Both of the authors believe their courses to have been quite successful as taught this way. There are a variety of criteria by which they came to this conclusion.

1. Material learned by students: As judged by frequent discussions with students, their lectures, and, in the case of the undergraduate course, the final, the students seemed to have absorbed as much material as they might have in a comparable conventional course.

2. Student motivation: In contrast to the usual motivation of exams and grades, the students here studied to understand the material so that they, in turn, could explain it. This not only produced a healthier atmosphere, but resulted in a sizeable fraction of the students in each class taking on additional work, on their own initiation, in the form of calculations, additional study, or laboratory problems. (Some undergraduates continued working on projects beyond the end of the course, and into the following semester.) Students in conventional courses often complain that they have insufficient time for even their required work.

3. Student evaluation: The students generally felt that they learned at least as much as in a conventional course, that the breadth of the material they learned was adequate, that they worked harder, and that they enjoyed it more. They felt a deeper involvement in and understanding of the material, and generally found the course to be more inspiring and interesting. They appreciated the independent learning aspects as well as the teaching experience. They all found much more give and take among the students, and the undergraduates found a more desirable student–teacher relationship. They generally appreciated the specific format of the course, in particular many found the pre-talk conferences extremely valuable.

There were, of course, drawbacks. The talks were generally adequate to bad, with only an occasional good one. Although this is surely undesirable, it is far less significant here, where everyone recognizes that the major learning takes place outside the lecture, than in a conventional course. Even so, the situation may be alleviated somewhat as the teacher gains experience in stimulating the intellectual give and take in class between students. This is somewhat difficult as the students are afraid of embarassing one another in front of the teacher, and requires a certain amount of delicacy on the teacher's part.

In addition, the undergraduates felt the need for assigned problems and worked out examples. They also felt that lectures should avoid tedious mathematical derivations if they could be left to notes, as in the graduate course. The teacher felt that the groups should meet with him twice, once fairly early to discuss problems of content, and once shortly before the talks for problems of presentation.

This type of course is apparently quite successful within the two contexts in which it has been tried. Its general viability is still not clear. Thus, both courses in which it was tried were advanced courses in special fields and not the prerequisite for any other course. In addition, they were both small classes, numbering a dozen students in each case. Within this context, the success of this format in setting a more desirable motivation for learning with only minimal drawbacks seems to make it an extremely valuable tool. Lower level courses, which treat material required to meet the needs of subsequent courses, may present greater problems, but as long as sufficiently small sections are available, we feel that exploration of the application of this format, appropriately modified, to even these courses, may yet prove to be extremely fruitful.

The questionnaire submitted to students follows. Questions marked with an asterisk were omitted for the undergraduates.

(1) How did the total amount of material you *learned* compare with that learned in other courses?
(2) Do you think you learned enough general material or is your knowledge too restricted to the material of your group's talks?
(3) Did you find this course more or less enjoyable than lecture courses you have taken?
(4) Did you work harder or less hard in this course than in others?
(5) Did you find this an effective and/or efficient way to learn?
(6) What specific virtues did this course have compared to others?
(7) What specific drawbacks did this course have compared to others?
(8) Please comment on each of the specific aspects of the course listed below:
 (a) Conferences prior to talks
 *(b) Method of selecting speakers
 (c) Your talk (or talks)
 (d) Everyone else's talks
 *(e) Notes (in both original and "appendix" format, both yours and others)
 *(f) Subsidiary topics pursued by some students
 (g) Method of staggered topics for each group (e.g., Topics 1, 4, 7, etc.)
 (h) General course organization
 (i) Actual classes (in particular, my behavior)
 (j) Lack of examinations
(9) Please carefully discuss the over-all impact of the course, including any strong or weak points, any changes (up to and including annihilation) that you would like to see made, whether you think this approach is best suited to an advanced (elementary) course like this, or lower (upper) level courses, etc. Please feel free to comment frankly and to suggest any innovations you think might be worth trying. Also mention anything worth mentioning that you haven't covered elsewhere in this questionnaire.

[1] The texts used for these courses were: P. G. deGennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, New York, 1966); J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, New York, 1964); and L. R. B. Elton, *Introductory Nuclear Theory* (Saunders, Philadelphia, 1966).

Boundary Conditions in Electrodynamics

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It is well known[1] that boundary conditions for the electromagnetic fields at the surface of discontinuity between two media, say, I and II, are

$$\mathbf{n}\cdot(\mathbf{D}_{\rm II}-\mathbf{D}_{\rm I})=4\pi\rho_s,\qquad \mathbf{n}\times(\mathbf{E}_{\rm II}-\mathbf{E}_{\rm I})=0,$$
$$\mathbf{n}\times(\mathbf{H}_{\rm II}-\mathbf{H}_{\rm I})=(4\pi/c)\mathbf{j}_s,\qquad \mathbf{n}\cdot(\mathbf{B}_{\rm II}-\mathbf{B}_{\rm I})=0,\quad (1)$$

where $\mathbf{n}$ is the unit vector perpendicular to the surface of discontinuity and in the direction I to II, subscripts I and II denote the values of the fields close to the boundary in the media I and II, respectively, and ρ_s and $\mathbf{j}_s$ are the surface densities of charge and current at the boundary. Equations (1) are usually derived from Maxwell's equations,

$$\nabla\cdot\mathbf{D}=4\pi\rho,\qquad \nabla\times\mathbf{E}+c^{-1}\dot{\mathbf{B}}=0,$$
$$\nabla\times\dot{\mathbf{H}}-c^{-1}\dot{\mathbf{D}}=(4\pi/c)\mathbf{j},\qquad \nabla\cdot\mathbf{B}=0,\quad (2)$$

by integrating them over an elementary volume around, or an elementary surface over, the surface of discontinuity. In this note, we give an easy derivation of Eqs. (1).

If we use superscripts (1) and (2) to denote fields in the media I and II, the fields have the form,

$$[\mathbf{D},\mathbf{E},\mathbf{B},\mathbf{H}]=[\mathbf{D}^{(1)},\mathbf{E}^{(1)},\mathbf{B}^{(1)},\mathbf{H}^{(1)}]\theta(-x)$$
$$+[\mathbf{D}^{(2)},\mathbf{E}^{(2)},\mathbf{B}^{(2)},\mathbf{H}^{(2)}]\theta(x),\quad (3)$$

where $x=\mathbf{n}\cdot\mathbf{x}$ and $\theta(x)$ is the unit step function which is unity for $x>0$ and zero for $x<0$. If ρ_s and $\mathbf{j}_s$ are the surface charge density and the surface current density at the boundary, we can write $\rho=\rho'+\rho_s\delta(x)$, $\mathbf{j}=\mathbf{j}'+\mathbf{j}_s\delta(x)$, where $\delta(x)$ is the Dirac delta function, and ρ' and $\mathbf{j}'$ are finite at the boundary. Then, on substituting Eqs. (3) in Eqs. (2) and on equating the terms involving $\delta(x)$ on both the sides of the equations thus obtained, we get the conditions (1) since $\mathbf{D}^{(1)}\delta(x)=\mathbf{D}_{\rm I}\delta(x)$, etc.

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[1] See, for example, M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1965), p. 4.

Maxwell's Equations in a Rotating Reference Frame

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Schiff[1] has derived the vector form of Maxwell's equations in a rotating coordinate system by the methods of general relativity. In this note it is shown how these equations may be derived classically.

In a rest system Maxwell's equations are

$$\nabla\cdot\mathbf{B}=0,\qquad \nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t,$$
$$\nabla\cdot\mathbf{E}=\rho,\qquad \nabla\times\mathbf{B}=(\partial\mathbf{E}/\partial t)+\mathbf{j}.\quad (1)$$

The force on a point charge q moving with velocity $\mathbf{v}$ is

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\quad (2)$$

In the presence of distributed charges and currents, the force density is

$$\mathbf{f}=\rho\mathbf{E}+\mathbf{j}\times\mathbf{B}.\quad (3)$$

According to an observer moving with velocity $\mathbf{v}$, the sources and fields are

$$\rho'=\rho,\qquad \mathbf{j}'=\mathbf{j}-\rho\mathbf{v},$$
$$\mathbf{B}'=\mathbf{B},\qquad \mathbf{E}'=\mathbf{E}+\mathbf{v}\times\mathbf{B}.\quad (4)$$

These transformations are Newtonian ($t'=t$) but not necessarily Galilean (the $\mathbf{v}$ need not be constant). The invariance of ρ follows from the general invariance of ρdV, the amount of charge in an element of volume dV, and the Newtonian invariance of dV; the transformation for $\mathbf{j}$ then follows from charge conservation. In Newtonian physics, force is invariant, and the transformation for $\mathbf{E}$ follows immediately from the invariance of $\mathbf{F}$ and q in Eq. (2). Similarly, the invariance of $\mathbf{B}$ is readily obtained from the invariance of $\mathbf{f}dV$ (and hence $\mathbf{f}$) and the transformation properties of ρ, $\mathbf{j}$, and $\mathbf{E}$. With

$$\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}\quad (5)$$

the expressions in Eq. (4) represent the relations between the sources and fields in a rotating system and those in the rest system.

The time rate of change of a vector $\mathbf{A}$ may be expressed in terms of its rate of change in the rotating system as[2]

$$d\mathbf{A}/dt=(d\mathbf{A}/dt)'+\boldsymbol{\omega}\times\mathbf{A}.\quad (6)$$

If the terms in Eq. (6) refer to a point fixed in the rotating frame, that is, if

$$(dA/dt)'=(\partial \mathbf{A}/\partial t)', \tag{7}$$

then the point moves with velocity $\mathbf{v}$ in the rest frame and

$$d\mathbf{A}/dt=(\partial \mathbf{A}/\partial t)+\mathbf{v}\cdot\nabla\mathbf{A}. \tag{8}$$

From Eqs. (6)–(8),

$$\partial\mathbf{A}/\partial t=(\partial\mathbf{A}/\partial t)'+\boldsymbol{\omega}\times\mathbf{A}-\mathbf{v}\cdot\nabla\mathbf{A}. \tag{9}$$

For uniform rotation, $\boldsymbol{\omega}$ is constant, and $\boldsymbol{\omega}\times\mathbf{A}=\mathbf{A}\cdot\nabla\mathbf{v}$ with $\nabla\cdot\mathbf{v}=0$, and Eq. (9) may be expressed in the form

$$(\partial\mathbf{A}/\partial t)+(\nabla\cdot\mathbf{A})\mathbf{v}=(\partial\mathbf{A}/\partial t)'+\nabla\times(\mathbf{v}\times\mathbf{A}). \tag{10}$$

In particular, since $\nabla\cdot\mathbf{B}=0$,

$$\partial\mathbf{B}/\partial t=(\partial\mathbf{B}/\partial t)'+\nabla\times(\mathbf{v}\times\mathbf{B}), \tag{11}$$

and, since $\nabla\cdot\mathbf{E}=\rho$,

$$(\partial\mathbf{E}/\partial t)+\rho\mathbf{v}=(\partial\mathbf{E}/\partial t)'+\nabla\times(\mathbf{v}\times\mathbf{E}). \tag{12}$$

Since $\mathbf{E}=\mathbf{E}'-\mathbf{v}\times\mathbf{B}$, Eq. (12) may be expressed in the form

$$(\partial\mathbf{E}/\partial t)+\rho\mathbf{v}=(\partial\mathbf{E}'/\partial t)'+\mathbf{i}', \tag{13}$$

where

$$\mathbf{i}'=-\mathbf{v}\times(\partial\mathbf{B}/\partial t)'+\nabla\times[\mathbf{v}\times(\mathbf{E}'-\mathbf{v}\times\mathbf{B})]. \tag{14}$$

Equations (11) and (13), together with the expressions for $\mathbf{j}'$ and $\mathbf{E}'$ given in Eq. (4), complete the transformations needed to determine Maxwell's equations in the rotating frame; all other terms (ρ, $\mathbf{B}$, t, $\mathbf{v}$, $\boldsymbol{\omega}$, $\mathbf{r}$, ∇) are the same in both systems.

Thus, $\nabla'\cdot\mathbf{B}'=\nabla\cdot\mathbf{B}'=\nabla\cdot\mathbf{B}$, and since $\nabla\cdot\mathbf{B}=0$,

$$\nabla'\cdot\mathbf{B}'=0. \tag{15}$$

That is, in the rotating system, the divergence of the magnetic field vanishes. Similarly, since $\mathbf{E}'=\mathbf{E}+\mathbf{v}\times\mathbf{B}$,

$$\nabla'\cdot\mathbf{E}'=\nabla\cdot\mathbf{E}+\nabla\cdot(\mathbf{v}\times\mathbf{B})$$

and since $\nabla\cdot\mathbf{E}=\rho$ $(=\rho')$,

$$\nabla'\cdot\mathbf{E}'=\rho'+\sigma', \tag{16}$$

where

$$\sigma'=\nabla\cdot(\mathbf{v}\times\mathbf{B})=2\boldsymbol{\omega}\cdot\mathbf{B}-\mathbf{v}\cdot(\nabla\times\mathbf{B}) \tag{17}$$

Also,

$$\nabla'\times\mathbf{E}'=\nabla\times\mathbf{E}+\nabla\times(\mathbf{v}\times\mathbf{B})$$
$$=-(\partial\mathbf{B}/\partial t)+\nabla\times(\mathbf{v}\times\mathbf{B}),$$

and hence, according to Eq. (11),

$$\nabla'\times\mathbf{E}'=-(\partial\mathbf{B}/\partial t)'. \tag{18}$$

Finally,

$$\nabla'\times\mathbf{B}'=\nabla\times\mathbf{B}=(\partial\mathbf{E}/\partial t)+\mathbf{j}$$
$$=(\partial\mathbf{E}/\partial t)+\mathbf{j}'+\rho\mathbf{v},$$

and hence, according to Eq. (13),

$$\nabla'\times\mathbf{B}'=(\partial\mathbf{E}'/\partial t)+\mathbf{j}'+\mathbf{i}'. \tag{19}$$

The relations given in Eqs. (15), (16), (18), and (19) are expressed completely in terms of primed and invariant terms and hence represent Maxwell's equations in the rotating frame. With the primes dropped, they are

$$\nabla\cdot\mathbf{B}=0,$$
$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t, \tag{20}$$
$$\nabla\cdot\mathbf{E}=\rho+\sigma,$$
$$\nabla\times\mathbf{B}=(\partial\mathbf{E}/\partial t)+\mathbf{j}+\mathbf{i}.$$

where the "extra" terms σ and $\mathbf{i}$ are given by

$$\sigma=\nabla\cdot(\mathbf{v}\times\mathbf{B})=2\boldsymbol{\omega}\cdot\mathbf{B}-\mathbf{v}\cdot(\nabla\times\mathbf{B}),$$
$$\mathbf{i}=\mathbf{v}\times(\nabla\times\mathbf{E})+\nabla\times[\mathbf{v}\times(\mathbf{E}-\mathbf{v}\times\mathbf{B})]. \tag{21}$$

These expressions agree with those of Schiff.

In the rest system potentials $\mathbf{A}$ and φ exist such that

$$\mathbf{B}=\nabla\times\mathbf{A},$$
$$\mathbf{E}=-\partial\mathbf{A}/\partial t-\nabla\varphi. \tag{22}$$

In this section it is shown that the fields in the rotating system, given by

$$\mathbf{B}'=\mathbf{B},$$
$$\mathbf{E}'=\mathbf{E}+\mathbf{v}\times\mathbf{B} \tag{23}$$

may be derived from potentials $\mathbf{A}'$ and φ' given by

$$\mathbf{A}'=\mathbf{A},$$
$$\varphi'=\varphi-\mathbf{v}\cdot\mathbf{A}. \tag{24}$$

The invariance of $\mathbf{B}$ follows immediately from the invariance of $\mathbf{A}$, whereas

$$\mathbf{E}'=-(\partial\mathbf{A}'/\partial t)'-\nabla\varphi'$$
$$=-(\partial\mathbf{A}/\partial t)'-\nabla\varphi+\nabla(\mathbf{v}\cdot\mathbf{A}), \tag{25}$$

where

$$\nabla(\mathbf{v}\cdot\mathbf{A})=\mathbf{v}\cdot\nabla\mathbf{A}+\mathbf{A}\cdot\nabla\mathbf{v}+\mathbf{v}\times(\nabla\times\mathbf{A})+\mathbf{A}\times(\nabla\times\mathbf{v}).$$

Since $\mathbf{A}\cdot\nabla\mathbf{v}=\boldsymbol{\omega}\times\mathbf{A}$, and $\mathbf{A}\times(\nabla\times\mathbf{v})=2\mathbf{A}\times\boldsymbol{\omega}$, it follows that (using $\nabla\times\mathbf{A}=\mathbf{B}$)

$$\nabla(\mathbf{v}\cdot\mathbf{A})=\mathbf{v}\cdot\nabla\mathbf{A}-\boldsymbol{\omega}\times\mathbf{A}+\mathbf{v}\times\mathbf{B},$$

and hence, from Eq. (9),

$$\nabla(\mathbf{v}\cdot\mathbf{A})-(\partial\mathbf{A}/\partial t)'=-(\partial\mathbf{A}/\partial t)+\mathbf{v}\times\mathbf{B}.$$

Thus, from Eq. (25),

$$\mathbf{E}'=-(\partial\mathbf{A}/\partial t)-\nabla\varphi+\mathbf{v}\times\mathbf{B}$$

or $\mathbf{E}'=\mathbf{E}+\mathbf{v}\times\mathbf{B}$, in agreement with Eq. (23).

Maxwell's first two equations

$$\nabla\cdot\mathbf{B}=0,$$
$$\nabla\times\mathbf{E}=\partial\mathbf{B}/\partial t \tag{26}$$

follow immediately from Eq. (22), that is, from the existence of the potentials. Since, as has been explicitly demonstrated, potentials exist in the rotating system, it follows that Maxwell's first two equations have the same form in both systems, as was found by somewhat different methods in the preceding section.

The last two relations in Eq. (20) imply immediately

$$(\partial/\partial t)(\rho+\sigma)+\nabla\cdot(\mathbf{j}+\mathbf{i})=0.$$

From Eq. (21) it is readily seen that

$$(\partial\sigma/\partial t)+\nabla\cdot\mathbf{i}=0$$

and that therefore, in the rotating frame,

$$(\partial\rho/\partial t)+\nabla\cdot\mathbf{j}=0$$

in agreement with the transformation given in Eq. (4).

There is no unique way to describe uniform rotation in general relativity. Schiff's transformation

$$x'=x\cos\omega t+y\sin\omega t,$$
$$y'=-x\sin\omega t+y\cos\omega t,$$
$$z'=z,$$
$$t'=t,$$

may be used in a classical analysis; it was shown that the results are the same.

Classically, it was shown that Maxwell's first two equations [Eq. (26)] have the same form in all (uniformly) rotating systems. According to general relativity, these two equations have the same form in all systems and for all coordinate transformations. This general result follows from the assumption that the vector components of the electric and magnetic fields form an antisymmetric tensor F_{ik} which satisfies

$$(\partial F_{ik}/\partial x^j)+(\partial F_{kj}/\partial x^i)+(\partial F_{ji}/\partial x^k)=0. \tag{27}$$

Classically, the invariance of the first two equations followed from the existence of scalar and vector potentials in the rotating systems. Correspondingly, in general relativity, their general invariance is a consequence of the existence of a four potential A_k such that

$$F_{ik}=(\partial A_k/\partial x^i)-(\partial A_i/\partial x^k)$$

from which it follows that F_{ik} satisfies Eq. (27).

* Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of The Rand Corp. or the official opinion or policy of any of its governmental or private research sponsors.

[1] L. I. Schiff, Proc. Natl. Acad. Sci. U. S. **25**, 391 (1939).
[2] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1950), p. 133.

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LETTERS TO THE EDITOR

Note on Azimuthal Angle and Angular Momentum in Quantum Mechanics

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Recently, a paper on angle and angular momentum operators in quantum mechanics has been published in this Journal.[1] The authors discuss some difficulties connected with the commutation relation

$$[L_z,\phi]=-i \tag{1}$$

for angle ϕ and angular momentum L_z ($\hbar=1$). By taking matrix elements of (1) between eigenstates f_m and f_n of L_z with eigenvalues m and n, respectively, one arrives at

$$(m-n)(f_m,\phi f_n)=-i\delta_{mn}. \tag{2}$$

Furthermore, according to most quantum mechanics textbooks, a commutation relation like (1) implies an uncertainty relation

$$\operatorname{var}L_z\cdot\operatorname{var}\phi\geq\tfrac{1}{4}. \tag{3}$$

As pointed out in Ref. 1, however, the relations (2) and (3) are simply wrong. This is immediately obvious for Eq. (2) (take, e.g., $m=n$). Likewise, the uncertainty relation (3) is violated for eigenstates of L_z since for such states $\operatorname{var}L_z=0$ and $\operatorname{var}\phi=\pi^2/3$.

The authors of Ref. 1 attempt to resolve this apparent paradox by the statement that the commutation relation (1) is also wrong. In fact, they claim that the domain of definition of the commutator $[L_z,\phi]$ does not contain any nontrivial state vector and that, therefore, Eq. (1) makes no sense at all. This last assertion, however, is not true, as I will now explain.

As is well known, observables have to be represented by self-adjoint operators. The only candidate for ϕ is