Gifford–McMahon cycle — a theoretical analysis

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A theoretical analysis of the Gifford–McMahon cycle is presented. Expressions for the ideal refrigeration produced and the figure of merit are developed. Various losses occurring in a real machine are considered and equations to account for these losses are derived. Results are presented in graphical form.

Keywords: Gifford–McMahon cycle; ideal refrigeration; loss analysis

Nomenclature†

A	Area
C	Capacity ratio
c_p	Specific heat of gas at constant pressure
c_v	Specific heat of gas at constant volume
D, d	Diameter
D_e, d_e	Equivalent diameter
G	Mass velocity
H, h	Enthalpy
h, h_c	Heat transfer coefficient
I	Regenerator inefficiency
k	Thermal conductivity
L, l	Length
M	Molecular weight
m	Mass
$\dot{m}$	Mass flow rate of gas
NTU	Number of transfer units
P_H	High pressure
P_L	Low pressure
PR	Pressure ratio
ΔP	Pressure drop
Q	Heat
Q_L	Refrigeration load
ΔQ	Refrigeration loss
R	Gas constant
S	Entropy; stroke length
ΔS	Entropy change
T	Temperature
T_L	Load temperature
T_{L1}	First stage temperature
T_{L2}	Second stage temperature
TR	Temperature ratio, T_0/T_L
TR1	First stage temperature ratio, T_0/T_{L1}
TR2	Second stage temperature ratio, T_0/T_{L2}
T_0	Ambient temperature
t	Time; thickness
V	Displaced volume
V_1	First stage displaced volume
V_2	Second stage displaced volume
VR	Volume ratio, V_1/V_2
V_{RV}	Regenerator void volume
V_{RV1}	First stage regenerator void volume
V_{RV2}	Second stage regenerator void volume
W_c	Work input to compressor
W_e	Work output of expander
x	Distance; void volume fraction; pressure drop fraction; regenerator void volume ratio, V_{RV}/V
y	First stage regenerator void volume ratio, V_{RV1}/V_1
z	Second stage regenerator void volume ratio, V_{RV2}/V_2

†Symbols used are defined in the text as and when they occur. The Nomenclature gives only a general list of terms used

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Nomenclature contd.

Dimensionless terms

Nu	Nusselts number, hD_e/k
Pr	Prandtl number, $c_p\mu/k$
Re	Reynolds number, GD_e/μ
St	Stanton number, h/Gc_p

Greek letters

ε	Heat exchanger efficiency, emissivity
γ	Ratio of specific heats, c_p/c_v
ρ	Density
σ	Stefan–Boltzmann constant
τ	Time
μ	Viscosity

In this Paper a thermodynamic analysis of the Gifford–McMahon (G–M) cycle is presented and expressions for the ideal refrigeration produced and figure of merit (FOM) are developed. The various losses reducing the refrigeration available in a real machine are then considered. Mathematical expressions to account for these losses are developed and the results shown in graphical form.

Analysis of the Gifford–McMahon cycle

Gifford–McMahon cycle cryorefrigerators

Gifford and McMahon[1] reported on two novel refrigeration systems which have found ready applications in small capacity cryorefrigerators. Both the systems are basically similar; one has a single expansion volume whereas the other has two volumes. Methods of computing the performance of the G–M cycle are reported by several authors[2–7].

Working of the double volume G–M cycle cryorefrigerator and ideal refrigeration produced

A schematic diagram of the system and the *P–V* diagram for the double volume G–M cycle refrigerator are shown in *Figures 1* and *2*, respectively. The refrigerator consists of a thin-walled stainless steel cylinder inside which is a moveable displacer made of material with low thermal conductivity, such as micarta. The two volumes thus formed are connected to each other through a regenerator. Gas enters and leaves the system through valves, as shown. This is a 'no work' cycle, since there is no net work output from the shaft. Instead, it operates as a heat pump, pumping the heat from lower volume to upper volume. The sequence of operation is as follows:

Process 1–2. The displacer is at its lowest position, the exhaust valve is closed and the inlet valve is open. Pressure in the upper expansion space and regenerator increases from P_1 to P_2.

Process 2–3. The inlet valve remains open and the displacer moves up, thus displacing the gas to the lower expansion space through the regenerator. If the regenerator is already cold from a previous cycle, the gas gets cooled in it and more gas will flow in through the inlet valve.

Process 3–4. When the displacer reaches the top of its stroke, the inlet valve is closed and the exhaust valve is open. At this point the gas in the expansion volume undergoes 'Joule expansion', i.e. the gas remaining in

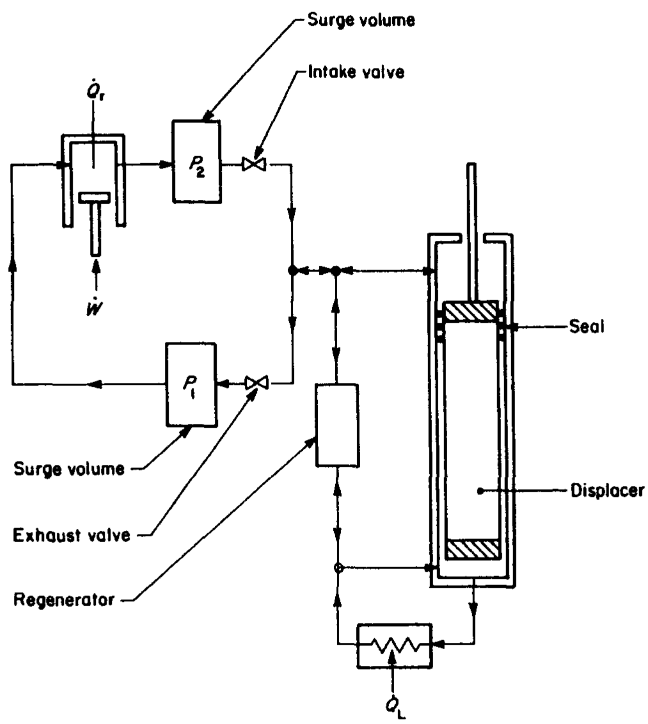

Figure 1 Double volume G–M cycle refrigerator

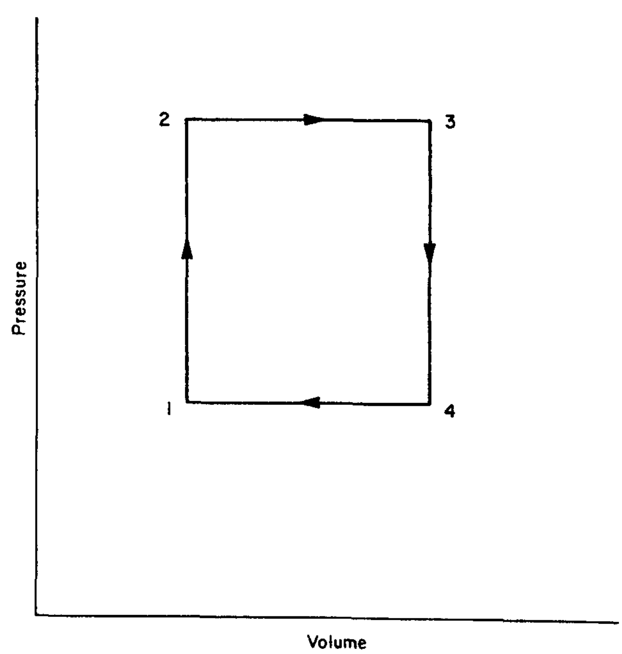

Figure 2 Pressure–volume diagram for double volume G–M cycle refrigerator

the volume after expansion has done work in pushing the gas that has gone out, thus producing cooling.

Process 4–5. The displacer now moves down, with the exhaust valve still open. The cold gas remaining in the lower expansion volume moves out, thus providing refrigeration.

Process 5–1. After providing refrigeration, the gas enters the regenerator and is warmed as it flows out, cooling the regenerator. This will precool the next batch of gas when the cycle is repeated.

G–M cycle cryorefrigerators have been built which provide refrigeration at 80 K in the first stage and at 30 and 15 K in the second and third stages[3]. The three stage cryorefrigerator was used to precool a separate Joule–Thomson (J–T) stream of high pressure helium in a small helium liquefier producing 0.5 dm^3 h^{-1} of liquid helium.

Salient features of the G–M cycle machines are summarized below.

1 They are highly reliable with mean time between failures (MTBF) of the order of 10 000 h, as there is less wear and tear of seals and other mechanisms due to slower running speeds (60–150 rev min^{-1}).
2 The compressor portion is isolated, unlike Stirling machines, which minimizes vibration and increases convenience and flexibility.
3 The valves and seals are at room temperature.
4 They have a simpler driving mechanism, as it is designed only to take into account the friction of the seals and the small pressure differential across the regenerator (in the double volume system).
5 The clearance between the cylinder and displacer is not as critical as in other machines. Clearance of ≈0.005–0.010 in (0.075–0.25 mm) is sufficient.
6 They make use of highly efficient thermal regenerators which are easy to build.
7 They are less sensitive to gas impurities because during the reverse flow in each cycle the condensed impurities are blown out.
8 Multistage G–M cycle machines are easy to build. Adding another stage simply involves addition of an expansion volume and a regenerator. The same inlet and exhaust valves can be used at room temperature. Stuart *et al*[8] have described the performance of a 4 K refrigerator in which they used a coaxial three stage G–M cycle machine to precool the J–T stream. Another three stage machine has produced 6.5 K in the third stage[9].

Efficiencies of these machines are quite low. For example, for practical 4.2 K refrigerators, the figure of merit (or per cent Carnot) of G–M cycle machines varies from ≈ 1 to 4%, as shown by Strobridge and Chelton[10]. G–M cycle refrigerators have still become quite popular because of their simplicity and high reliability. A detailed survey of commercially available cryogenic refrigerators given by Crawford[11] contains additional data on G–M cycle refrigerators.

For the sake of analysis, it is assumed that the regenerator is 100% efficient, that there are no void volumes in the regenerator or at the cold end, and that the working gas behaves like an ideal gas. Then, referring to *Figure 3* and considering the control volume 1, if the *I* Law[12] is applied

$$Q_L = W + (H_{out} - H_{in})$$

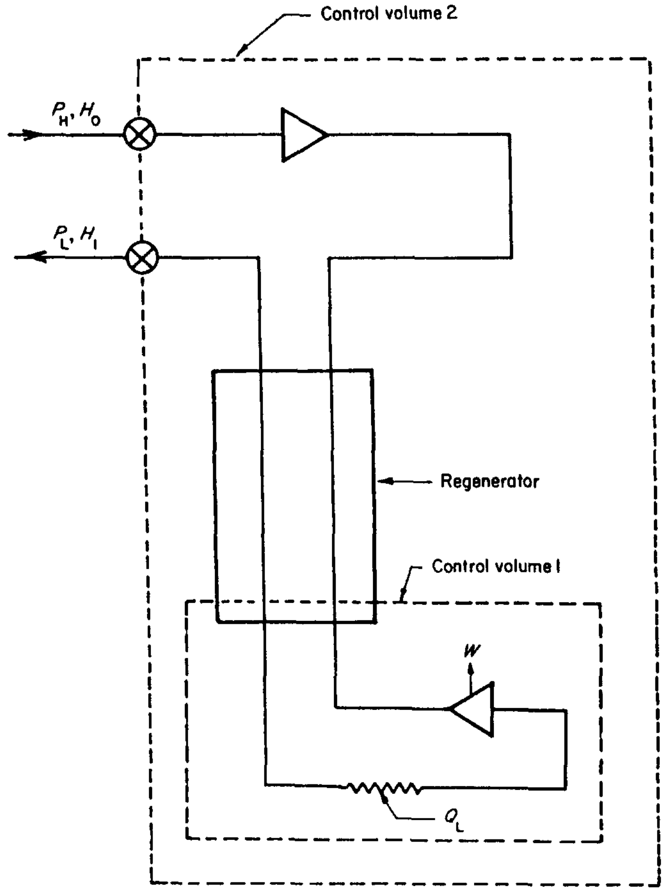

Figure 3 G–M cycle control volumes

where:

Q_L = refrigeration load;
W = work of expansion;
H_{out} = enthalpy of gas going out from control volume 1; and
H_{in} = enthalpy of gas going into control volume 1.

But $(H_{out} - H_{in}) = 0$, since there is no temperature difference at the cold end of the regenerator for a 100% efficient regenerator. For a perfect gas, enthalpy is only a function of temperature and does not depend upon pressure. Therefore

$$Q_L = W$$

where W is equal to the area traced on the P–V diagram, $= (P_H - P_L) \cdot V$. This is also seen in *Figure 2*.

$$Q_L = (P_H - P_L)V \tag{1}$$

This is the expression for the ideal refrigeration produced per cycle.

Next, applying the I Law to the control volume 2

$$Q_L - W = m(H_1 - H_0)$$

where:

Q_L = refrigeration load;
W = work into or from the system (zero in this case);
H_1 = enthalpy of gas leaving;
H_0 = enthalpy of gas entering; and
m = mass flow into and out of the refrigerator.

Thus

$$Q_L = m(H_1 - H_0)$$
$$= m c_p (T_1 - T_0)$$

Thus T_1 will always be higher than T_0. Mass flow per cycle is clearly equal to the difference between the mass at the cold end volume (just at the conclusion of the intake stroke, i.e. at pressure P_H and temperature T_L) and the mass in the system at the beginning of the cycle, i.e. at P_L and T_1. Or

$$m = \frac{P_H V}{R T_L} - \frac{P_L V}{R T_1} \text{ (per cycle)} \tag{2}$$

This Equation assumes that the void volume in the regenerator is zero (which is not strictly correct) and the other void volumes at the cold end or connecting piping are negligible.

From the above analysis, it may be noted that the G-M cycle is a heat pump cycle and the heat extracted at the cold end appears as the enthalpy increase at the warm end. Further, theoretically, for a given machine, the refrigeration available depends only on the pressure difference and is given by $(P_H - P_L) V$ and does not depend on cold end temperature. This is a unique feature of this cycle as compared to the Stirling or Brayton cycle.

Figure of merit

To calculate the figure of merit (FOM), which is the performance of this cycle compared to the ideal Carnot cycle, W/Q_L should first be calculated. Assuming an isothermal compression which is valid for low pressure ratios and good cooling arrangements, the following can be stated

$$W_{iso} = -m R T_0 \ln(P_H/P_L)$$

where m is the mass per cycle given by Equation (2). The negative sign indicates that the work is applied to the system. Thus

$$W_{iso} = -R T_0 \frac{V}{R}\left[\frac{P_H}{T_L} - \frac{P_L}{T_1}\right]\ln\left[\frac{P_H}{P_L}\right]$$

and the work required per unit of refrigeration is given by

$$-W/Q_L = T_0 V [(P_H/T_L) - (P_L/T_1)] \ln(P_H/P_L) / [V(P_H - P_L)]$$

$$= (T_0/T_1) P_L [(P_H/P_L)(T_1/T_L) - 1]$$

$$\times \ln(P_H/P_L) / \{P_L [(P_H/P_L) - 1]\}$$

It can be assumed that $T_0/T_1 \approx 1.0$ when both the temperatures are expressed as absolute temperatures. Then it can be written

$$-(W/Q_L)_{GM} = [(PR \times TR - 1) \ln PR]/(PR - 1) \tag{3}$$

where:

PR = pressure ratio = P_H/P_L; and
TR = temperature ratio = $T_1/T_L = T_0/T_L$.

Also

$$-(W/Q_L)_{Carnot} = (T_0 - T_L)/T_L$$

and

$$FOM = (W/Q_L)_{Carnot}/(W/Q_L)_{GM} \tag{4}$$

Figure 4 shows the isothermal compressor work requirement for helium gas against pressure ratio, for three different flow rates (750, 700 and 650 dm^3 min^{-1}). *Figure 5* shows a plot of W/Q_L for a G-M cycle with various values of pressure ratio at temperature ratios of 4, 10 and 15. *Figure 6* shows the FOM against pressure ratio. The performance of the cycle (FOM) increases slightly as the cold end temperature decreases, for a given pressure ratio.

G-M cycle on the temperature–entropy diagram

When considering the analysis of a cycle, it is often convenient to draw the cycle on the temperature–entropy (*T-S*) diagram, since the area on the *T-S* diagram directly indicates the quantities of heat or work. However, in the case of the G-M cycle, it is difficult to depict this cycle on the *T-S* diagram since each batch of gas follows a different path and there will be many overlapping areas for the whole cycle.

The general nature of the cycle may be shown on the *T-S* diagram by following a small portion of the gas involved. *Figure 7* shows the *T-S* diagram for the first batch of gas which enters the refrigerator[5]. This gas enters the valve (point 1) at high pressure, P_H, and ambient temperature, T_0. It expands through the valve to a low pressure, P_L into the volume, V_1, (point 2) and then gets compressed adiabatically by the addition of more gas to a

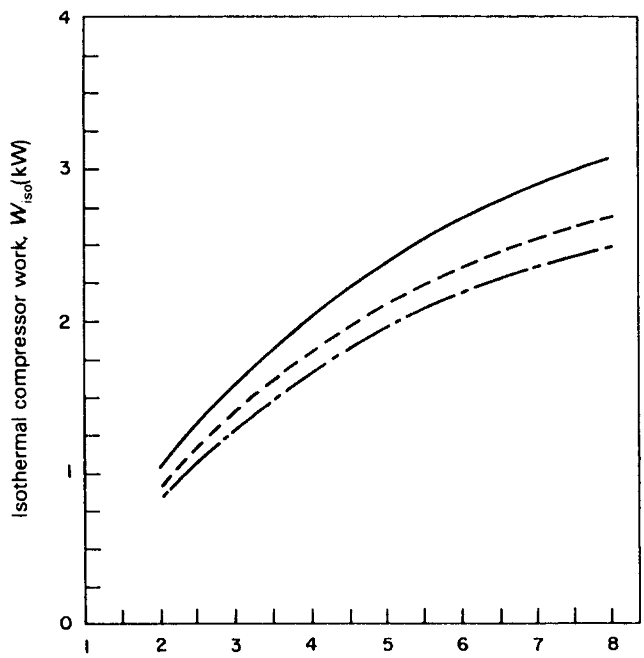

Figure 4 Isothermal compressor work for helium gas, W_{iso}(= $\dot{m}RT_0$lnPR). T_0 = 300 K. Flow rate (dm^3 min^{-1}): ——, 750; – – –, 700; – · –, 650

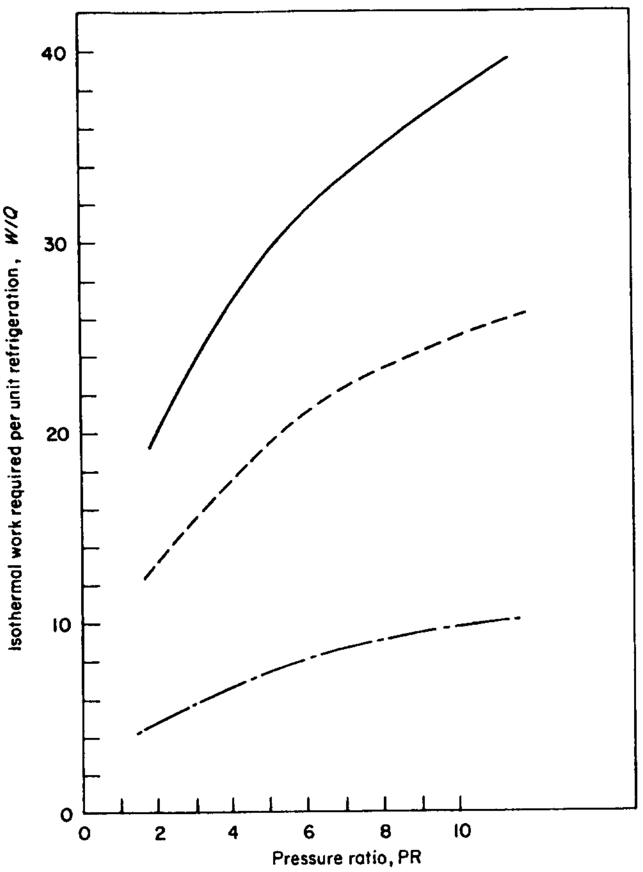

Figure 5 Isothermal work required per unit refrigeration *versus* pressure ratio for a G–M cycle cryorefrigerator. TR(= T_0/T_L): ——, 15; – – –, 10; – · –, 4

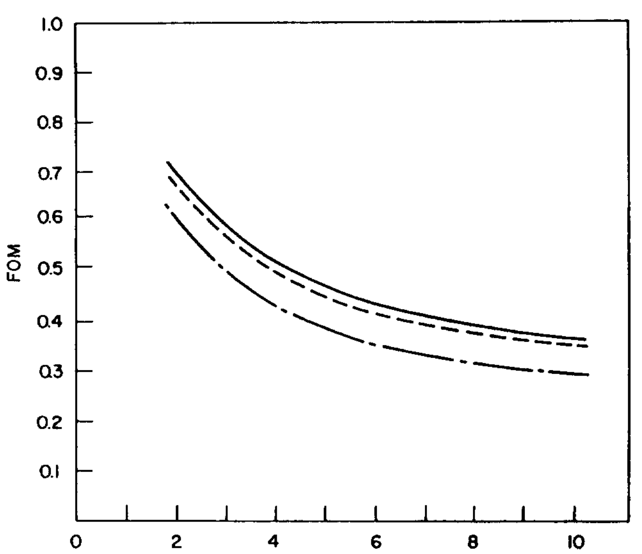

Figure 6 Figure of merit *versus* pressure ratio for a G–M cycle cryorefrigerator. TR(= T_0/T_L): ——, 15; – – –, 10; – · –, 4

high pressure, P_H, (point 3). When the gas is being transferred to the cold end volume, V_2, through the regenerator, some more gas from the supply source comes in and gets mixed with this batch of compressed gas. Let us say that the resultant temperature of the mixture is T_1 (point 4). It is further cooled in the regenerator to the load temperature, T_L, (point 5). Next, this cold high pressure

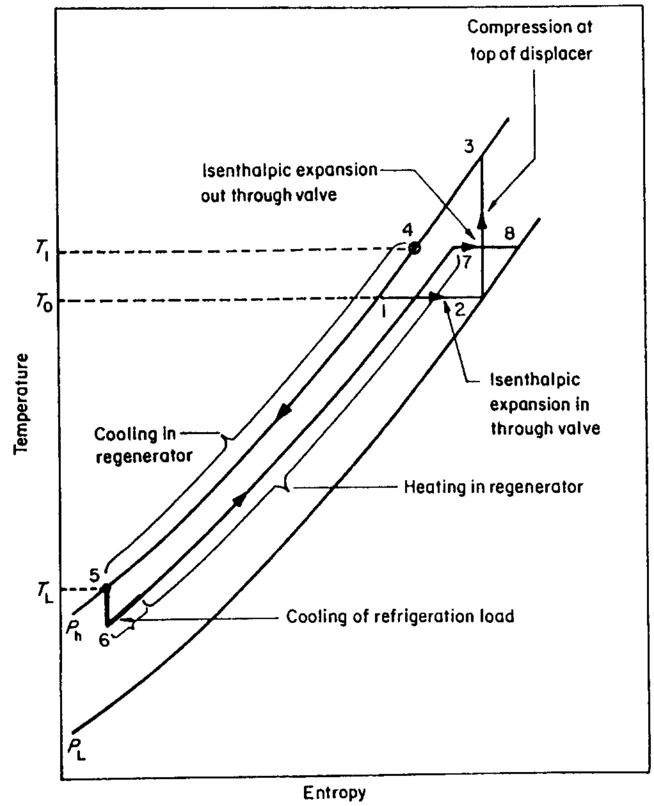

Figure 7 Temperature–entropy diagram for the first batch of gas to enter the refrigerator

gas is emitted through the valve and gets cooled (point 6). The refrigeration effect is provided by this cold gas, as shown. The gas is heated in the regenerator and is then emitted through the valve (point 8) at a temperature close to the mixing temperature, T_1. The next batch of gas entering the refrigerator undergoes a similar cycle but would be represented by a different *T–S* diagram; a complete set of lines for all the 'segments' of gas would give many overlapping areas and it would be difficult to compute the performance by integrating the effects of the different portions of the gas. Instead, the overall operation of the volumes V_1 and V_2 are considered and the total ideal refrigeration produced is computed as

$$Q_L = \oint V \mathrm{d}P = V(P_H - P_L)$$

Analysis of losses in a real machine

Ideal refrigeration capacity can be computed as outlined above. However, the actual refrigeration capacity in a real machine is always much less because of the various losses that invariably occur.

Losses occurring in a real machine may be considered under the following headings:

1 losses due to regenerator inefficiency;
2 losses due to void volumes at the cold end;
3 pressure drop losses;
4 losses due to non-ideal nature of the gas;
5 regenerator void volume losses;
6 motional heat losses (or Shuttle losses);
7 loss due to gas leak past the O ring seal;
8 conduction losses through the cylinder and displacer;

9 radiation losses; and
10 losses due to imperfect heat transfer at the cold end to the load. These various losses will now be considered in turn.

Losses due to regenerator inefficiency

The function of the regenerator is to cool the incoming gas to the temperature of the cold end volume. Let us say that the gas enters the regenerator at its warm end at a temperature T_H and let the steady state cold end temperature be T_L. Then, for a 100% efficient regenerator, the incoming gas gets cooled from T_H to T_L and the emitted gas gets warmed from T_L to T_H. The refrigeration produced is supplied isothermally at T_L. However, if the regenerator is not 100% efficient, as is the case in practice, the gas reaching the cold end will not be at the temperature T_L but at a slightly higher temperature, say T'_L. So, part of the refrigeration produced will be wasted in cooling this gas from T'_L to T_L and will thus be unavailable. So, the refrigeration 'lost' due to regenerator inefficiency per cycle will be

$$\Delta Q = m c_p (T'_L - T_L)$$

where m = mass of gas reaching the cold end (at P_H and T_L) through the regenerator. Thus

$$\Delta Q = m c_p I (T_H - T_L)$$

where:

I = regenerator inefficiency
= 1 − efficiency; and
c_p = specific heat of gas at constant pressure.

Thus

$$\Delta Q = [P_H V/(R T_L)]\, c_p I (T_H - T_L) \tag{5}$$

further

$$Q_L = (P_H - P_L) V$$

and

$$\Delta Q/Q_L = (P_H/P_L)(T_H/T_L) c_p I (1 - T_L/T_H) / [(P_H/P_L - 1)R]$$

$$= \mathrm{PR\,TR}\,[\gamma/(\gamma-1)]\, I(1 - 1/\mathrm{TR})/(\mathrm{PR} - 1)$$

since

$$c_P = [(\gamma/\gamma - 1)]\, R$$

therefore,

$$\Delta Q/Q_L = [\gamma/(\gamma-1)]\, I\, \mathrm{PR}(\mathrm{TR} - 1)/(\mathrm{PR} - 1) \tag{6}$$

In *Figure 8* $\Delta Q/Q$ is plotted against pressure ratio for various values of I, for TR values of 4 and 10. *Figure 9* shows a plot of $\Delta Q/Q$ against TR for various values of I at PR values of 2 and 4. From these two plots, four points may be noted.

1 For a real machine ($I > 0$), the refrigeration output is greater at higher pressure ratios. So, at higher pressure ratios, lower temperatures will be attained.
2 At a given pressure ratio, the refrigeration loss increases as the regenerator inefficiency increases. It can be seen that at PR = 2 and TR ≈ 6, practically all the refrigeration produced will be lost if the regenerator inefficiency, I, reaches a value of 4%.
3 For a given regenerator inefficiency and at a given TR, the losses reduce as the pressure ratio increases.
4 At given values of TR and I, increasing the pressure ratio beyond ≈ 4 does not seem to be very advantageous.

Of course, in plotting these graphs only the regenerator losses are considered and other losses are neglected. This is done to demonstrate the effect of only this loss. It is clear that the regenerator should have as high a value of efficiency as possible (i.e. as low a value of I as possible).

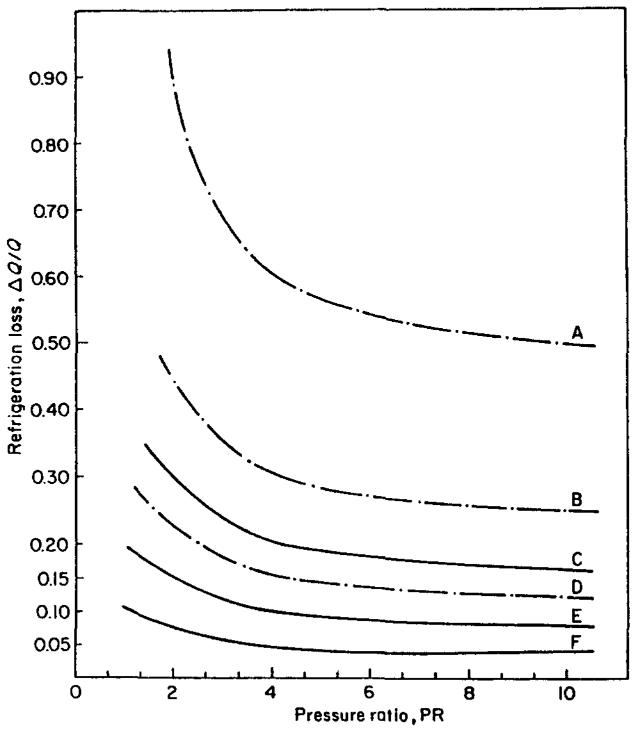

Figure 8 Pressure ratio *versus* refrigeration loss due to regenerator inefficiency in a G–M cycle cryorefrigerator. TR(= T_0/T_L); - · -, 10.0; ——, 4.0. Curves A and C, I = 2.0%; B and E, I = 1.0%; D and F, I = 0.5%

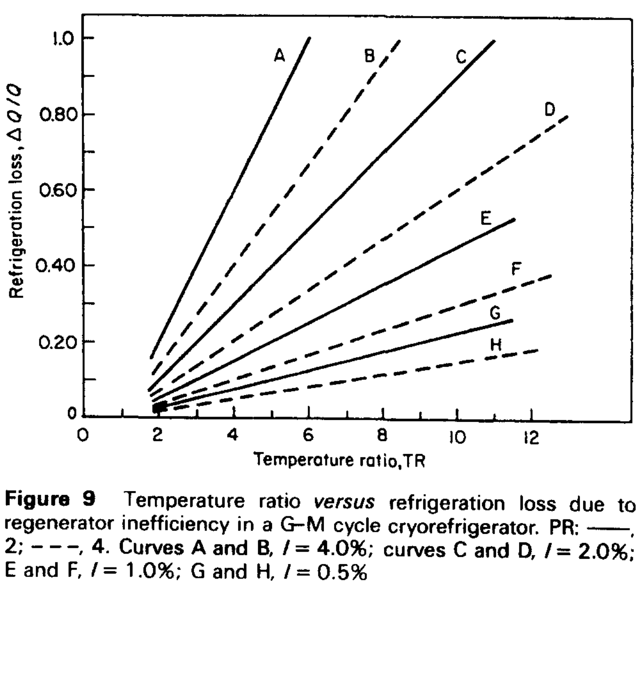

Figure 9 Temperature ratio *versus* refrigeration loss due to regenerator inefficiency in a G–M cycle cryorefrigerator. PR: ——, 2; – – –, 4. Curves A and B, I = 4.0%; curves C and D, I = 2.0%; E and F, I = 1.0%; G and H, I = 0.5%

Generally, the regenerator used in G–M cycle machines operating up to ≈70 K consists of a stack of punched copper or phosphor bronze wire screen discs, fitted snugly in a regenerator tube. The regenerator tube is made of a material with low thermal conductivity such as thin walled stainless steel. Sometimes, the displacer itself houses the regenerator. For lower temperatures, ≈20 K, lead balls (≈0.1 mm dia.) are used as regenerator material. These regenerators normally have an efficiency value of > 99%.

Losses due to void volume at the cold end

Ideal refrigeration produced is equal to $(P_H—P_L)\ V$, i.e. proportional to the cold end volume. If there is any void volume at the cold end, the gas which remains in this void volume does not contribute to the supply of refrigeration load and thus contributes to the loss. The void volume at the cold end is formed because of the connecting pipe between the regenerator and the cold volume, and also because of the radial gap between the cylinder and the displacer.

Let the void volume be V' and the displaced cold end volume, V. Then, refrigeration loss, ΔQ, is given by

$$\Delta Q = V'(P_H - P_L)$$

and ideal refrigeration, Q_L, is given by

$$Q_L = (V + V')(P_H - P_L)$$

and

$$\Delta Q/Q_L = V'/(V' + V)$$

If x is defined as V'/V = cold end void volume fraction, the following can be stated

$$\Delta Q/Q_L = x/(1 + x) \tag{7}$$

Figure 10 shows a plot of $\Delta Q/Q_L$ against x. It can be seen that as the void volume increases, the fraction of refrigeration lost also increases.

Losses due to pressure drops

As already shown for the ideal case, the refrigeration produced is $(P_H—P_L)\ V$ and the cycle on the P–V diagram is a square one. However, in practical cases, there will be pressure drops in the inlet and exhaust valves and in the regenerator. Thus the actual expansion will not be from P_H to P_L but from P'_H to P'_L, where $P'_H < P_H$ and $P'_L > P_L$. This reduces the refrigeration. Ackermann[13] has shown that the pressure–displacement losses are ≈50% of the ideal refrigeration produced. This loss accounts for the major fraction (71%) of the total losses while operating at 77 K. These losses become more acute when the cold end temperature decreases, since at that time the mass flow to the refrigerator increases and the time given for the extra gas to fill the volume is not sufficient. This results in the displaced volume not being fully pressurized.

For the sake of analysis, let us assume that the fractional pressure drops in the high pressure and low pressure side are equal, say x. If the displaced volume is V, it can be writen that refrigeration loss, ΔQ, is given by

$$\Delta Q = x\,(P_H + P_L)\,V$$

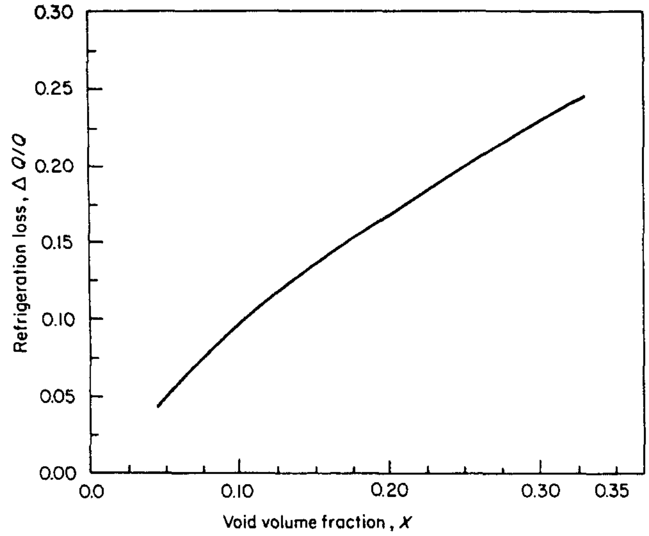

Figure 10 Refrigeration loss *versus* void volume fraction for a G–M cycle cryorefrigerator

and ideal refrigeration, Q_L, is given by

$$Q_L = (P_H - P_L)\,V$$

Therefore

$$\Delta Q/Q_L = x(P_H + P_L)/(P_H - P_L)$$

or

$$\Delta Q/Q_L = x(\mathrm{PR}+1)/(\mathrm{PR}-1) \tag{8}$$

where:

$x = \Delta P_H/P_H$
$= \Delta P_L/P_L$
= pressure drop fraction.

Figure 11 shows the refrigeration loss due to pressure drop against the pressure drop fraction for various values of pressure ratios. It is interesting to note that at lower pressure ratios the effect of pressure drop on refrigeration loss is more pronounced.

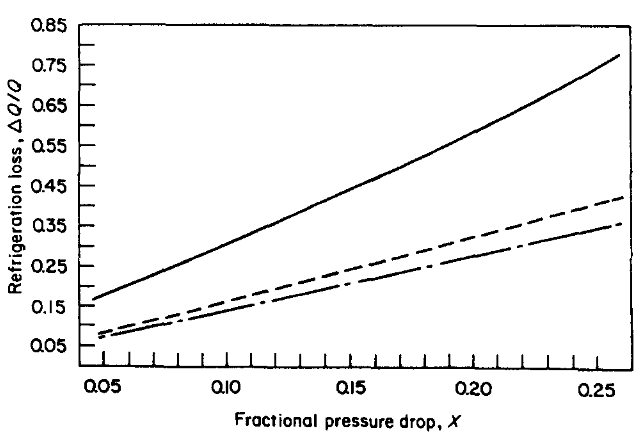

Figure 11 Refrigeration loss *versus* fractional pressure drop for a G–M cycle cryorefrigerator. PR: ——, 2; – – –, 4; – · –, 6

Losses due to non-ideal nature of the gas

If the derivation of Equation (1) for the ideal refrigeration produced is considered

$$Q_L = (P_H - P_L)V + (H_{out} - H_{in})$$

where:

H_{out} = enthalpy of the low pressure gas exhausting, and
H_{in} = enthalpy of the high pressure gas entering.

($H_{out} - H_{in}$) was set equal to zero for an ideal gas since in that case the enthalpy depends only on temperature and not on pressure. Temperature at the warm end of the regenerator is the same for both the streams for a 100% efficient regenerator. However, in reality for helium the gas is not perfectly ideal and the value of ($H_{out} - H_{in}$) is negative. So there is a loss in available refrigeration. It is not difficult to calculate H_{in} because the high pressure gas enters the cold volume at a constant temperature, T_L. However, there is some difficulty in calculating H_{out} as the pressure of the emitted gas varies with time, and density and enthalpy of gas vary with pressure. So

$$H_{out} = \int_3 h\,dm + h_0 m_4 \text{ (J)} \tag{9}$$

where the term under the integral is for the third stage when the gas is being emitted and h_0 and m_4 refer to the following stage, when the pressure in the cold volume is steady at P_L and the temperature is T_L. For constant T_L it can be written that

$$H_{out} = \int_{P_H}^{P_L} Vh\,d\rho + Vh_0\rho_0$$

Here, h and ρ refer to enthalpy and density, respectively, and are evaluated at constant temperature, T_L, but at varying pressures. The subscript 0 refers to the state at P_L and T_L.

As an example, if an operating temperature of 60 K and operating pressures of P_H = 18 atm and P_L = 8 atm are considered, for a displacement volume of 35.54 cm^3, the following is obtained

$$(P_H - P_L)V = 35.54 \text{ (J cycle}^{-1})$$

$$\int_{18\text{ atm}}^{8\text{ atm}} Vh\,d\rho + Vh_0\rho_0 = 0.1548 \text{ (J cycle}^{-1})$$

H_{in} = 0.1553 J cycle^{-1} and ($H_{out} - H_{in}$) = −0.0005 J cycle^{-1}.

Values of h and ρ are taken at small intervals of 1 atm from work by McCarty[14]. At higher pressure levels, this enthalpy loss is less than the loss at lower pressure levels, e.g. at P_H = 5 atm and P_L = 2 atm, $H_{out} - H_{in}$ = −0.001873 J cycle^{-1}.

Loss due to regenerator void volume

In the derivation of the Equation for FOM, it was assumed that there is no void volume in the regenerator. However, this is not true, as the regenerator is made of punched copper screens. The void volume of the regenerator can be considerable and may be even two to three times the displacement volume, V. For a 200 × 200 mesh copper wire screen matrix, the porosity, p, is ≈ 0.686, i.e. of the total regenerator volume, only a fraction equal to (1 − 0.686) is occupied by the metal and the balance (a fraction of 0.686) is the void volume. In each cycle, the regenerator void volume is also pressurized and depressurized. This contributes to reduced refrigeration production, but the mass circulated by the compressor increases. Thus the compressor work requirement increases and the FOM decreases.

The mass circulated in each cycle will be

$$m = \frac{P_H V}{RT_L} - \frac{P_L V}{RT_H} + \frac{2(P_H - P_L)V_{RV}}{R(T_H + T_L)} \tag{10}$$

In Equation (10) the last term accounts for the extra mass required due to the regenerator void volume, V_{RV}. This volume is assumed to be at a mean temperature of ($T_H + T_L$)/2, and the isothermal compressor work requirement is

$$W = -mRT_0 \ln(P_H/P_L)$$

and ideal refrigeration, Q_L, is given by

$$Q_L = (P_H - P_L)V$$

Substituting and simplifying the following is obtained

$$W/Q_L = [\text{PR} \times \text{TR} - 1]\,(\ln \text{PR})/(\text{PR} - 1)$$
$$+ 2y(\ln \text{PR})\,\text{TR}/(\text{TR} - 1) \tag{11}$$

where $y = V_{RV}/V$.

Comparing this with Equation (3), it can be seen that the second term represents the increase in work requirement due to regenerator void volume. *Figure 12* shows a plot of FOM *versus* PR at TR values of 4 and 10, for

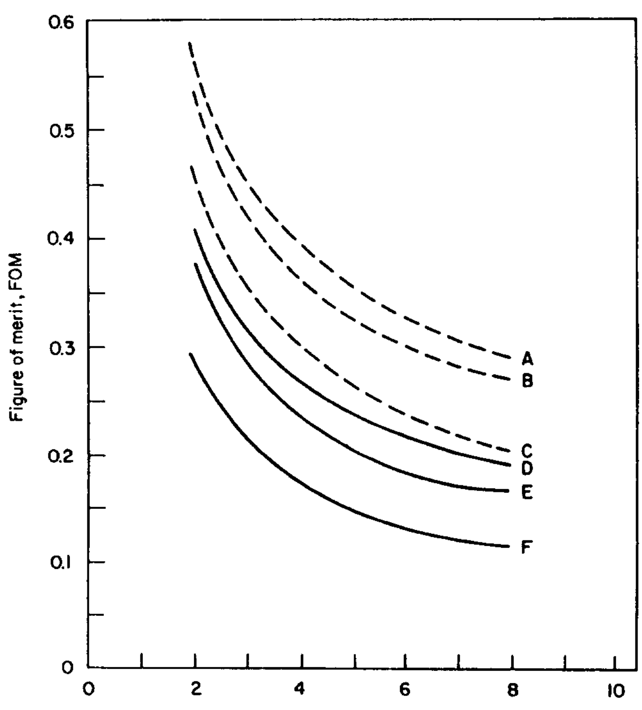

Figure 12 FOM *versus* pressure ratio for various values of y(= V_{RV}/V), showing the effect of regenerator void volume in a G–M cycle cryorefrigerator. TR: ——, 4; – – –, 10. Curves A and D, y = 2; B and E, y = 3; C and F, y = 5

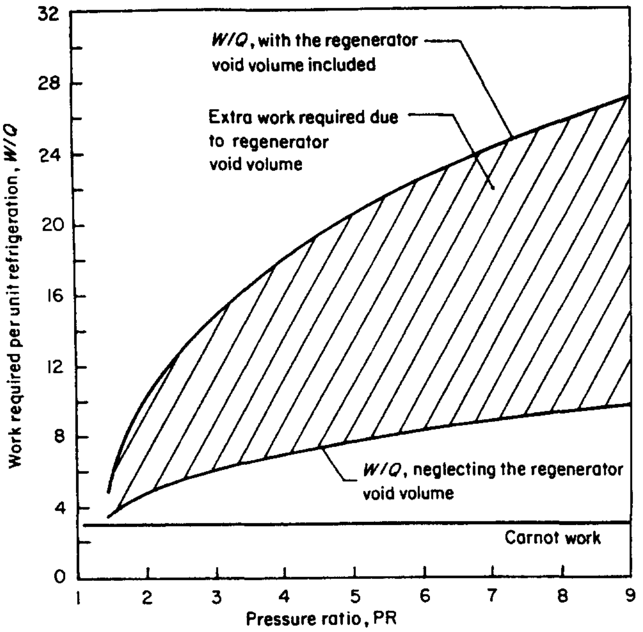

Figure 13 Effect of regenerator void volume in a single stage G–M cycle cryorefrigerator. T_0 = 300 K; TR(= T_0/T_L) = 4; y(= V_{RV}/V) = 3

values of y of 2, 3 and 5. It can be seen that: at lower temperatures the effect of regenerator void volume becomes more pronounced, and FOM decreases as void volume is increased.

Figure 13 shows the effect of regenerator void volume for the specific case of a G–M cycle cryorefrigerator operating at 75 K, i.e. TR = 4; $y = V_{RV}/V$ is assumed to be equal to 3. Here W/Q is plotted against PR. For reference, $(W/Q)_{Carnot}$ is also shown. The shaded portion represents the extra work requirement due to regenerator void volume.

Motional heat losses (or shuttle losses)

Motional heat losses (or shuttle losses) have been explained by Ackermann and Gifford[6]. These arise due to the travel of the displacer to and fro inside the cryocylinder. When the displacer is at the warm end, i.e. the cold end volume is at its maximum, the longitudinal temperature distribution on the displacer is below that of the wall and, therefore, heat is conducted through the intermediate helium gas from the wall to the displacer. When the displacer travels to the cold end, i.e. the cold end volume is at its minimum, the displacer faces the cylinder wall which has a temperature distribution lower than its own, and heat is conducted from the displacer to the cold end of the cylinder through the intermediate helium gas. Thus, there is a continuous heat pumping action from the warm end to the cold end. Obviously, the amount of heat pumped will be a function of the thermal conductivity of the gas, length of travel (or stroke) of the displacer, and time allowed at the end of each stroke for the heat transfer to occur. Ackermann and Gifford have suggested the following Equation to compute the motional heat leak, Q_M

$$Q_M = k_g \pi D S^2 C_T (T_H - T_L)/(2Lt) \qquad (12)$$

where:

k_g = thermal conductivity of He gas;
D = diameter of the displacer;
S = stroke length;
C_T = fraction of cycle in which heat is transferred at the extreme piston positions;
L = length of displacer; and
t = clearance between cylinder wall and displacer.

It can be seen from this Equation that the motional heat leak is proportional to the square of stroke; it is therefore preferable to have a smaller stroke.

Losses due to gas leak past the O ring seal

Any gas leaking past the O ring seal which is situated at the warm end will affect the lowest temperature obtained. Of course, the advantage of a double volume G–M machine is that the pressure difference across the seal is only that due to the pressure drop in the regenerator, which is of the order of a few p.s.i. Also, the helium gas in the radial clearance along the length of the cylinder acts as a regenerator and the warm gas leaking past the gas gets cooled as it approaches the cold end volume. Quantitative values for the leakage past the O ring seal have to be found by experiment. However, this loss is quite small in magnitude compared to other losses.

Conduction losses

Conduction losses occur through the cylinder as well as through the displacer. Conduction loss is given by

$$Q_{cond} = -kA\,dT/dx \qquad (13)$$

where:

k = thermal conductivity of the material;
A = cross-sectional area through which heat flows; and
dT/dx = temperature gradient over a length dx.

To reduce the conduction loss, the value of thermal conductivity, k, should be low and the heat flow length, dx, should be large. For a G–M cycle machine, thin walled stainless steel is used as the material for the cylinder and perspex is used as the material for the displacer. The thermal conductivity, k, of a material also depends on temperature. So, values for thermal conductivity integrals, i.e.

$$\int_{T_0}^{T} k\,dT$$

which are readily available in tubular form for various materials, can be used.

$$Q_{cond} = (A/L)\int_{T_L}^{T_H} k\,dT \qquad (14)$$

$$\int_{T_L}^{T_H} k\,dT = \int_{T_0}^{T_H} k\,dT - \int_{T_0}^{T_L} k\,dT$$

Values of thermal conductivity integrals

$$\int_{T_0}^{T} k\,dT,$$

from T_0 = 4 K to various temperatures, T, are available in standard texts (see Reference 2, for example).

Knowing the values of area of cross-section, A; length of heat flow, L; the value of $\int k\,dT$ for the appropriate material; and the appropriate temperature range, conduction losses are calculated using Equation (14).

Radiation losses

Radiation losses occur because of heat exchange between the wall of the vacuum enclosure and the cold end of the cryorefrigerator. The radiation loss for concentric cylinders is given by

$$Q_{\text{rad}} = \frac{\sigma A_1 (T_2^4 - T_1^4)}{[1/\varepsilon_1 + (A_1/A_2)(1/\varepsilon_2 - 1)]} \tag{15}$$

where:

σ = Stefan–Boltzmann constant = 5.67×10^{-8} W m^{-2} K^{-4};
A_1 = area of inner or enclosed surface;
A_2 = area of outer surface;
T_1 = absolute temperature of surface 1;
T_2 = absolute temperature of surface 2;
ε_1 = emissivity of inner surface; and
ε_2 = emissivity of outer surface.

When $A_2 \gg A_1$

$$Q_{\text{rad}} = \sigma \varepsilon_1 A_1 (T_2^4 - T_1^4) \tag{16}$$

Generally, radiation loss is much lower than other losses.

Losses due to imperfect heat transfer at cold end[12]

In a G–M machine, cold is produced at the cold end by expansion of the high pressure gas. Normally, the cold end is made of copper and the temperature is measured by fixing the probe onto this copper cap. The load to be cooled is fixed onto this cap using screws. Generally, it is assumed that the temperature, as measured on this cold end, T_L, is the temperature of the expanded gas, T_C. However, on close examination it would appear that this may not be so, as the temperature of the copper cap would be determined by the effectiveness of heat transfer from the cold gas to the copper surface. So, the gas has necessarily to be at a lower temperature than the measured surface temperature of the cold end.

The effectiveness, ε_C, of heat exchange between the cold end and the gas can be calculated in the usual way. First the heat transfer coefficient, h, is calculated, and then the NTU is calculated from

$$\text{NTU} = hA/(\dot{m} c_p)$$

Here, A is the surface area, $\dot{m}$ is the mass flow rate across this surface and c_p is the specific heat of gas at constant pressure. Then, assuming that the measured temperature at the cold end, T_L, is constant, and that the cold gas temperature is T_C, and treating this case as that of a heat exchanger in which one of the streams is at a constant temperature, T_L, (say, as in a condenser), it can be written that[15]

$$\varepsilon_C = 1 - e^{-\text{NTU}} \tag{17}$$

From the definition of effectiveness it can be written that

$$\varepsilon_C = Q_{\text{actual}}/Q_{\text{max}}$$

$$= Q/[Q + c_p \dot{m}(T_L - T_c)]$$

and

$$(T_L - T_C) = [(1-\varepsilon_C)/\varepsilon_C]\,[Q/(c_p \dot{m})]$$

Substituting

$$Q = (P_H - P_L)V$$

and

$$c_p = \gamma R/(\gamma - 1)$$

the following is obtained

$$T_L/T_C = 1 + [(1-\varepsilon_C)/\varepsilon_C]\,[(\gamma-1)/\gamma]\,[(\text{PR}-1)/\text{PR}] \tag{18}$$

where:

ε_C = effectiveness at the cold end;
PR = pressure ratio; and
γ = ratio of specific heats, c_p/c_v.

Figure 14 shows a plot of Equation (18) for three values of pressure ratio, i.e. for PR = 2, 4 and 6. It can be seen from this Figure that the efficiency of heat exchange between the gas and the cold end is very important to reduce the temperature difference between them.

Analysis of two stage G–M cycle cryorefrigerator

The two stage G–M cycle cryorefrigerator with a concentric arrangement, as shown in *Figure 15a*, will now be considered. This is the type of refrigerator actually built by the present authors. There are three volumes, V_1, V_2 and V_3, connected to each other through two regenerators as shown. V_1 is assumed to be equal to V_3 for analysis. The first stage temperature, T_{L1} is obtained in V_1 and second stage temperature, T_{L2}, in V_2. The first stage regenerator void volume is V_{RV1} and second stage regenerator void volume is V_{RV2}.

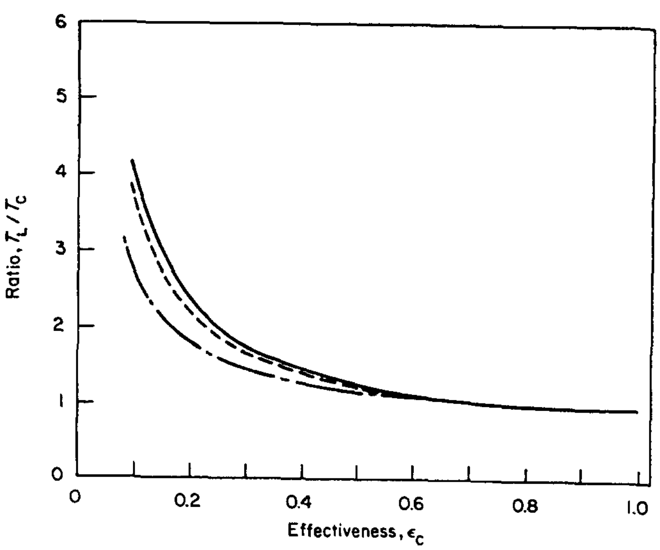

Figure 14 T_L/T_C *versus* ε_C for various values of pressure ratio. PR: ——, 6; – – –, 4; – · –, 2

Ideal refrigeration produced

As for a single stage G–M cycle cryorefrigerator, ideal refrigeration produced, Q_L, is given by

$$Q_L = (P_H - P_L)V_2 \tag{19}$$

Isothermal compressor work requirement

The isothermal compressor work requirement, W_{iso}, is given by

$$W_{iso} = -mRT_0 \ln(P_H/P_L) \tag{20}$$

where:

m = mass flow per cycle

$$= \frac{P_H V_1}{RT_{L1}} + \frac{P_H V_2}{RT_{L2}} - \frac{P_L V_1}{RT_0} \tag{21}$$

This assumes that the gas emitted is at a temperature very nearly equal to that of the ambient air, T_0. It is also assumed that regenerator void volumes are equal to zero. If instead, the regenerator void volumes are taken into account, it can be written that

$$m = \frac{P_H V_1}{RT_{L1}} + \frac{P_H V_2}{RT_{L2}} - \frac{P_L V_1}{RT_0} + \frac{2(P_H - P_L)V_{RV1}}{R(T_0 + T_{L1})} + \frac{2(P_H - P_L)V_{RV2}}{R(T_{L1} + T_{L2})} \tag{22}$$

Work requirement per unit of refrigeration produced

When regenerator void volumes are neglected. When the void volumes are neglected, from Equations (19), (20) and (21), it can be written that

$$\left[\frac{W_{iso}}{Q_L}\right]_{\text{two stage G–M}} = \frac{[\mathrm{PR} \times \mathrm{VR} \times \mathrm{TR}_1 + \mathrm{PR} \times \mathrm{TR}_2 - \mathrm{VR}] \ln(\mathrm{PR})}{(\mathrm{PR} - 1)} \tag{23}$$

where:

$\mathrm{PR} = P_H/P_L$;

$\mathrm{VR} = V_1/V_2$;

$\mathrm{TR}_1 = T_0/T_{L1}$; and

$\mathrm{TR}_2 = T_0/T_{L2}$.

When regenerator void volumes are considered. When the void volumes are considered, it can be written that

$$\left[\frac{W_{iso}}{Q_L}\right]_{\text{two stage G–M}} = \frac{[\mathrm{PR} \times \mathrm{VR} \times \mathrm{TR}_1 + \mathrm{PR} \times \mathrm{TR}_2 - \mathrm{VR}] \ln(\mathrm{PR})}{(\mathrm{PR} - 1)} + \left[\frac{2y \times \mathrm{VR} \times \mathrm{TR}_1}{(\mathrm{TR}_1 + 1)} + \frac{2z \times \mathrm{TR}_1 \times \mathrm{TR}_2}{(\mathrm{TR}_1 + \mathrm{TR}_2)}\right] \ln(\mathrm{PR}) \tag{24}$$

where:

$y = V_{RV1}/V_1$; and

$z = V_{RV2}/V_2$.

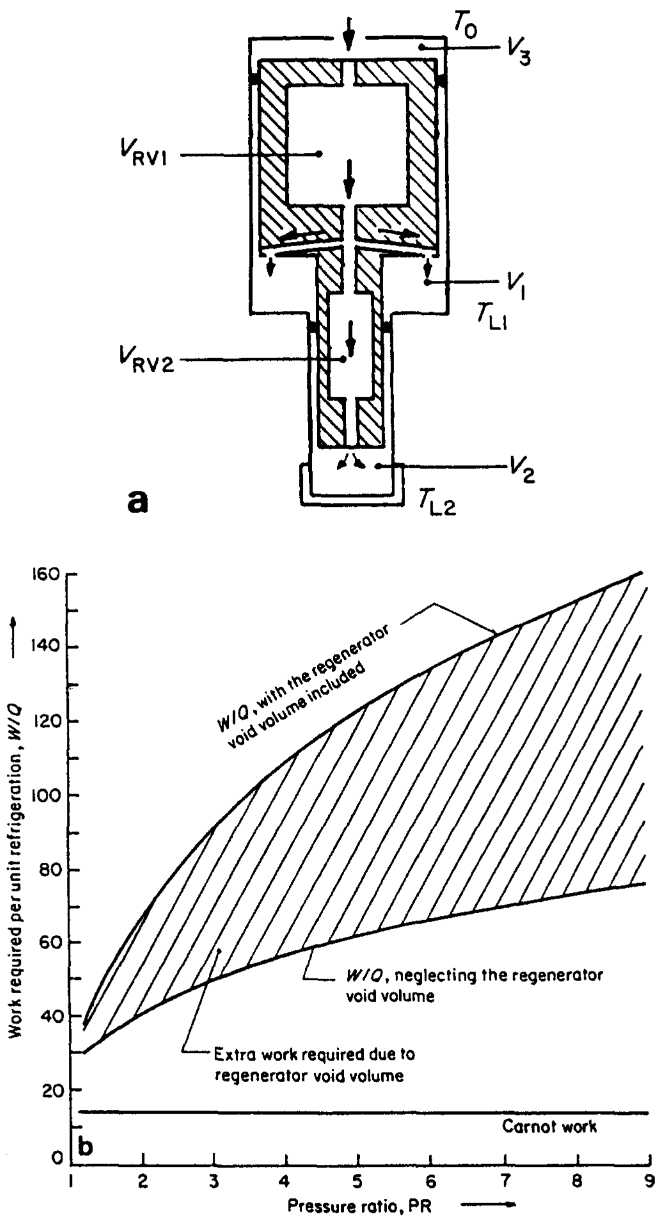

Figure 15 (a) Two stage G–M cryorefrigerator with a concentric arrangement. (b) Effect of regenerator void volume for a two stage G–M cycle cryorefrigerator. Data: T_0 = 300 K; VR = 4; TR_1 = 4; TR_2 = 15; $y(= V_{RV1}/V_1)$ = 3; $z(= V_{RV2}/V_2)$ = 3

Comparing Equation (23) and Equation (24), the second term in Equation (24) represents the extra work requirement due to regenerator void volumes.

Figure 15 shows a plot of Equations (23) and (24). Here, W/Q is plotted against PR for the specific case of a cryorefrigerator with the second stage at 20 K and the first stage at 75 K. The various parameters are: T_0 = 300 K; TR_1 = 4; TR_2 = 15; VR = 4; y = 3 and z = 3.

In this plot, for reference $(W/Q)_{\mathrm{Carnot}}$ for T_L = 20 K is also shown. The shaded area represents the extra work requirement due to regenerator void volumes. The effect of regenerator void volumes becomes more prominent at higher pressure ratios.

Conclusions

A theoretical analysis of the G–M cycle has been presented. Expressions for the ideal refrigeration developed and the various losses occurring in a real machine have been derived. Results have been presented in graphical form.

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