Solar Cells, 8 (1983) 283 - 296

THE FILL FACTOR OF A SOLAR CELL FROM A MATHEMATICAL POINT OF VIEW

ALEXIS DE VOS

Laboratorium voor Elektronica en Meettechniek, Rijksuniversiteit te Gent, Sint-Pietersnieuwstraat 41, B-9000 Gent (Belgium)

(Received June 1, 1982; accepted October 4, 1982)

Summary

In this paper the problem of finding the fill factor of a solar cell given the values of the network elements, *i.e.* the photogenerated current, the dark saturation current, the diode quality factor, the shunt resistance and the series resistance, is tackled.

Cases without shunt or series losses are treated first and then cases with small shunt and series losses are considered. Most of the proposed solutions to the problem are presented in the form of infinite expressions so that the user can obtain the desired accuracy by truncation of these expressions.

1. Introduction

The problem of calculating the fill factor FF (or the curve factor) of a solar cell has often been treated in the literature. As this calculation involves the solution of a transcendental equation, no exact values of FF can be found. Therefore the literature only provides approximate solutions in the form of a graph or an analytical expression. In most cases the accuracy of the solutions is not mentioned and no method is provided of enhancing the accuracy if the reader should need to.

In the present paper, various mathematical approaches to the problem will be presented. Solutions will be given in the form of continued logarithms and series expansions. These infinite expressions have the advantage that they can be truncated in function of the desired accuracy. Some finite approximations will be described, but the reader can easily construct his own finite formula in accordance with his proper needs.

First, I will treat the simple case of an ideal solar cell. Figure 1(a) shows the equivalent electrical circuit of such a cell. It consists of a current source I_1 (the light-generated current) and a diode with dark saturation current I_0. No series resistance and no shunt resistance are present: $R_s = 0$

0379-6787/83/0000-0000/$03.00

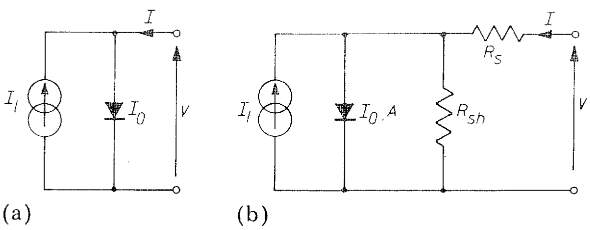

Fig. 1. Equivalent networks (a) of an ideal solar cell and (b) of an actual solar cell.

and $R_{sh} = +\infty$. The quality factor A of the diode equals unity. A detailed analysis of this ideal case will be presented in Section 2.

In Section 3, actual solar cells will be studied. Figure 1(b) shows the equivalent network of such a cell. It contains a finite series resistance R_s and a finite shunt resistance R_{sh}. The diode factor A can have an arbitrary value.

2. The ideal solar cell

The equivalent circuit of Fig. 1(a) leads to the well-known classical $I-V$ characteristic of an ideal solar cell:

$$I(V) = I_0\left\{\exp\left(\frac{qV}{kT}\right) - 1\right\} - I_l \tag{1}$$

This immediately gives us expressions for the short-circuit current and the open-circuit voltage:

$$I_{sc} = -I_l \tag{2}$$

and

$$V_{oc} = \frac{kT}{q}\ \ln\left(\frac{I_l}{I_0} + 1\right) \tag{3}$$

However, the maximum power P_m (and thus also the efficiency) of the cell can only be found by maximizing the power $P = -VI(V)$ with respect to V. This gives rise to the well-known transcendental equation

$$\left(1 + \frac{qV_m}{kT}\right)\ \exp\left(\frac{qV_m}{kT}\right) = \frac{I_l}{I_0} + 1 \tag{4}$$

This is solved for V_m, the maximum power voltage, and this solution must then be substituted into eqn. (1) in order to yield $I_m = I(V_m)$, the maximum power current, and finally $P_m = -V_m I_m$, the maximum power.

Traditionally, the maximum power is expressed in terms of the short-circuit current and the open-circuit voltage:

$$P_m = -\mathrm{FF} I_{sc} V_{oc} \tag{5}$$

Here the FF of the cell is the dimensionless number

$$\mathrm{FF} = \frac{V_\mathrm{m} I_\mathrm{m}}{V_\mathrm{oc} I_\mathrm{sc}} \tag{6}$$

For convenience, we introduce the dimensionless voltages $v = qV/kT$, $v_\mathrm{oc} = qV_\mathrm{oc}/kT$ and $v_\mathrm{m} = qV_\mathrm{m}/kT$. The characteristic eqn. (1) can then be rewritten

$$\frac{I(v)}{I_\mathrm{sc}} = \frac{\exp(v_\mathrm{oc}) - \exp(v)}{\exp(v_\mathrm{oc}) - 1} \tag{7}$$

The condition eqn. (4) for maximum power becomes

$$v_\mathrm{m} + \ln(v_\mathrm{m} + 1) = v_\mathrm{oc} \tag{8}$$

If this relation is taken into account, eqn. (7) yields

$$\frac{I_\mathrm{m}}{I_\mathrm{sc}} = \frac{\exp(v_\mathrm{oc})}{\exp(v_\mathrm{oc}) - 1} \, \frac{v_\mathrm{m}}{v_\mathrm{m} + 1} \tag{9}$$

FF is found as follows:

$$\mathrm{FF} = \frac{v_\mathrm{m}}{v_\mathrm{oc}} \, \frac{I_\mathrm{m}}{I_\mathrm{sc}} \tag{10}$$

It has been stressed often that FF is a function of v_oc *only*. Indeed, eqns. (8), (9) and (10) give rise to

$$\begin{aligned} v_\mathrm{oc} &= v_\mathrm{m} + \ln(v_\mathrm{m} + 1) \\ \mathrm{FF} &= \frac{v_\mathrm{m}^2 \exp(v_\mathrm{m})}{\{(v_\mathrm{m} + 1)\exp(v_\mathrm{m}) - 1\}\{v_\mathrm{m} + \ln(v_\mathrm{m} + 1)\}} \end{aligned} \tag{11}$$

This set of equations can be interpreted as a parameter representation of the function $\mathrm{FF}(v_\mathrm{oc})$ where v_m is the parameter. Varying v_m from 0 to $+\infty$ enables us to draw a curve of FF *versus* v_oc. The great theoretical and practical importance of this function $\mathrm{FF}(v_\mathrm{oc})$ explains why so many authors have published graphs of this function [1 - 8]. Figure 2 shows this useful curve once more.

In the present paper, I want to raise the problem of the explicit analytical function which expresses the relation between FF and v_oc. For such an explicit expression we have to solve eqn. (8) analytically for v_m and substitute into eqns. (9) and (10). Now as eqn. (8) is transcendental, it has no solution where v_m is a finite expression and contains only what are recognized as "elementary" functions of v_oc. The finite formulae which have been published by some authors [8, 9] have only limited accuracy. For correct formulae, only infinite expressions can be constructed. Appendix A gives an example of such an approach which yields interesting computer algorithms for solving the transcendental equation. In contrast, we can construct an asymptotic series of the function $v_\mathrm{m}(v_\mathrm{oc})$. Then v_m is expressed as the sum of an infinite number of terms, where each term is infinitesimally small compared with the previous term when v_oc tends to $+\infty$.

The following asymptotic expansion is derived in Appendix B:

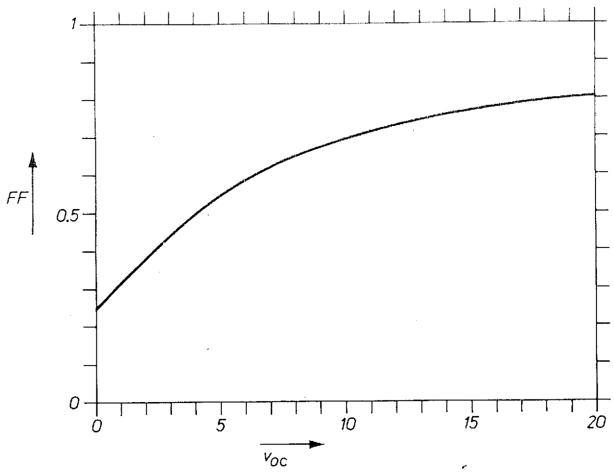

Fig. 2. The fill factor FF of an ideal solar cell as a function of the dimensionless open-circuit voltage $v_{oc} = qV_{oc}/kT$.

$$v_m = v_{oc} - \ln v_{oc} + \frac{1}{v_{oc}}(\ln v_{oc} - 1) +$$
$$+ \frac{1}{v_{oc}^2}\left\{\frac{1}{2}(\ln v_{oc})^2 - 2\ln v_{oc} + \frac{3}{2}\right\} +$$
$$+ \frac{1}{v_{oc}^3}\left\{\frac{1}{3}(\ln v_{oc})^3 - \frac{5}{2}(\ln v_{oc})^2 + 5\ln v_{oc} - \frac{17}{6}\right\} +$$
$$+ \frac{1}{v_{oc}^4}\left\{\frac{1}{4}(\ln v_{oc})^4 - \frac{17}{6}(\ln v_{oc})^3 + \frac{19}{2}(\ln v_{oc})^2 - 13\ln v_{oc} + \frac{73}{12}\right\} + \ldots \tag{12}$$

By examination of the process which leads to this result, we can prove that

$$v_m = v_{oc} - \ln v_{oc} + \sum_{k=1}^{+\infty} \frac{1}{v_{oc}^k} P_k(\ln v_{oc}) \tag{13}$$

where $P_k(x)$ are polynomials of degree k.

The various approximations

$$\left(\frac{v_m}{v_{oc}}\right)_{app} = 1 - \frac{1}{v_{oc}}\ln v_{oc} + \sum_{k=1}^{n} \frac{1}{v_{oc}^{k+1}} P_k(\ln v_{oc}) \tag{14}$$

which result from the truncation of the series given by eqn. (13) introduce errors $(v_m/v_{oc}) - (v_m/v_{oc})_{app}$; these are shown in Fig. 3 for $n = 1, 2$ and 3. We have noticed that for $v_{oc} = e$ the error equals zero again and again. Indeed, in this particular case any truncation of the expansion given by eqn. (13) gives

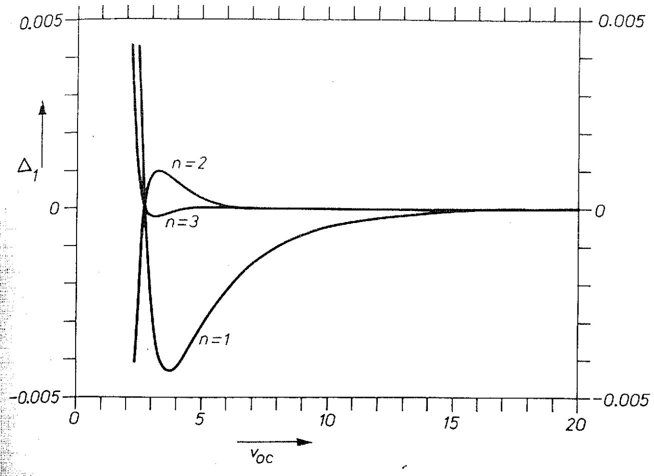

Fig. 3. Error $\Delta_1 = v_m/v_{oc} - (v_m/v_{oc})_{app}$ which was introduced by using the truncated series given by eqn. (14) in order to calculate the (dimensionless) maximum power voltage v_m of an ideal solar cell.

rise to the correct value $v_m = e - 1$, as all $P_k(1) = 0$. For $v_{oc} > e$, which is the case in practical cell design (v_{oc} is of the order of 15 - 40 for most photovoltaic devices), Fig. 3 illustrates the fast convergence of eqn. (12). The absolute convergence for sufficiently large values of v_{oc} can be proved by making use of the Cauchy and Rouché theorems concerning the analytical functions of complex variables [10].

Substituting eqn. (12) into eqns. (9) and (10) leads to the following expansions:

$$\frac{I_m}{I_{sc}} = \frac{\exp(v_{oc})}{\exp(v_{oc}) - 1}\left[1 - \frac{1}{v_{oc}} - \frac{1}{v_{oc}^2}(\ln v_{oc} - 1) - \right.$$
$$- \frac{1}{v_{oc}^3}\{(\ln v_{oc})^2 - 3\ln v_{oc} + 2\} -$$
$$- \frac{1}{v_{oc}^4}\left\{(\ln v_{oc})^3 - \frac{11}{2}(\ln v_{oc})^2 + 9\ln v_{oc} - \frac{9}{2}\right\} -$$
$$\left. - \frac{1}{v_{oc}^5}\left\{(\ln v_{oc})^4 - \frac{25}{3}(\ln v_{oc})^3 + \frac{47}{2}(\ln v_{oc})^2 - 27\ln v_{oc} + \frac{65}{6}\right\} + \ldots\right] \tag{15}$$

and

$$\mathrm{FF} = \frac{\exp(v_{oc})}{\exp(v_{oc}) - 1}\left[1 - \frac{1}{v_{oc}}(\ln v_{oc} + 1) + \frac{1}{v_{oc}^2}\ln v_{oc} + \right.$$
$$+ \frac{1}{v_{oc}^3}\left\{\frac{1}{2}(\ln v_{oc})^2 - \ln v_{oc} + \frac{1}{2}\right\} +$$

$$+ \frac{1}{v_{oc}^4}\left\{\frac{1}{3}(\ln v_{oc})^3 - \frac{3}{2}(\ln v_{oc})^2 + 2\ln v_{oc} - \frac{5}{6}\right\} +$$
$$+ \frac{1}{v_{oc}^5}\left\{\frac{1}{4}(\ln v_{oc})^4 - \frac{11}{6}(\ln v_{oc})^3 + 4(\ln v_{oc})^2 - 4\ln v_{oc} + \frac{19}{12}\right\} + \ldots\Bigg] \tag{16}$$

It turns out that for $v_{oc} \geqslant 2.5$, the series (16) can be truncated after the term $(\ln v_{oc})/v_{oc}^2$ without introducing errors in excess of 0.0014. The resulting approximate formula for FF becomes very condensed

$$\mathrm{FF}(v_{oc}) \approx \left\{\left(1 - \frac{\ln v_{oc}}{v_{oc}}\right)\left(1 - \frac{1}{v_{oc}}\right)\right\}\{1 - \exp(-v_{oc})\}^{-1} \tag{17}$$

Of the three factors involved, the factor $1 - (\ln v_{oc})/v_{oc}$ deviates most from unity, whereas $1 - \exp(-v_{oc})$ is closest to unity. To take an example, for $v_{oc} = 24$ (*i.e.* for $V_{oc} = 600$ mV) eqn. (17) yields $0.867\,58 \times 0.958\,33/1.000\,00 = 0.831\,42$, compared with the exact value $\mathrm{FF}(24) = 0.831\,61$.

If values of FF for v_{oc}s smaller than 2.5 are required (which, however, is very unlikely), then the expansion given by eqn. (16) can be replaced by the following power series (see Appendix C):

$$\mathrm{FF} = \frac{1}{4} + \frac{1}{16}v_{oc} + \frac{1}{256}v_{oc}^2 - \frac{1}{1024}v_{oc}^3 + \ldots \tag{18}$$

In order to have an expression for FF which can be used for all v_{oc} values ranging from 0 to $+\infty$, two partial sums can be combined:

$$\mathrm{FF}(v_{oc}) \approx \begin{cases} \dfrac{1}{4} + \dfrac{1}{16}v_{oc} + \dfrac{1}{256}v_{oc}^2 - \dfrac{1}{1024}v_{oc}^3 & \text{for } v_{oc} \leqslant 4.17 \\ \left\{\left(1 - \dfrac{\ln v_{oc}}{v_{oc}}\right)\left(1 - \dfrac{1}{v_{oc}}\right)\right\}\{1 - \exp(-v_{oc})\}^{-1} & \text{for } v_{oc} \geqslant 4.17 \end{cases} \tag{19}$$

The inner boundary has been chosen to be 4.17 in order that the approximation is a continuous function of v_{oc}. For all values of the argument v_{oc} from 0 to $+\infty$ the approximate formula introduces an absolute error which is smaller than 0.0015. Figure 4 shows the graphs of the two partial sums which were combined to construct the approximation and Fig. 5 shows the error introduced by it. For small arguments the error is $\mathrm{O}(v_{oc}^4)$, whereas for large arguments the error is $\mathrm{O}\{(\ln v_{oc})^2/v_{oc}^3\}$.

3. Influence of non-ideality of the solar cell

From a technical point of view it is interesting to know the FF of a non-ideal solar cell, *i.e.* a cell where $A \neq 1$, $R_s \neq 0$ and $R_{sh} \neq +\infty$. Such a cell has the following I–V characteristic, derived from Fig. 1(b):

$$I(V) = I_0\left\{\exp\left(\frac{qV - qR_sI}{AkT}\right) - 1\right\} - I_1 + \frac{V - R_sI}{R_{sh}} \tag{20}$$

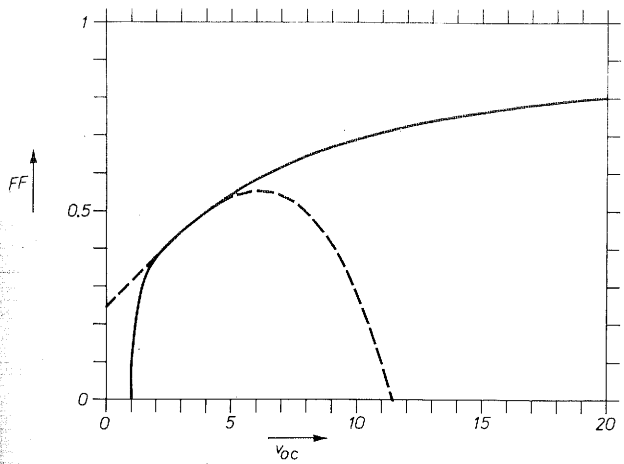

Fig. 4. Approximations for the function $\mathrm{FF}(v_{oc})$: ——, $\{1-(\ln v_{oc})/v_{oc}\}(1-1/v_{oc})/\{1-\exp(-v_{oc})\}$; – – –, $1/4+(1/16)v_{oc}+(1/256)v_{oc}^2-(1/1024)v_{oc}^3$.

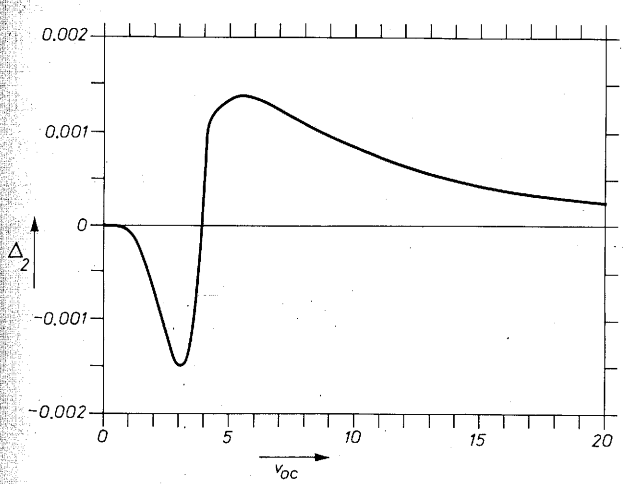

Fig. 5. Error $\Delta_2 = \mathrm{FF} - \mathrm{FF_{app}}$ which was introduced by using the approximate function given by eqn. (19) in order to calculate FF for an ideal solar cell.

The extension of the results of Section 2 for $A \neq 1$ is trivial, as the thermal voltage kT/q only has to be replaced by an effective thermal voltage AkT/q. So, all the results of Section 2 remain valid on the condition that the dimensionless voltages v, v_{oc} and v_m are now interpreted as qV/AkT, qV_{oc}/AkT and qV_m/AkT.

The extension of the results derived in Section 2 to cases where R_s and R_{sh} have finite values is far more complicated. It is traditional to limit oneself to first-order corrections. Therefore the dimensionless series resistance r_s and the dimensionless shunt conductance g_{sh} are introduced [7] as

$$r_s = -\frac{I_{sc}}{V_{oc}} R_s \tag{21}$$

and

$$g_{sh} = -\frac{V_{oc}}{I_{sc}} \frac{1}{R_{sh}} \tag{22}$$

In a well-designed photovoltaic cell $r_s \ll 1$ and $g_{sh} \ll 1$. Therefore the terms in r_s^2, g_{sh}^2, $r_s g_{sh}$, r_s^3 etc. may be neglected when the characteristic values for the cell, such as I_{sc}, V_{oc}, V_m and FF, are calculated. For example, the reader can easily derive from eqn. (20)

$$I_{sc} \approx -I_1 + I_0 \ln\left(\frac{I_1}{I_0} + 1\right) r_s \tag{23}$$

and

$$V_{oc} \approx \frac{AkT}{q} \ln\left(\frac{I_1}{I_0} + 1\right) - \frac{AkT}{q} \frac{I_1}{I_1 + I_0} g_{sh} \tag{24}$$

First-order calculations of V_m and FF are more laborious. In the present section only the main results will be given without going into the details of the calculations.

The first-order expansion of FF will be expressed as follows:

$$\mathrm{FF} \approx \mathrm{FF}_0(v_{oc}) - C(v_{oc}) r_s - D(v_{oc}) g_{sh} \tag{25}$$

The notation $C(v_{oc})$ has been chosen in accordance with ref. 5. After the necessary calculations, it is possible to find the following parameter representation of $\mathrm{FF}_0(v_{oc})$, $C(v_{oc})$ and $D(v_{oc})$:

$$\begin{aligned} v_{oc} &= v_{m0} + \ln(v_{m0} + 1) \\ \mathrm{FF}_0 &= \frac{v_{m0}^2 \exp(v_{m0})}{\{(v_{m0} + 1)\exp(v_{m0}) - 1\}\{v_{m0} + \ln(v_{m0} + 1)\}} \\ C &= \frac{v_{m0}^2 \{\exp(v_{m0}) - 1\}}{\{(v_{m0} + 1)\exp(v_{m0}) - 1\}(v_{m0} + 1)} \\ D &= \frac{v_{m0}[\{v_{m0}^2 - \ln(v_{m0} + 1)\}\exp(v_{m0}) + \ln(v_{m0} + 1)]}{\{(v_{m0} + 1)\exp(v_{m0}) - 1\}\{v_{m0} + \ln(v_{m0} + 1)\}^2} \end{aligned} \tag{26}$$

The parameter has been denoted v_{m0} instead of v_m as in eqn. (11) because it no longer has the physical meaning of qV_m/AkT or even of the first-order approximation of qV_m/AkT, but instead is the zeroth-order approximation of this qV_m/AkT.

When we compare the first two equations of equation set (26) with equation set (11), we can conclude that $\mathrm{FF}_0(v_{oc})$ is the FF function $\mathrm{FF}(v_{oc})$ of the idealized cell. Thus, Fig. 2 represents the curve $\mathrm{FF}_0(v_{oc})$. In order to construct the curves $C(v_{oc})$ and $D(v_{oc})$, we have to vary the value of the parameter v_{m0}. The result is shown in Fig. 6. Curve $C(v_{oc})$ has been published

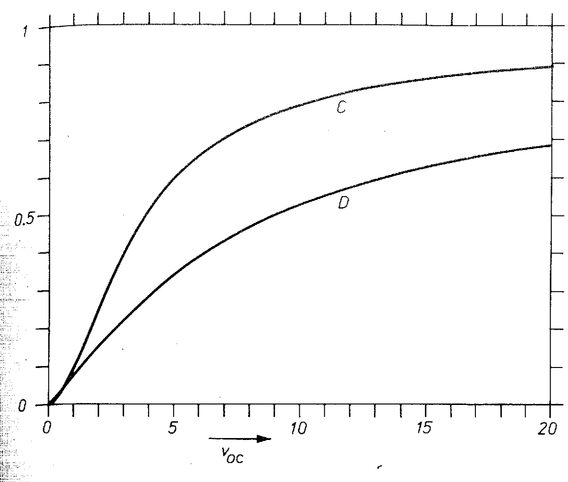

Fig. 6. The specific FF losses C and D of a solar cell, resulting from series resistance and shunt resistance respectively as a function of the dimensionless open-circuit voltage $v_{oc} = qV_{oc}/AkT$.

before [5, 8], but curve $D(v_{oc})$ has not, as far as I have been able to establish.

Explicit and finite analytical expressions for the functions $FF_0(v_{oc})$, $C(v_{oc})$ and $D(v_{oc})$ do not exist. For large values of the argument, asymptotic expansions must again be constructed. For $FF_0(v_{oc})$, the expansion given by eqn. (16) still holds. For the functions C and D we have new expressions.

If we neglect terms containing $\exp(-v_{oc})$, $\exp(-2v_{oc})$ etc., we obtain

$$FF_0(v_{oc}) = 1 - \frac{1}{v_{oc}}(\ln v_{oc} + 1) + \\ + \frac{1}{v_{oc}^2}\ln v_{oc} + \frac{1}{v_{oc}^3}\left\{\frac{1}{2}(\ln v_{oc})^2 - \ln v_{oc} + \frac{1}{2}\right\} + \ldots \tag{27}$$

$$C(v_{oc}) = 1 - \frac{2}{v_{oc}} - \frac{1}{v_{oc}^2}(2\ln v_{oc} - 3) - \\ - \frac{1}{v_{oc}^3}\{2(\ln v_{oc})^2 - 8\ln v_{oc} + 6\} + \ldots \tag{28}$$

and

$$D(v_{oc}) = 1 - \frac{1}{v_{oc}}(2\ln v_{oc} + 1) + \\ + \frac{1}{v_{oc}^2}\{(\ln v_{oc})^2 + 2\ln v_{oc} - 1\} - \\ - \frac{1}{v_{oc}^3}\{(\ln v_{oc})^2 - \ln v_{oc} + 2\} + \ldots \tag{29}$$

Truncation after the terms in $1/v_{oc}^2$ introduces no errors in excess of 0.001 if $v_{oc} \geqslant 15$.

4. Conclusions

In the present paper the problem of calculating the FF of an *ideal* solar cell was investigated in detail. Several ways to solve the problem have been put forward which express FF as a function of the open-circuit voltage V_{oc}: a parameter representation; a continued logarithm representation; an asymptotic series expansion in $1/V_{oc}$; a Taylor expansion in V_{oc}. The various methods can be used for various applications such as curve tracing, computer calculations, approximations for large V_{oc} and, finally, approximations for small V_{oc}.

The calculation of FF for an *actual* solar cell, showing the series and shunt losses, is a harder problem. The case of greatest technological interest (large open-circuit voltage V_{oc}, small series resistance R_s and large shunt resistance R_{sh}) has been treated by the parameter representation method and the asymptotic expansion method.

References

1 W. Shockley and H. Queisser, Detailed balance limit of efficiency of p–n junction solar cells, *J. Appl. Phys., 32* (1961) 510 - 519.
2 J. Lindmayer, Theoretical and practical fill factors in solar cells, *COMSAT Tech. Rev., 2* (1972) 105 - 121.
3 H. Hovel, *Solar Cells,* Vol. 11, *Semiconductors and Semimetals,* New York, 1975, p. 61.
4 M. Green, General solar cell curve factors including the effects of ideality factor, temperature and series resistance, *Solid-State Electron., 20* (1977) 265 - 266.
5 A. Rothwarf and A. Barnett, Design analysis of the thin-film CdS–Cu_2S solar cell, *IEEE Trans. Electron Devices, 24* (1977) 381 - 387.
6 P. Landsberg, Recombination in solar cells: theoretical aspects, in F. Cardon, W. Gomes and W. Dekeyser (eds.), *Photovoltaic and Photoelectrochemical Solar Energy Conversion,* Plenum, New York, 1981, pp. 1 - 66.
7 R. Hall, Silicon photovoltaic cells, *Solid-State Electron., 24* (1981) 595 - 616.
8 C. Singal, Analytical expressions for the series-resistance-dependent maximum power point and curve factor for solar cells, *Sol. Cells, 3* (1981) 163 - 177.
9 J. Parrott, Analysis of an edge-illuminated graded-gap solar cell, *Solid-State Electron Devices, 2* (1978) S79 - S82.
10 N. De Bruijn, *Asymptotic Methods in Analysis,* Dover, New York, 1981, pp. 21 - 32.

Appendix A

From the transcendental equation

$$v_m + \ln(v_m + 1) = v_{oc} \tag{A1}$$

we construct a new simpler equation by applying the substitutions

$$w_{oc} = v_{oc} + 1 \tag{A2}$$

and

$$w_m = v_m + 1 \tag{A3}$$

This yields

$$w_m + \ln w_m = w_{oc} \tag{A4}$$

or

$$w_m = w_{oc} - \ln w_m \tag{A5}$$

Substituting the w_m on the right-hand side of eqn. (A5) by the complete right-hand side of eqn. (A5) gives

$$w_m = w_{oc} - \ln(w_{oc} - \ln w_m) \tag{A6}$$

Substituting again the w_m on the right-hand side of the equation by $w_{oc} - \ln w_m$, we obtain

$$w_m = w_{oc} - \ln\{w_{oc} - \ln(w_{oc} - \ln w_m)\} \tag{A7}$$

If the above substitution is applied over and over again, this leads to

$$w_m = w_{oc} - \ln(w_{oc} - \ln[w_{oc} - \ln\{\ldots(w_{oc} - \ln w_{oc})\ldots\}]) \tag{A8}$$

or

$$v_m = v_{oc} - \ln(v_{oc} + 1 - \ln[v_{oc} + 1 - \ln\{\ldots(v_{oc} + 1 - \ln v_{oc})\ldots\}]) \tag{A9}$$

These expressions could be called "infinite continued logarithms" by analogy with the infinite continued fractions. They give rise to a very simple algorithm for the computation of v_m with a desired accuracy a for a given v_{oc} (see Table A1). The larger the argument v_{oc} is, the faster is the convergence. For example, if $v_{oc} = 24$ (or $V_{oc} = 600$ mV), this gives rise to a value of v_m with an accuracy $a = 10^{-10}$ after only ten iterations: $v_m = 20.912\,923\,426\,4$.

TABLE A1

```
1 input v_oc
2 w_oc = v_oc + 1
3 w_m = w_oc
4 w_m = w_oc - log(w_m)
5 if abs{w_m - w_oc + log(w_m)} > a goto 4
6 v_m = w_m - 1
7 output v_m
```

Appendix B

In order to solve

$$v_m + \ln(v_m + 1) = v_{oc} \tag{B1}$$

for large v_{oc}, we must first point out that, for large v_{oc}, v_m is also large and therefore dominates $\ln(v_m + 1)$. Thus, we have to a first approximation

$$v_m \approx v_{oc} \tag{B2}$$

From this we can deduce that $\ln(v_m + 1)$ behaves like $\ln v_{oc}$ and that it can therefore be regarded as a small correction to eqn. (B2):

$$v_m = v_{oc} - \ln(v_m + 1) \tag{B3}$$

Now we substitute the v_m on the right-hand side of eqn. (B3) by the complete right-hand side of eqn. (B3):

$$v_m = v_{oc} - \ln\{v_{oc} - \ln(v_m + 1) + 1\} \tag{B4}$$

We can split off the dominant part of the second term

$$v_m = v_{oc} - \ln v_{oc} - \ln\left\{1 - \frac{1}{v_{oc}}\ln(v_m + 1) + \frac{1}{v_{oc}}\right\} \tag{B5}$$

and use the expansion

$$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

in order to find

$$v_m = v_{oc} - \ln v_{oc} + \frac{1}{v_{oc}}\{\ln(v_m + 1) - 1\} + \\ + \frac{1}{2}\frac{1}{v_{oc}^2}[\{\ln(v_m + 1)\}^2 - 2\ln(v_m + 1) + 1] + \ldots \tag{B6}$$

Again substituting each v_m of the new right-hand side by $v_{oc} - \ln(v_m + 1)$ and again expanding the logarithmic functions, we obtain

$$v_m = v_{oc} - \ln v_{oc} + \frac{1}{v_{oc}}(\ln v_{oc} - 1) + \\ + \frac{1}{v_{oc}^2}\left\{\frac{1}{2}(\ln v_{oc})^2 - \ln v_{oc} - \ln(v_m + 1) + \frac{3}{2}\right\} + \ldots \tag{B7}$$

Proceeding in this way, we will finally arrive at the asymptotic expansion eqn. (12).

Appendix C

If we realize that $\ln v_{oc}$ can also be written $-\ln(1/v_{oc})$, it becomes clear that the expansions (12), (15) and (16) are expansions in $1/v_{oc}$ and are therefore only interesting for small values of $1/v_{oc}$ and thus for large values of v_{oc}. Therefore, for small v_{oc}, an expansion of FF in terms of v_{oc} itself instead of in terms of $1/v_{oc}$ is necessary.

For this purpose, we proceed as follows. For $v_{oc} \to 0$ we see from eqn. (8) that $v_m \to 0$. Therefore the left-hand side of the transcendental equation eqn. (8) is expanded as a power series in v_m:

$$v_{oc} = 2v_m - \frac{1}{2}v_m^{\,2} + \frac{1}{3}v_m^{\,3} - \frac{1}{4}v_m^{\,4} + \ldots \tag{C1}$$

In order to reverse this formula, each v_m on the right-hand side of eqn. (C1) is substituted by

$$a_1 v_{oc} + a_2 v_{oc}^{\,2} + a_3 v_{oc}^{\,3} + a_4 v_{oc}^{\,4} + \ldots \tag{C2}$$

which yields

$$v_{oc} = 2a_1 v_{oc} + \left(2a_2 - \frac{1}{2}a_1^{\,2}\right) v_{oc}^{\,2} + + \left(2a_3 - a_1 a_2 + \frac{1}{3}a_1^{\,3}\right) v_{oc}^{\,3} + \ldots \tag{C3}$$

If the corresponding coefficients of equal powers of v_{oc} on both sides of this equation are equated, this leads consecutively to the values of a_1, a_2, a_3 etc. We then have v_m expanded in powers of v_{oc}:

$$v_m = \frac{1}{2}v_{oc} + \frac{1}{16}v_{oc}^{\,2} - \frac{1}{192}v_{oc}^{\,3} - \frac{1}{3\,072}v_{oc}^{\,4} + + \frac{13}{61\,440}v_{oc}^{\,5} - \frac{47}{1\,474\,560}v_{oc}^{\,6} - \frac{73}{41\,287\,680}v_{oc}^{\,7} + \ldots \tag{C4}$$

We can substitute this result into eqns. (9) and (10), taking into account that

$$\frac{\exp(v_{oc})}{\exp(v_{oc}) - 1} = 1 + \sum_{k=0}^{+\infty} \frac{B_k}{k!} v_{oc}^{\,k-1} = \frac{1}{v_{oc}} + \frac{1}{2} + \frac{1}{12}v_{oc} - \frac{1}{720}v_{oc}^{\,3} + \ldots \tag{C5}$$

where B_k are the Bernouilli numbers. This finally gives rise to

$$\mathrm{FF} = \frac{1}{4} + \frac{1}{16}v_{oc} + \frac{1}{256}v_{oc}^{\,2} - \frac{1}{1\,024}v_{oc}^{\,3} - - \frac{7}{73\,728}v_{oc}^{\,4} + \frac{37}{1\,474\,560}v_{oc}^{\,5} + \frac{287}{141\,557\,760}v_{oc}^{\,6} + \ldots \tag{C6}$$

It is worth mentioning that only the first five terms of series (C4) and (C6) were calculated "by hand". I committed the computation of the higher order terms, an inhumanly laborious task, to the charge of a computer using suitably ingenious programs [C1].

Reference for Appendix C

C1 A. Nijenhuis and H. Wilf, *Combinatorial Algorithms*, Academic Press, New York, 1975, pp. 134 - 142.