The theory of oscillating-vane vacuum gauges

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This paper sets out to review some of the theories which have been published concerning oscillating vane and fibre molecular vacuum gauges. The theories of gas damping of a vane or fibre produce results which differ only in a numerical constant. The reasons for the differences in these constants are discussed.
The effective area of a fibre contributing to damping is considered and the result used to reconcile two apparently contradictory results. Departures from simple theory are discussed and a comparison is made of some measured and calculated constants for some gauges, including a comment on an absolute gauge.

Introduction

The principle of the oscillating vane or fibre vacuum gauge is that of a simple pendulum in which the damping is due mainly to the gas. The vane or fibre is set in motion by some means and its rate of amplitude decay measured, usually by observing the time taken for the amplitude of the oscillation to fall to one half of its initial value, this time being denoted the half-life $t_{0\cdot 5}$. Since the damping is a function of the pressure and the half-life is a function of the damping, a measure of $t_{0\cdot 5}$ is an indication of pressure.

A quartz fibre gauge was first suggested by Langmuir[1] who used the device in investigating the clean-up of gas in tungsten lamps. The fibre was allowed to swing freely from one end, oscillation being initiated by shaking the lamp in which the fibre was situated. The half-life was observed and found to increase over a period of time, indicating a decrease in pressure. Langmuir gave no analysis or calibration for the gauge which was used merely to show that a decrease in pressure had occurred.

An approximate theory

Assuming there is no friction at the hinge of the vane or fibre and that the angular displacement θ is small, the torque equation will be

$$I\frac{d^2\theta}{dt^2} + G\frac{d\theta}{dt} + wgr\theta = 0 \qquad (1)$$

where I = moment of inertia of the fibre, G = co-efficient of damping, w = mass of fibre, r = radius of gyration, and g = gravitational acceleration. Equation (1) has the solution

$$\theta = \theta_0 e^{-at}\cos\omega t \qquad (2)$$

where $a = G/2I$ and ω is the frequency of oscillation which for $G \ll I$ will be given by $\sqrt{g/r}$.

If θ_1 and θ_2 are two consecutive amplitudes in the same direction and τ is the period of oscillation then from (2) we have

$$\theta_2/\theta_1 = e^{-\frac{G\tau}{2I}} = e^{-\delta} \qquad (3)$$

where δ is defined as the logarithmic decrement and is thus given by

$$\delta = \frac{G\tau}{2I} = \log_e\frac{\theta_1}{\theta_2}, \qquad (4)$$

and will be related to the half-life by

$$t_{0\cdot 5} = \frac{\tau}{\delta}\log_e 2. \qquad (5)$$

Consider an infinitesimal area dA of the vane at a distance l from the hinge. The damping torque to this area is $dG.d\theta/dt = dG.u/l$ where u is the linear velocity of the area; but the torque is also $l.P_d.dA$ where P_d is the difference in pressure between the front and back of the vane. Hence $dG = P_d\, l^2 dA/u$.
The total damping coefficient due to the entire vane will be

$$G = \frac{P_d}{u}\int l^2\, dA = \frac{P_d}{u}.J \qquad (6)$$

where J may be defined as the second moment of projected area normal to the direction of motion. The ratio P_d/u is unaffected by the integration since it will be shown below to be independent of the vane dimensions.

A quartz fibre gauge was used by Haber and Kerschbaum[2] who gave an approximate theory which was apparently not contradicted for almost 40 years. The pressure exerted on a plate located in the gas is given by

$$P = \frac{1}{3}mnC^2 \qquad (7)$$

where m = mass of a gas molecule, n = molecule density and C = r.m.s. velocity of gas molecules. If the plate is moving with a velocity u then a pressure difference occurs between the front and back of the plate which is assumed by Haber and Kerschbaum to be

$$P_d = \frac{1}{3}\,m\,n\,(C+u)^2 - \frac{1}{3}\,m\,n\,(C-u)^2 = \frac{4}{3}\,m\,n\,u\,C. \qquad (8)$$

We may write that $m = Mk/R_0$, $n = P/kT$ and $C = \sqrt{3R_0T/M}$ where M = molecular mass, k = Boltzmann's constant, R_0 = universal gas constant, T = absolute temperature and P = gas pressure. Substituting in (8) gives

$$\frac{P_d}{u} = \frac{4}{\sqrt{3}}P\sqrt{\frac{M}{R_0T}}. \qquad (9)$$

It follows from (5), (6) and (9) that

$$\frac{1}{t_{0\cdot 5}} = \frac{4}{\sqrt{3}}\cdot\frac{J}{2I\log_e 2}\cdot P\sqrt{\frac{M}{R_0T}} \qquad (10)$$

from which it is apparent that the half-life is inversely proportional to the pressure and the square root of molecular mass of the gas.

An exact theory

The theory of Haber and Kerschbaum, reiterated by many others since, is in error in equation (8) where the plate velocity u is combined with the root mean square velocity C of the gas molecules. This approach is only an approximation; thus it is necessary to consider the velocity v of the individual molecules. The method is also in error in that the effective area of the fibre is taken as to be $2r$ per unit length where r is the radius of the fibre, assumed to be of circular cross-section. It will be shown later that the effective area for damping must be determined by considering the rate of change of momentum of a molecule striking the surface and that this will depend on the shape of the surface.

Anderson[3] assumes a Maxwell-Boltzmann distribution for this velocity, in say, the x direction which is given by

$$dn = n\left(\frac{m}{2\pi\, kT}\right)^{\frac{1}{2}} e^{-\frac{mv^2}{2kT}} \,.\, dv. \tag{11}$$

If a plate lying in a plane normal to the x direction is moving in the x direction with a velocity u, then molecules strike the plate with velocities $(v\text{–}u)$ at the back and $(v + u)$ at the front. The numbers of molecules striking the front and back surfaces per unit time are $(v + u)\, dn$ and $(v - u)\, dn$ respectively. The pressure exerted on the vane is twice the rate of change of momentum in the x direction, so that on substituting for dn from equation (11) we obtain the pressure difference between front and back as

$$P_d = n\left(\frac{2\, m^3}{\pi kT}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{mv^2}{2kT}} [(v + u)^2 - (v - u)^2]\, dv. \tag{12}$$

This yields the result, after substituting for m, n and k of

$$P_d = 4\sqrt{\frac{2}{\pi}} \,.\, uP\sqrt{\frac{M}{R_0 T}} \tag{13}$$

which may be compared with equation (9). It will be seen that these expressions differ only in the numerical constants, being dimensionally similar.

This theory has been confirmed by Carter[4] using a different approach. Consider unit area of the plate and let the density of molecules in the space around it be n. The probability of a molecule striking the plate at an angle ϕ with the normal to the surface is $\frac{1}{2} \sin \phi \; d \phi$. If the plate is moving with a velocity u along the normal the molecules striking unit area in unit time must come from a parallelepiped of length $(v + u/\cos \phi)$ and base $\cos \phi$, ie, from a volume $(v \cos \phi + u)$. If the number of molecules with a velocity v is dn, then the number striking unit area per unit time at an angle ϕ will be $\frac{1}{2} (v \cos \phi + u) \sin \phi \; d \phi \; dn$ at the front and $\frac{1}{2} (v \cos \phi - u) \sin \phi \; d \phi \; dn$ at the back. Since the pressure is twice the rate of change of momentum the front and back pressures will be, respectively,

$$P_f = \int_0^\infty \int_0^{\pi/2} 2\; m\; (v\; \cos\; \phi\; + u) \,.\, \tfrac{1}{2}(v\; \cos\; \phi\; + u)\; \sin\; \phi \,.\, d\phi \,.\, dn. \tag{14}$$

$$P_b = \int_0^\infty \int_0^{\pi/2} 2\; m\; (v \cos\; \phi\; - u) \,.\, \tfrac{1}{2}(v\; \cos\; \phi\; - u)\; \sin\; \phi \,.\, \mathrm{d}\phi \,.\, dn. \tag{15}$$

The pressure difference will therefore be

$$P_d = \int_0^\infty \int_0^{\pi/2} 4\, m\, u\, v \sin \phi \,.\, \cos \phi \,.\, d\phi \,.\, dn = 2\, m\, n\, u\, \bar{v}, \tag{16}$$

where $\bar{v}$ is the average velocity of the molecules. This result may be compared with equation (8), and on substituting for m, n, and $\bar{v}$ will yield equation (13).

The theories of Mielenz and Schonheit

The theory of quartz-fibre vacuum gauges has been discussed by Mielenz and Schonheit[5] in a paper published in 1953, in which they describe both an "approximate" and a "strict" theory.

The approximate theory is similar to that of Haber and Kerschbaum except that the velocity of the gas molecules in equation (8) is taken as

$$\bar{v}_x = \sqrt{\frac{kT}{2\pi\, m}} = \tfrac{1}{4}\bar{v} \tag{17}$$

and is based on the number of molecules crossing unit area per second which is

$$n_r = \tfrac{1}{4}n\bar{v} = \frac{n\; C}{\sqrt{6\pi}} \,. \tag{18}$$

This results in a numerical multiplying constant in equation (9) of $4/(3\sqrt{2\pi})$ which is rather small. The method is of course open to the same objections as that of Haber and Kerschbaum.

In their strict theory, Mielenz and Schonheit consider a fibre of circular cross-section which is struck by molecules travelling in an arbitrary direction. By assuming a Maxwell-Boltzmann distribution for the molecular velocities and by considering components of molecular and fibre velocities in a three-dimensional frame of reference they obtain a value for the multiplying constant of $\sqrt{2\pi}$. The effective area per unit length of fibre with this value of constant is again twice the radius of the fibre, ie the projected area of the fibre. In this analysis such a value for the effective area appears to be correct.

Comparison of theories

The four values of multiplying constant so far mentioned are shown for ease of comparison in Table 1 together with the system assumed in their derivation and the system for which they would be applicable. The two approximate values were derived on the assumption of a projected area of twice the radius of a circular fibre and as such would only be applicable in fact to a flat vane of the same area. Since they are also approximate in the respect of molecular velocity they will not be considered further.

The exact values have been derived for a circular fibre and a flat vane respectively and should therefore be compatible. That they are so may easily be shown by considering the force acting on a stationary fibre of circular cross-section and radius r, as in Figure 1. Let gas molecules strike the surface of the fibre with a velocity v in the $-\, x$ direction, the angle between this velocity and the normal to the surface of the fibre being θ. For an elastic collision the new velocity of the molecule is v at angle 2θ to the x-axis, the component of velocity in the x direction being $v \cos 2\, \theta$. The change of momentum is thus $mv\, (1 + \cos 2\theta)$, and the force acting on an infinitesimal area $r.\, d\theta.\, dl$ will be

$$dF = 2\, m \,.\, \frac{dv}{dt}\, (1 + \cos 2\theta)\; r \,.\, d\theta \,.\, dl. \tag{19}$$

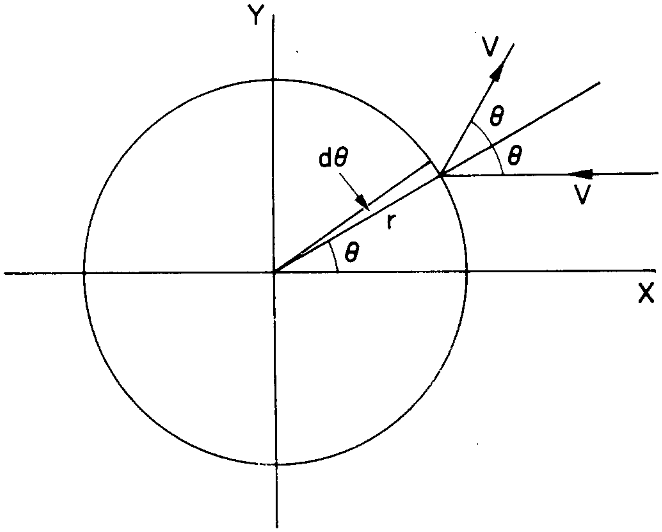

Figure 1

The force acting on length *d* of the fibre is thus

$$F = \int_{-\pi/2}^{2/\pi} 2\,m\,.\,\frac{dv}{dt}\,(1 + \cos 2\theta)\,rd\theta\,dl = 2\pi\,rm\,\frac{dv}{dt}\,.\,dl. \tag{20}$$

For a flat plate of width $2r$ the force acting on a length dl is, since in this case the change of momentum is $2\,mv$,

$$F = 8\,rm\,\frac{dv}{dt}\,.\,dl. \tag{21}$$

It follows that the damping effect of gas molecules on a flat plate of width $2r$ is $4/\pi$ times that on a circular fibre of radius r, and that the ratio of the Anderson-Carter constant to the Mielenz-Schonheit constant should be $4/\pi$, which is indeed seen to be the case. Alternatively one can say that the Anderson-Carter constant of $4\sqrt{2/\pi}$ may be used for circular fibres provided that the effective area for damping is taken as $\pi r/2$ per unit length, ie as one half of the actual area exposed to collision with molecules, and not the projected area $2r$ as is usually assumed.

Table 1. Numerical multiplying constants produced by different analyses of vibrating vane and fibre gauges.

Analysis	Constant		System	
			Assumed	Applicable
Haber & Kerschbaum	$4/\sqrt{3}$	2.31	Circular fibre	Flat vane
Mielenz & Schonheit				
Approx	$4/3\sqrt{2\pi}$	0.532	Fibre	Flat vane
Strict	$\sqrt{2\pi}$	2.51	Fibre	Circular fibre
Anderson Carter	$4/\sqrt{2\pi}$	3.19	Flat vane	Flat vane

Departures from simple theory

The foregoing theories assume that the oscillating vane or fibre is situated in an infinitely large volume. In practice this can never be so although in the case of a very thin fibre the distance from the fibre to the walls of the housing may be such that little effect of the proximity of the walls is noticed. An automatic vane gauge developed by the author and described elsewhere[6] was found to exhibit a gas damping greater than the calculated value. Since the vane used was large and the walls of the housing were in close proximity to the vane, the number of molecules striking the vane in unit time would be greater than the free space value due to reflection from the walls resulting in an increased damping factor. This explanation may account for the fact that most gauges of this type are unsuitable as absolute devices and require calibration against a standard.

The stiffness of the hinge or suspension of the vane will limit the lowest pressure that can be measured and may be expressed by the constant k_2 such that

$$\frac{\log_e 2}{t_{0\cdot5}} = k_1 P + k_2. \tag{22}$$

If $t_{0\cdot5}$ is measured and the suspension damping is ignored then the apparent pressure will be

$$P_a = \frac{\log_e 2}{k_1\,t_{0\cdot5}} \tag{23}$$

The true pressure will be lower than this since from equation (17) we obtain

$$P = P_a - k_2/k_1. \tag{24}$$

It follows that as P_a approaches k_2/k_1 in magnitude, equation (19) gives the small difference between two comparable numbers with consequent inaccuracy. The ratio k_2/k_1 may be regarded as the pressure equivalent of the suspension damping, and the lower limit of pressure measurement will be some fraction of this depending on the error that can be tolerated.

The theory also assumes that the pressure is low enough for molecular flow to prevail. At higher pressures where viscous flow occurs there is a departure from the $P\sqrt{M}$ law. Coolidge[7] has investigated the higher pressure range experimentally, indicating that the departure from linearity occurs for $P\sqrt{M} > 0.1$, for pressure in torr. Darwin[8] has considered a ribbon gauge theoretically in order to obtain one equation which applies to both the molecular and viscous regions and there seems to be no reason why a similar argument should not be applied to the oscillating vane arrangement.

Comparison of gauges

It will be apparent from equation (19) that the basic requirement of a vane of fibre gauge is that the suspension damping pressure equivalent k_2/k_1 must be as small as possible. This means that the suspension stiffness must be reduced to as small a value as possible but with a mechanical suspension, some limit must inevitably be reached.

From equations (6) and (13) it is apparent that k_1 is proportional to the ratio J/I. For a simple vane hinged along one edge, J and I will be related by the density ρ of the vane material and the thickness t of the vane since

$$J/I = 1/\rho t \tag{25}$$

from which one may make the otherwise obvious deduction that the vane should be made from a thin sheet of light material. Suitable materials are quartz or aluminium, having comparable densities. In the gauge used by the author the vane was made from aluminium of 0.005 mm thickness, but the value of J/I obtained was much less than $1/\rho t$ due to the complex configuration of the vane necessitated by the requirements of automatic control.

A comparison of several fibre and vane gauges is given in Table 2 showing values of J/I together with measured and calculated values of $t_{0\cdot5}$. The table is unfortunately incomplete due to a lack of information in some of the published papers. The values of J/I and $t_{0\cdot5}$ were calculated from the gauge data where

Table 2. A comparison of some fibre and vane gauges.

Gauge	Ref	J/I cm² – g⁻¹	p torr	$t_{0\cdot5}$ secs Calc	 Meas
Langmuir	1	—	1.8×10^{-5}	—	3960
Haber & Kerschbaum	2	24	4.2×10^{-4}	955	415
Andrews	9	109	—	—	—
Bruche	10	60	2×10^{-3}	69	25
Hurd & Corrin	11	6.4	5×10^{-5}	3×10^4	—
Anderson	3	17.9	1×10^{-3}	266	330
Neher	12	740	5×10^{-5}	261	—
Christian & Leck	6	373	2.8×10^{-5} (neon)	1050	353

this was available, the suspensions being ignored. The measured values of $t_{0\cdot5}$ and the pressures, in air or nitrogen, are those quoted by the individual authors. The effect of suspension damping is to reduce the value of $t_{0\cdot5}$ and this is generally seen to be so. In the case of the author's gauge, the greater part of the reduction was due to the increased sensitivity already discussed.

Of all the gauges listed in Table 2 only one of these is claimed to be an absolute instrument. This is the gauge due to Neher[12] which consists of an aluminium vane of width b and length $2\,l$ attached through its centre to a torsional suspension so producing the maximum value of J/I for the material used. Neher gives the lotharithmic decrement as

$$\delta = 4\, m\, n\, C\, b\, l^3\, \tau/9I, \tag{26}$$

apparently based on the theory of Haber and Kerschbaum. The gauge was compared with an ionisation gauge in air at a pressure of 5×10^{-5} torr, the corresponding value obtained from the vane gauge being 6.8×10^{-5} torr. The gauge appears to be in error by +36 per cent if equation (21) is used. However, from equations (8) and (16) we have

$$\delta = \frac{K J \tau}{2\,I} = \frac{2\, m\, n\, \bar{v}\, J\tau}{2\,I}, \tag{27}$$

and since $J = 2bl^3/3$ then the decrement becomes

$$\delta = 2\, m\, n\, \bar{v}\, b\, l^3\, \tau/3I, \tag{28}$$

which is the more accurate expression. If the decrement is measured and the pressure calculated using equations (21) and (23), denoted by P_{21} and P_{23} respectively, then the relation between these values is seen to be

$$P_{21} = \sqrt{\frac{6}{\pi}}\,.\,P_{23} = 1.382\, P_{23}. \tag{29}$$

The pressure calculated from equation (21) is thus in error by +38.2 per cent. Since the ionization gauge used by Neher could probably be in error by as much as +2.2 per cent, this would explain the error of +36 per cent which he obtained. This gauge would thus appear to be an absolute device provided equation (23) is used.

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References

[1] I Langmuir, *J Amer, Chem Soc, 35, 1913, 105.*
[2] F Haber, and F Kerschbaum *Z Elektrochem, 20, 1914, 20.*
[3] J R Anderson, *Rev Scient Instrum, 29, 1958, 1073.*
[4] G Carter, *Private Communication, Liverpool University, 1964.*
[5] K D Mielenz and E Schonheit, *Z Angew Phys, 5, 1953, 90.*
[6] R G Christian, *M Eng Thesis, Liverpool University, 1965.*
R G Christian and J H Leck, *to be published.*
[7] A S Coolidge, *J Amer Chem Soc, 45, 1923, 1637.*
[8] H W Darwin, *Vakuumtechnik, 2, 1962, 45.*
[9] M R Andrews, *J Phys Chem, 30, 1926, 1497.*
[10] E Bruche, *Physik, Z 26, 1925, 717.*
[11] D T Hurd and M L Corrin, *Rev Scient Instrum, 25, 1954, 1126.*
[12] H V Neher, *Rev Scient Instrum, 33, 1962, 803.*