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Extension of the application of conformal mapping techniques to coplanar lines with finite dimensions

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Conformal mapping techniques are used to obtain closed-form expressions for the coplanar line characteristic impedance and its effective dielectric constant in the general case in terms of all finite line dimensions and substrate thickness. Also, closed-form expressions are presented for the case where there is a ground plane on the other substrate surface. These expressions, that can be calculated manually, are in very good agreement with published experimental results.

1. Introduction

Calculations of wave characteristic impedance and effective dielectric constant of coplanar lines by conformal mapping techniques were first presented by Wen (1969). Although his analysis leads to simple analytical expressions for the line parameters, they are valid only for the case of infinite substrate thickness and infinite dimensions for the two side ground strips.

Davis *et al.* (1973) extended Wen's analysis to the case of finite substrate thickness, still assuming infinite width to the two ground strips. The expressions obtained by them were not simple ones that can be solved analytically. They concluded that Wen's simple expressions are valid for substrate thickness greater than twice the slot width, a condition that can hardly be achieved for lines having high characteristic impedances. Several trials were made to calculate line parameters taking into account the effect of finite substrate thickness and finite line dimensions using numerical methods like the relaxation method proposed by Hatsuda (1975) or involving Green's function as presented by Houdart (1976).

These computations are iterative in nature and no closed-form expressions for line parameters in the general case can be extracted from the literature.

Here, we present closed-form expressions for the coplanar line impedance and its effective dielectric constant in the general case in terms of finite line dimensions and substrate thickness using conformal mapping techniques. Also closed-form expressions are presented for the case when there is a ground plane on the other substrate surface; a case that, to our knowledge, has not been treated before.

2. General analysis

Still within the zeroth-order approximation of a quasi-TEM structure and referring to the line configuration shown in Fig. 1, our analysis is based on the assumption that when the line dimensions a and h are finite, the line capacity between the centre strip and the two ground strips is equal to the sum of the line capacity C_1 in the absence of the dielectric and the line capacity C_2 when assuming that all the electric field is concentrated in the dielectric whose relative permittivity is $(\epsilon_r - 1)$.

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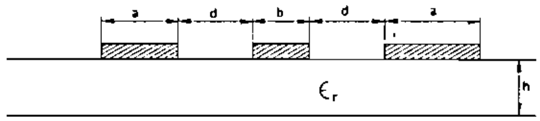

Figure 1. Coplanar surface strip transmission line.

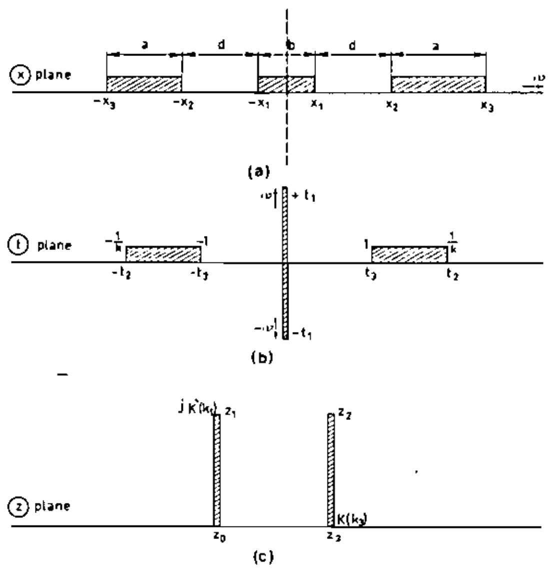

Figure 2. Conformal mapping transformations for the calculation of the line capacitance C_1 in the absence of the dielectric.

2.1. *Calculation of the capacity C_1*

As a first step towards the solution of this boundary-value problem, we will map the boundary of the x plane (Fig. 2(a)) into the real axis in the t plane (Fig. 2(b)) using the mapping function:

$$t=\frac{x}{x_3}\sqrt{[(x_3^2-x_1^2)/(x^2-x_1^2)]} \qquad \text{for } |x|>|x_1| \tag{1 a}$$

and

$$t=j\frac{x}{x_3}\sqrt{[(x_3^2-x_1^2)/(x_1^2-x^2)]} \qquad \text{for } |x|\leqslant|x_1| \tag{1 b}$$

noting that:

$$x_1=\frac{b}{2},\ x_2=\frac{b}{2}+d \quad \text{and} \quad x_3=\frac{b}{2}+d+a$$

This configuration in the t plane in turn can be mapped into a rectangle in the z plane (fig. 2(c)) using the mapping function:

$$t=\operatorname{sn}(z,k_1) \tag{2}$$

where $\mathrm{sn}(z, k_1)$ is the elliptic function and k_1 is given by:

$$k_1 = \frac{t_3}{t_2} = \frac{x_3}{x_2}\sqrt{\frac{x_2^2 - x_1^2}{x_3^2 - x_1^2}} \tag{3 a}$$

$$k_1 = \left(1 + \frac{2a}{2d+b}\right)\sqrt{\frac{\left(1+\frac{b}{d}\right)}{\left(1+\frac{a}{d}+\frac{b}{d}\right)\left(1+\frac{a}{d}\right)}} \tag{3 b}$$

Hence, the capacitance per unit length of the coplanar line is four times that of the mapping of one quarter section in the z plane, i.e.:

$$C_1 = 4\epsilon_0 \frac{K'(k_1)}{K(k_1)} \tag{4}$$

where $K(k_1)$ is the complete elliptic integral of the first kind, and $K'(k) = K(k')$ where $k' = \sqrt{(1-k^2)}$.

2.2. *Calculation of the capacity* C_2

The boundary problem in the x plane (Fig. 3 (a)) can be transformed in another boundary value problem in the t plane (Fig. 3 (b)) using the mapping function

$$t = \sin\mathrm{h}\left(\frac{\pi x}{2h}\right)$$

This boundary value problem in the t plane can be mapped into a rectangle in a final W' plane using the same transformations utilized for the structure of Fig. 2. Hence.

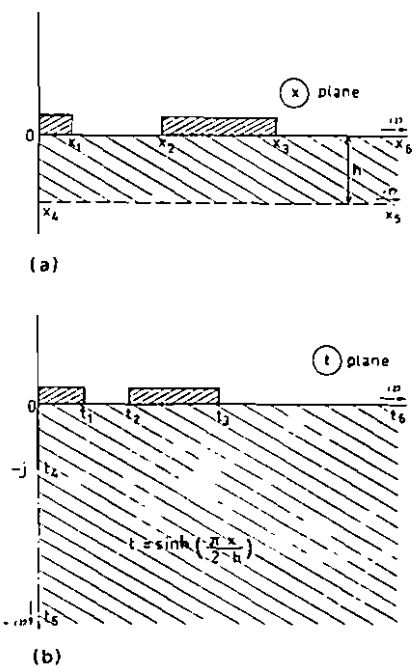

Figure 3. Conformal mapping transformation for the calculation of the line capacity C_2.

the capacitance per unit length of the coplanar line is twice that of the mapping of the half section in the W' plane considering that all the electric field lines are concentrated in the dielectric whose relative permittivity is $(\epsilon_r - 1)$, i.e.:

$$C_2 = 2\epsilon_0(\epsilon_r - 1)\frac{K'(k_2)}{K(k_2)} = 2\epsilon_0(\epsilon_r - 1)\frac{K(k'_2)}{K'(k'_2)} \tag{5}$$

where

$$k_2 = \frac{t_3}{t_1}\sqrt{\frac{t_2^2 - t_1^2}{t_3^2 - t_1^2}} \quad \text{and} \quad k'_2 = \sqrt{(1 - k_2^2)} \tag{6}$$

after several simplifications, k'_2 can be put on the form:

$$k'_2 = \frac{\sinh\left(\frac{\pi b}{4h}\right)}{\sinh\left[\frac{\pi}{2h}\left(\frac{b}{2} + d\right)\right]}\sqrt{\frac{\sinh^2\left[\frac{\pi}{2h}\left(\frac{b}{2} + d + a\right)\right] - \sinh^2\left[\frac{\pi}{2h}\left(\frac{b}{2} + d\right)\right]}{\sinh^2\left[\frac{\pi}{2h}\left(\frac{b}{2} + d + a\right)\right] - \sinh^2\left(\frac{\pi b}{4h}\right)}} \tag{7}$$

Hence, the effective dielectric constant of the coplanar line can be defined as:

$$\epsilon_{eff} = \frac{C_1 + C_2}{C_1} = 1 + \frac{C_2}{C_1} \tag{8}$$

Substituting from eqns. (4) and (5) into (8), we obtain:

$$\epsilon_{eff} = 1 + \frac{(\epsilon_r - 1)}{2}\frac{K'(k_2)}{K(k_2)}\frac{K(k_1)}{K'(k_1)} \tag{9}$$

From transmission line theory, it is known that

$$u_{ph} = \frac{c}{\sqrt{\epsilon_{eff}}} \quad \text{and} \quad Z_0 = \frac{1}{(C_1 + C_2)u_{ph}} \tag{10}$$

Where c is the velocity of wave propagation in free space and u_{ph} is the phase propagation velocity.

Hence, the line characteristic impedance Z_0 can be put in the form:

$$Z_0 = \frac{30\pi}{\sqrt{\epsilon_{eff}}}\frac{K(k_1)}{K'(k_1)} \tag{11}$$

Equations (9) and (11) give closed-form expressions for the coplanar line parameters which can be manually calculated using Hilberg's (1969) approximate expressions for the general function $[K(k)/K'(k)]$:

$$\frac{K(k)}{K'(k)} \simeq \frac{2}{\pi} ln\left[2\sqrt{\frac{1+k}{1-k}}\right] \qquad \text{for } \frac{1}{\sqrt{2}} \leqslant k \leqslant 1 \tag{12a}$$

$$\frac{K(k)}{K'(k)} \simeq \frac{(\pi/2)}{ln\left[2\sqrt{\frac{1+k'}{1-k'}}\right]} \qquad \text{for } 0 \leqslant k \leqslant \frac{1}{\sqrt{2}} \text{ and } k' = \sqrt{(1-k^2)} \tag{12b}$$

Hence, Z_0 and ϵ_{eff} can be calculated easily either manually or by the aid of a simple hand calculator for any general line configuration and substrate thickness.

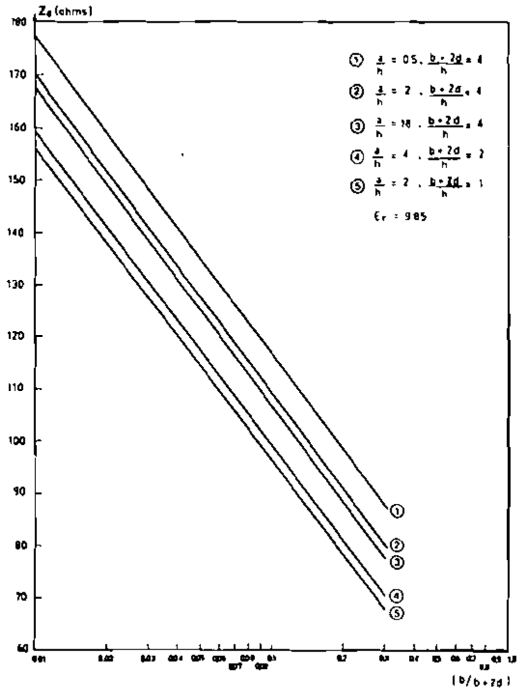

Figure 4. Line characteristic impedance as a function of the ratio $b/(b+2d)$ for the shown values for a/h and $(b+2d)/h$.

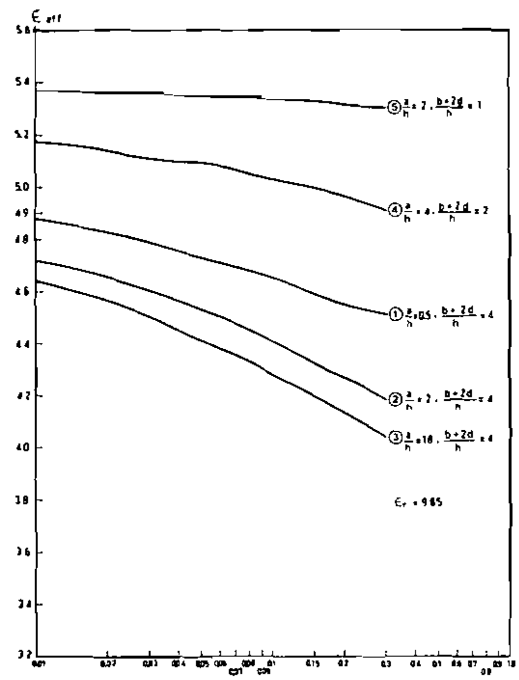

Figure 5. Line effective relative dielectric constant as a function of the ratio $b/(b+2d)$ for the shown values for a/h and $(b+2d)/h$.

Figures (4) and (5) give the characteristic impedance and the effective dielectric constant for the shown line parameters. These results differ by less than 1·5% from that published by Houdart (1976) which were calculated with the aid of a digital computer program. We have calculated the effective relative permittivities of the coplanar lines realised by Müller (1977), whose dimensions are shown in his figures 1 and 2, using our simple closed-form expressions and compared them with the measured ones at a frequency of 4 GHz. This comparison is shown in the table; from it we see that the calculated values differ by less than 4% from the measured ones.

Fig. 1 line	ϵ_{eff} (experimental)	ϵ_{eff} (calculated)	Error %	Fig. 2 line	ϵ_{eff} (experimental)	ϵ_{eff} (calculated)	Error %
1	4·737	4·777	0·84%	1	4·75	4·811	1·26%
2	4·237	4·282	1·06%	2	4·5	4·555	1·22%
3	3·627	3·545	2·26%	3	4·17	4·154	0·38%
4	3·00	2.866	4·477%	4	3·7	3·670	0·81%
5	4·41	4·357	1·2 %	5	3·23	3·159	2·19%

Table.

3. Coplanar lines with additional ground plane

In order to fix our attention on the effect of the existence of another ground plane on the other substrate surface, we will assume that $a=\infty$ and the boundary value problem in the x plane will be as shown in Fig. 6(a).

The total line capacity can be considered to be equal to the sum of the capacity of the line C_1 in the half free space and its capacity C_2 in the other half space filled with the dielectric. Hence C_1 is calculated in the same manner described in § 2·1 noting that C_1 will be only twice that of the mapping of one quarter section in the z plane, i.e.:

$$C_1 = 2\epsilon_0 \frac{K'(k_1)}{K(k_1)} \tag{13}$$

and when $a=\infty$, k_1 is given by:

$$k_1 = \left(1+\frac{b}{d}\right)^{1/2} \Big/ \left(1+\frac{b}{2d}\right) \tag{14}$$

To calculate C_2, the original boundary problem in the x plane can be transformed into a symmetrical boundary value problem in a W plane using the following intermediate mapping functions:

boundary value problem in t plane: $t=\cosh^2\left(\frac{\pi x}{2h}\right)$

boundary value problem in z plane: $z=t-\lambda$

boundary value problem in W plane $W=\frac{z(z^2+\alpha\lambda^4)}{\lambda^3(1+\alpha z^2)k_3}$

where:

$$\lambda = \frac{1}{2}\cosh^2\left[\frac{\pi}{2h}\left(\frac{b}{2}+d\right)\right] \tag{15}$$

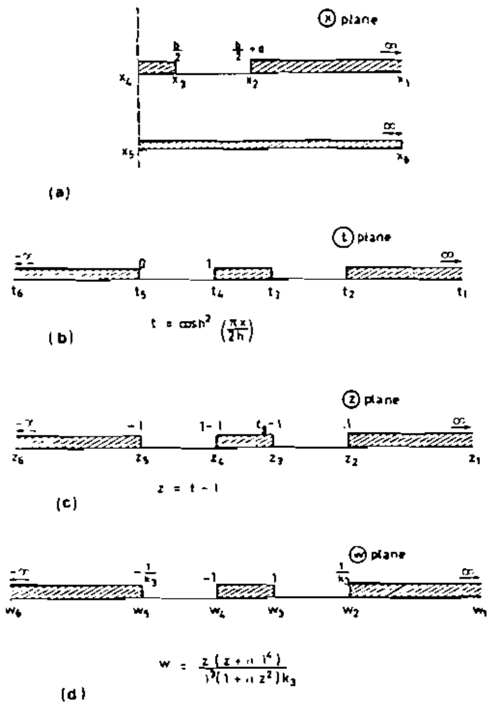

Figure 6. Conformal mapping transformations for the calculation of the coplanar line parameters when there is a ground plane on the other substrate surface.

α is the solution of the simple second order equation:

$$(\lambda^4 z_3 z_4)\alpha^2 + (\lambda^4 + z_3^2 z_4^2)\alpha + (z_3^2 - z_3 z_4 + z_4^2) = 0 \tag{16}$$

and

$$k_3 = \frac{z_3(z_3^2 + \alpha\lambda^4)}{\lambda^3(1 + \alpha z_3^2)} \tag{17}$$

This final boundary value problem in the W plane can be mapped into a rectangle in a final W' plane using the method described in §2.1. Hence, C_2 is given by:

$$C_2 = 2\epsilon_0 \epsilon_r \frac{K(k_3)}{K'(k_3)} \tag{18}$$

from the relation $C_1 + C_2 = 2C_1\epsilon_{eff}$, using eqns. (13) and (18), we obtain:

$$\epsilon_{eff} = \frac{1}{2}\left[1 + \epsilon_r \frac{K(k_3)}{K'(k_3)} \frac{K(k_1)}{K'(k_1)}\right] \tag{19}$$

Using eqn. (10), the characteristic impedance can be given by:

$$Z_0 = \frac{30\pi}{\sqrt{\epsilon_{eff}}} \frac{K(k_1)}{K'(k_1)} \tag{20}$$

Again, eqns. (19) and (20) represent closed-form expressions for line parameters that can be easily calculated. Figs 7 and 8 show the characteristic impedance and the

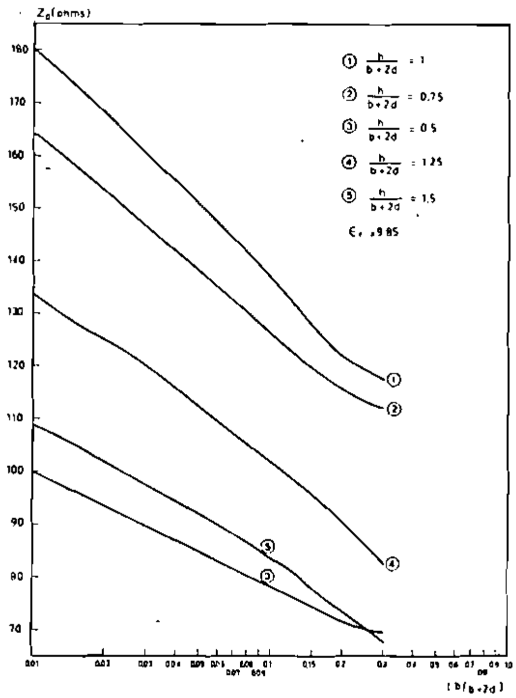

Figure 7. Line characteristic impedance as a function of the ratio $b/(b+2d)$, taking $h/(b+2d)$ as a parameter for $a=\infty$ and for the case where there is another ground plane.

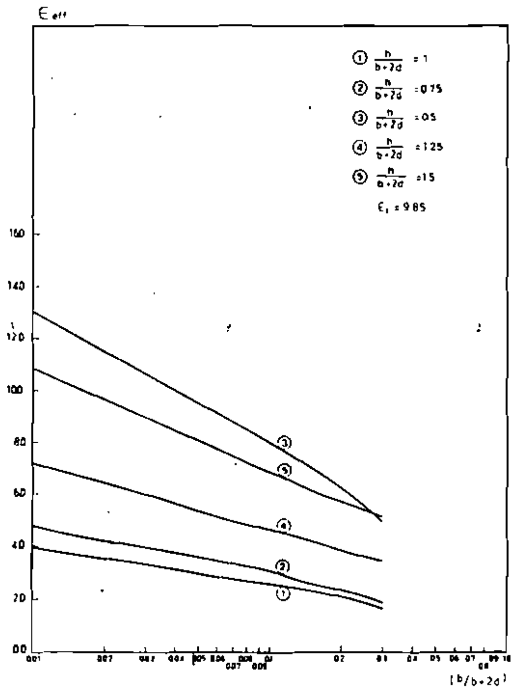

Figure 8. Line effective relative dielectric constant as a function of the ratio $b/(b+2d)$ taking $h/(b+2d)$ as a parameter for $a=\infty$ and for the case where there is another ground plane.

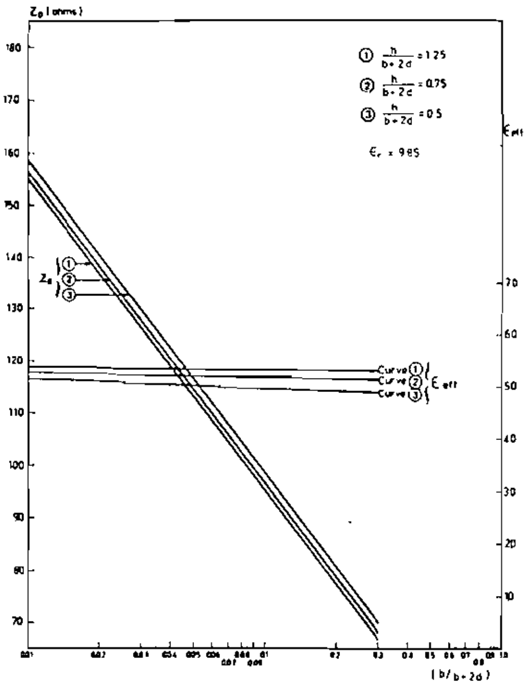

Figure 9. Characteristic impedance and effective dielectric constant for a coplanar line as a function of the ratio $b/(b+2d)$ taking $h/(b+2d)$ as a parameter for the case $a=\infty$.

effective relative dielectric line constant respectively. For purpose of comparison, the parameters of a coplanar line, without this additional ground plane for which $a=\infty$, are shown in Fig. 9. As is expected, the existence of another metallic plane, on the other substrate surface, allows the propagation of a wave of the microstrip type that is superimposed on the coplanar propagating wave, and the result is a complex propagating TEM wave. Increasing the substrate thickness h, while keeping the line dimensions b and d constant, has opposite effects on the propagation of the coplanar and the microstrip modes, so the characteristic impedance and the effective relative dielectric constant have maximum and minimum values respectively. In addition, the change in ε_{eff} as a function of line geometry is a drastic one and not like the case of a simple coplanar line without a ground plane.

4. Conclusion

The closed-form expressions obtained for the coplanar line characteristic impedance and its effective dielectric constant in the general case in terms of all line dimensions and substrate thickness using conformal mapping techniques represent a new and an easier way of calculating line parameters that agree with published experimental results. The addition of a ground plane on the other substrate surface allows the propagation of a complex TEM wave and the line parameters obtained in this case will be highly dependent on the ratio between line dimensions and substrate thickness.

References

Davis, M. E., Williams, E. W., and Celestini, A. C., 1973, *I.E.E.E. Trans. microw. Theory Tech.*, **21**, 594.

Hatsuda, T., 1975, *I.E.E.E. Trans. microw. Theory Tech.*, **23**, 795.

Hilberg, W., 1969, *I.E.E.E. Trans. microw. Theory Tech.*, **17**, 259.

Houdart, M., 1976, *Proceedings of the sixth European Microwave Conference*, p. 49.

Müller, E., 1977, *Electron. Lett.*, **13**, 729.

Wen, C. P., 1969, *I.E.E.E. Trans. microw. Theory Tech.*, **17**, 1087.