$\begin{aligned} \alpha_1[|i'|] &= \alpha_0[h] \\ \alpha_2[|i'|] &= 2\alpha_1[h]. \end{aligned} \quad (40)$$

The central moments of $|i'|$, i.e. the moments about the mean $m = \alpha_1[|i'|] = \alpha_0[h]$, are more conveniently related to those of the function $h_m(x)$ by

$$h_m(x) = \begin{cases} i(x) & x \leq m \\ i(x) - 1 & x > m. \end{cases} \quad (41)$$

The definition of $h_m(x)$ is similar to that of $h(x)$, except that the discontinuity point is displaced from $x = 0$ to $x = m$. Denoting by μ_n the central moment of order n, it is found analogously to (40) that

$$\mu_1[|i'|] = \mu_0[h_m] = 0, \quad (42a)$$

$$\mu_2[|i'|] = 2\mu_1[h_m] = \sigma^2. \quad (42b)$$

It follows from (42a) that the point $x = m$ has the property that the area bounded by $h_m(x)$ and the x axis is the same on either side of $x = m$. According to (42b), σ^2 is equal to twice the first-order moment of h_m about this point. The use of these properties can make the evaluation of a profile easier.

† The first-order moment $\alpha_1[i+]$ of the positive side of $i(x)$ for $v_s = \infty$ was evaluated previously[13], obtaining $\frac{1}{2}Lz'$, but with the normalization of i to $I(0, z') = \frac{1}{2}I(-\infty, z')$. With the normalization to $I(-\infty, z')$ used here, $\alpha_1[i+] = \frac{1}{4}Lz'$ and $\alpha_1[h] = \frac{1}{2}Lz'$, because of the odd symmetry of $h(x)$ for $v_s = \infty$. Hence $\alpha_2[|i'|] = Lz'$, in agreement with the value given in Table 1.

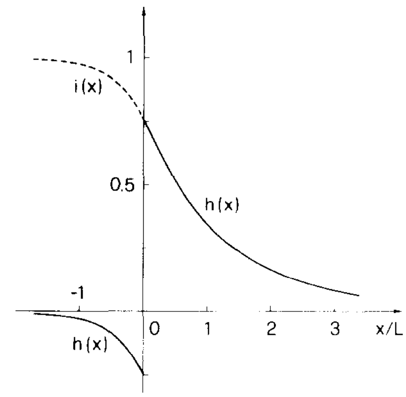

Fig. 5. Plot of the function $h(x)$ of eqn (39) for $v_s = 0$, $z'/L = 0.5$. The dashed line illustrates the relation to $i(x)$.

4. THE CASE OF ARBITRARY v_s

The use of polar coordinates, though appropriate for solving the diffusion problem (1) with $v_s = 0$, does not seem to be of advantage when v_s is finite. In fact, (3c) would become

$$\frac{1}{r}\frac{\partial p}{\partial \theta}\bigg|_{\theta=0} = \frac{v_s}{D}p\bigg|_{\theta=0} \quad (43)$$

and because of the factor $1/r$ in the boundary condition (43) the eigenfunction method is no longer applicable (see [7, p. 1039]). A possibility is to look for an approximate solution or to resort to more powerful but more complex methods like the Wiener–Hopf technique.

However, the expression of the profile as a convolution [eqn (17) or eqn (28)] and the consequent relations (37) and (38) between the moments of $|i'|$, $-c'$ and j are valid for any v_s. This fact and the knowledge of the moments of $|i'|$ for $v_s = \infty, 0$ give some indication, for instance, on the expression of σ^2 for finite v_s, even though the corresponding function $c(x) = I(x, 0)$ is not known.

In fact, eqns (37) and (38) show that the variance $\sigma^2 = \alpha_2 - \alpha_1^2$ of $|i'|$ is in general equal to the sum of the variances of $-c'$ and j. Denoting by σ_s^2 the variance of $-c'$ and using the result of eqn (C3), we may write

$$\sigma^2 = \sigma_s^2 + Lz'. \quad (44)$$

The term σ_s^2 will be a function of v_s and L, but not of z', since $c(x)$ does not contain this variable. Since

σ_s^2 has the dimensions of the square of a length, it can be written as

$$\sigma_s^2 = wL^2/2, \quad (45)$$

where w is a dimensionless function of v_s and L, and the factor $\frac{1}{2}$ has been introduced for convenience. Thus eqn (44) becomes

$$\sigma^2 = wL^2/2 + Lz'. \quad (46)$$

The results of Table 1 show that $w = 0$ for $v_s = \infty$ and $w = 1$ for $v_s = 0$; for finite v_s, w is likely to assume intermediate values. For dimensional reasons, w is expected to depend upon v_s and L only through the dimensionless product $v_s L/D$, but the explicit dependence can only be established by solving the diffusion problem for arbitrary v_s.

Nevertheless, eqn (46) [or eqn (44)] can be used to deduce L from profile variance measurements. In fact, the unknown factor w in eqn (46) is in any case independent of z'; therefore, it is sufficient to determine σ^2 at two different beam energies (i.e. at two known generation depths z_1', z_2') and use eqn (46) twice to eliminate w. Thus L can be found from the expression

$$L = \frac{\sigma_2^2 - \sigma_1^2}{z_2' - z_1'}, \quad (47)$$

where $\sigma_1^2(\sigma_2^2)$ is the variance of $|i'|$ when the generation depth is $z_1'(z_2')$. If the dependence of w upon v_s were known, eqn (46) would also yield a method of evaluating v_s. A similar procedure has been actually applied to evaluate both L and v_s in a different configuration, where the expression of the profile for arbitrary v_s was known [14].

5. DISCUSSION AND CONCLUSIONS

The basic assumptions of the model described here are essentially those made by Ioannou and Dimitriadis[4] for the case $v_s = \infty$, and the reader is referred to that paper for a detailed discussion. The present analysis deals with the complementary case $v_s = 0$ and also gives some indication about some expected profile properties for arbitrary v_s.

In [4] the diffusion length of some samples was evaluated by fitting the profile tail with the law $x'^{-3/2}\exp(-x'/L)$, i.e. by assuming $v_s = \infty$ (see (20)); in practice the value of L was obtained from the slope of the straight portion of the plot of $\ln(Ix'^{3/2})$ vs x'. A fit with the opposite assumption $v_s = 0$, i.e. with the law $x'^{-1/2}\exp(-x'/L)$ of (24) would obviously yield a different value of L. It is not difficult to see that by applying the decay law valid for $v_s = \infty$ to a hypothetical sample with $v_s = 0$ one obtains an apparent diffusion length L_a, which exceeds the true value L by a percentage amount L_a/x', x' being the point at which the slope is evaluated. For instance, at $x' \cong 3L_a$ the overestimate would be $\cong 30\%$. This example illustrates that, in the absence of information about v_s, the values of L obtained by the straightforward slope analysis can be affected by a systematic error.

The method introduced in Section 4 allows the evaluation of L without requiring the knowledge or the determination of v_s. An additional advantage of the moment method is that use is made of profile data at low scan distances, where the current is higher and hence usually known with greater relative accuracy. Current values at large scan distances, which are required to evaluate the moments but are frequently not available, can be estimated with sufficient accuracy by extrapolation of the profile tails[13, 14].

The method of Section 4 is based on the determination of σ^2 at two beam energies. The reason for using σ^2 instead of α_2 (or α_1) is that σ^2 is an intrinsic property of the profile, whereas the value of α_2 (or α_1) is related to the choice of the origin. In absence of additional information, there is no simple criterion to establish the location of the point $x = 0$ from an experimental profile. The next central moment of $|i'|$ is that of third order μ_3, but it is not as useful as σ^2. In fact, μ_3 varies between zero ($v_s = \infty$) and L^3 ($v_s = 0$), being independent of z'; therefore the elimination of the unknown weight factor, which will appear in the expression of μ_3 for finite v_s, by two measurements at different values of z' is no longer possible. The use of higher-order moments is not convenient, since usually they can be evaluated only with lower accuracy.

The values of the moments given in Table 1 are based on the simple scheme of point generation of minority carriers. The generalization to a generation along a line is straightforward; according to the discussion of [13], it is sufficient to interprete z' as the center of gravity of the electron depth–dose function; in silicon $z' = 0.41\ R$, R being the primary electron range[13].

Taking into account the finite lateral extension of the generation would require a further convolution of $i(x')$ (or $i'(x')$) of Section 3.2 with the function describing the lateral distribution of the generation. Because of the additivity property of the variance of convolutions (see (38)), this would increase the σ^2 values of Table 1 by a term equal to the variance of the lateral distribution function. Since this variance is of the order of z'^2, we see that as long as $z' \ll L$ this effect introduces only minor changes in σ^2. Therefore, with the above restriction, the use of more realistic but more complex generation schemes would not modify significantly the expressions given in Section 4.

Acknowledgements—The author wishes to thank the reviewers for useful comments on the manuscript; in particular he is grateful to one of them for making available the relevant papers of [16, 17].

NOTE ADDED IN PROOF

Since this work was submitted for publication, two papers on the same subject have appeared. Boersma *et al.*[16] solve the mixed boundary value problem of Section 2 for arbitrary v_s by means of the

Wiener–Hopf technique, and give an analytical expression for the related current profile. In particular, they consider the limiting case $v_s = 0$ and obtain an expression, which differs from that derived here but is actually equivalent, as proved in Appendix D.

In a following paper[17] Kuiken and van Opdorp discuss the use of asymptotic expansions of the expression by Boersma et al. for the determination of L and v_s from an experimental profile. The method to evaluate L proposed in [17] relies on the analysis of the profile decay and is essentially different from that suggested here, where integral properties of the profile are considered.

REFERENCES

1. H. J. Leamy, *J. Appl. Phys.* **53**, R51 (1982).
2. R. O. Bell and J. I. Hanoka, *Ann. Rev. Mater. Sci.* **11**, 353 (1981).
3. D. E. Ioannou and S. M. Davidson, *J. Phys. D: Appl. Phys.* **12**, 1339 (1979).
4. D. E. Ioannou and C. A. Dimitriadis, *IEEE Trans. Electron Dev.* **ED-29**, 445 (1982).
5. O. von Roos, *Solid-St. Electron.* **21**, 1069 (1978).
6. O. von Roos, *Solid-St. Electron.* **23**, 177 (1980).
7. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*. McGraw-Hill, New York (1953).
8. D. Naylor, *Proc. Camb. Phil. Soc.* **62**, 753 (1966).
9. J. D. Jackson, *Classical Electrodynamics*, Sect. 3.11. J. Wiley & Sons, New York (1975).
10. C. Donolato, *Appl. Phys. Lett.* **46**, 270 (1985).
11. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*. Academic Press, New York (1980).
12. M. A. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. Dover Publications, New York (1965).
13. C. Donolato, *IEEE Trans. Electron Dev.* **ED-31**, 121 (1984).
14. C. Donolato, *Appl. Phys. Lett.* **43**, 120 (1983).
15. V. Mangulis, *Handbook of Series*. Academic Press, New York (1965).
16. J. Boersma, J. J. Indenkleef and H. K. Kuiken, *J. Eng. Math.* **18**, 315 (1984).
17. H. K. Kuiken and C. van Opdorp, *J. Appl. Phys.* **57**, 2077 (1985).
18. F. Oberhettinger, *Fourier Transforms of Distributions and Their Inverses*. Academic Press, New York (1973).

APPENDIX A

Proof of (11).

For $\theta' = 0$, i.e. when the point source is on the positive x axis, (10) becomes

$$I(r',0) = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}r}{r}\sum_{n=0}^{\infty}(-1)^n q I_q(\lambda r_<)K_q(\lambda r_>). \tag{A1}$$

The addition theorem for modified Bessel functions of order $q = n + \frac{1}{2}$ ([15, p. 24, eqn (16)]) yields the identity

$$(1/R)\exp(-\lambda R) = 2(rr')^{-1/2}\sum_{n=0}^{\infty} q I_q(\lambda r_<)K_q(\lambda r_>)P_n(\cos\phi), \tag{A2}$$

where $R = (r^2 + r'^2 - 2rr'\cos\phi)^{1/2}$ and P_n is the Legendre polynomial of degree n. For $\cos\phi = -1$, being $P_n(-1) = (-1)^n$, (A2) gives

$$\sum_{n=0}^{\infty}(-1)^n q I_q(\lambda r_<)K_q(\lambda r_>) = \tfrac{1}{2}(rr')^{1/2}\frac{\exp[-\lambda(r+r')]}{r+r'}. \tag{A3}$$

Substituting this expression into (A1), we obtain

$$I(r',0) = (1/\pi)r'^{1/2}\int_0^\infty r^{-1/2}\frac{\exp[-\lambda(r+r')]}{r+r'}\,\mathrm{d}r. \tag{A4}$$

From [12], p. 302, eqn (7.4.9), we see that (A4) is just an integral representation of the complementary error function of argument $(\lambda r')^{1/2}$; this proves (11).

APPENDIX B

Proof that $I(x', z')$ satisfies (12) with the conditions (13).

Let G be a Green's function of (12); then I can be expressed in general through the Green's theorem[7] as

$$I(x',z') = -\int_{-\infty}^{+\infty}\left(G\left.\frac{\partial I}{\partial z}\right|_{z=0} - I\left.\frac{\partial G}{\partial z}\right|_{z=0}\right)\mathrm{d}x. \tag{B1}$$

Suppose that we choose G so that it satisfies the homogeneous version of (13a). Then the contribution of the positive x axis to the integral of (B1) vanishes, since both the normal gradient of I and G are zero for $x > 0$. On the negative x axis $G = 0$ and $I = 1$; hence (B1) gives

$$I(x',z') = \int_{-\infty}^{0}\left.\frac{\partial G}{\partial z}\right|_{z=0}\mathrm{d}x. \tag{B2}$$

It is easy to recognize that the function G as defined above is the same (apart from a factor D which does not appear in the final result) as the function p defined by (1) with (2). Therefore the definitions (9) and (B2) coincide and the above statement is proved.

APPENDIX C

Evaluation of moments of $j(x)$, $-c'_\infty(x)$ and $-c'_0(x)$.

According to the definitions of (30b), (25), and (19), we have

$$j(x) = (\lambda z'/\pi)\exp(\lambda z')\frac{K_1[\lambda(x^2+z'^2)^{1/2}]}{(x^2+z'^2)^{1/2}} = -(1/\pi)\exp(\lambda z')\frac{\partial}{\partial z'}K_0[\lambda(x^2+z'^2)^{1/2}]. \tag{C1}$$

We only need to evaluate $\alpha_2[j]$. Since [11, p. 705, eqn (6.596.3)]

$$\int_{-\infty}^{+\infty}x^2K_0[\lambda(x^2+z'^2)^{1/2}]\,\mathrm{d}x = (\pi/\lambda^3)\exp(-\lambda z')(1+\lambda z'), \tag{C2}$$

and using (C1) and (C2) in the definition of $\alpha_2[j]$, we have

$$\alpha_2[j] = \int_{-\infty}^{+\infty}x^2 j(x)\,\mathrm{d}x = Lz'. \tag{C3}$$

For $v_s = \infty$, one has immediately from (33)

$$\alpha_1[-c'_\infty] = 0, \qquad \alpha_2[-c'_\infty] = 0. \tag{C4}$$

The moments of $-c'_0$ can be evaluated starting from the identity

$$\int_0^\infty x^{-1/2}\exp(-\lambda x)\,dx = (\pi/\lambda)^{1/2}, \qquad (C5)$$

which is obtained by integrating (34) from zero to infinity. By using the property

$$\alpha_n[-c'_0] = (\lambda/\pi)^{1/2}\int_0^\infty x^{n-1/2}\exp(-\lambda x)\,dx$$
$$= (\lambda/\pi)^{1/2}(-1)^n\frac{\partial^n}{\partial\lambda^n}(\pi/\lambda)^{1/2} \qquad (C6)$$

it is easy to see that

$$\alpha_1[-c'_0] = L/2, \quad \alpha_2[-c'_0] = \tfrac{3}{4}L^2. \qquad (C7)$$

APPENDIX D

Proof of the equivalence between the present expression for the current profile and that given by Boersma *et al.*[16] for $v_s = 0$.

It seems easier to prove the equivalence between the expressions for the profile derivative; since both profiles approach zero for $x' \to \infty$, the equivalence of the derivatives entails that of the profiles themselves. Using dimensionless coordinates $X = \lambda x$, $Z = \lambda z$ without prime marks, the absolute value of the profile derivative $D(X)$ for $v_s = 0$ can be written as (see Section 3.2)

$$D(X) = d(X) * J(X), \qquad (D1)$$

where the asterisk denotes the convolution, and

$$d(X) = \begin{cases} 0 & X \leqslant 0 \\ (2\pi)^{1/2}(\pi X)^{-1/2}\exp(-X) & X > 0, \end{cases} \qquad (D2)$$

$$J(X) = \frac{(Z/\pi)K_1\left[(X^2+Z^2)^{1/2}\right]}{(X^2+Z^2)^{1/2}}. \qquad (D3)$$

The special form of (D1) suggests considering the Fourier transform of $D(X)$, which can be expressed through the convolution theorem [7, p. 464] as

$$\tilde{D}(k) = \tilde{d}(k)\cdot\tilde{J}(k), \qquad (D4)$$

where the tilde denotes the Fourier transform. From the tables in [18] we have that

$$\tilde{d}(k) = 2^{-1/2}\left[(\mu+1)^{1/2} + i(\mu-1)^{1/2}\right]/\mu, \qquad (D5)$$

$$\tilde{J}(k) = (2\pi)^{-1/2}\exp(-\mu Z), \qquad (D6)$$

where $\mu = (k^2+1)^{1/2}$. Hence

$$\tilde{D}(k) = \tfrac{1}{2}\pi^{-1/2}\exp(-\mu Z)\left[(\mu+1)^{1/2} + i(\mu-1)^{1/2}\right]/\mu. \qquad (D7)$$

It is shown now that the Fourier transform of the expression for $D(X)$ given by Boersma *et al.*[16] is equal to (D7); the identity of the transforms proves that of the original functions. In fact, by using the substitution $h = (t^2-1)^{1/2}$ and performing the derivative with respect to X in eqns (5.9) of [16], the following expressions for $D(X)$ are obtained

$$D(X) = \frac{1}{\pi}\int_0^\infty \frac{h\exp(-\nu X)}{\nu(\nu-1)^{1/2}}\cos(hZ)\,dh \qquad X \geqslant 0,$$
$$D(X) = \frac{1}{\pi}\int_0^\infty \frac{h\exp(\nu X)}{\nu(\nu+1)^{1/2}}\sin(hZ)\,dh \qquad X \leqslant 0, \qquad (D8)$$

where $\nu = (h^2+1)^{1/2}$. These integrals can be evaluated explicitly with the aid of the identities 3.962, [11, p. 498]; the resulting expression for $D(X)$ valid for any X is

$$D(X) = (2\pi)^{-1/2}(R+X)^{1/2}\exp(-R)/R, \qquad (D9)$$

where $R = (X^2+Z^2)^{1/2}$. To perform the Fourier transform of (D9) it is convenient to divide $D(X)$ into its even $(+)$ and odd $(-)$ parts $D_\pm(X)$:

$$D_\pm(X) = \tfrac{1}{2}[D(X) \pm D(-X)]. \qquad (D10)$$

Simple calculations show that

$$D_\pm(X) = \tfrac{1}{2}\pi^{-1/2}(R \pm Z)^{1/2}\exp(-R)/R. \qquad (D11)$$

The Fourier transform of $D(X)$ can be written as

$$\tilde{D}(k) = (2/\pi)^{1/2}\left[\int_0^\infty D_+(X)\cos(kX)\,dX + i\int_0^\infty D_-(X)\sin(kX)\,dX\right]. \qquad (D12)$$

Substituting eqn (D11) into eqn (D12) and performing the integrations by using again the identities 3.962 of [11], we obtain eqn (D7).