2.A.2 *Nuclear Physics* **A261** (1976) 130—140; © *North-Holland Publishing Co., Amsterdam*

AN EXPANSION OF CONTINUUM WAVE FUNCTIONS IN TERMS OF RESONANT STATES

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Received 1 April 1975
(Revised 15 December 1975)

Abstract: The poles and residues of the complete outgoing Green function in the complex momentum plane are used to obtain, in the case of finite range potentials, an eigenfunction expansion of the continuum wave solution. It is found that in the region $r < a$ the wave solution may be expressed as an infinite sum of discrete terms involving the bound, antibound and resonant states of the problem. At the boundary radius $r = a$ a different expansion is obtained. In this case, in order to get an infinite discrete sum, one has to introduce two subtraction terms. Otherwise the expansion is given by a finite sum of discrete terms and an integral contribution.

1. Introduction

The introduction of complex eigenvalues of the Schrödinger equation to deal with nuclear problems is as old as quantum mechanics. In fact almost fifty years ago, in 1928, Gamow [1]) made use of complex eigenvalues in his studies on the decay of α-particles.

From a physical point of view, complex eigenvalues are used to describe a situation in which a particle stays for a finite time in a certain state of the system and then decays. From a mathematical point of view this requires of the absence of incoming waves in the asymptotic solutions of the Schrödinger equation. It follows from unitarity that this is only possible for complex energies. Consequently, the corresponding radial functions grow exponentially with distance and the usual methods of normalization and eigenfunction expansions are not valid in this case.

Although complex eigenvalues and related eigenfunctions apply in a natural way to the description of resonances, in view of the difficulties mentioned above, it is not surprising to see that the first nuclear reaction theories did not utilize them.

In fact the formalism developed by Kapur and Peierls [2]) was based on a procedure in which the eigenfunctions had to form a complete orthonormal set and therefore they were defined by a boundary condition which contained a fixed real momentum, instead of the complex momentum appropriate to the complex energy of each eigenfunction. In practice this fixed momentum was identified with the momentum for which the scattering was calculated.

The formalism due to Wigner and Eisenbund [3]) uses real eigenvalues, determined by imposing a real boundary condition. Therefore the states used in the description

do not depend on the energy of the scattering process. However the eigenvalues do not give directly the positions and widths of the resonances.

The idea of using resonant states as defined from a physical point of view has been so appealing that work along these lines has continued since the early work by Gamow.

Another relevant contribution is the work by Siegert [4]) in 1939, who showed that the properties of an isolated sharp resonance follow directly from the corresponding eigenfunction. Peierls [5]) in 1959, obtained a representation for the S-matrix as an expansion in terms of its poles in the complex energy (or momentum) plane.

In 1961, Humblet and Rosenfeld published in a series of papers [6]) a theory of nuclear reactions based on these physically defined eigenvalues and eigenfunctions. Since their formalism rests on the mathematical device of the Mittag-Leffler expansion of the scattering or reaction amplitude, the problem related to the normalization and eigenfunction expansions of the resonant states is not considered. Therefore although the dispersion theory is very general, it is not well suited to deal with problems that would require knowledge of the eigenfunctions.

In recent years a number of authors [7-11]) have emphasized the question of the normalization and eigenfunction expansions of these complex eigenfunctions. The idea is to obtain a consistent procedure in order to use these states in the description of a number of processes where resonances play an important role, as for instance, stripping to unstable states.

Although at present there are already several normalization procedures for these states, the question of eigenfunction expansions seems to be not fully studied. In this context the well-known relationship between the poles of the scattering matrix, or more generally, of the complete outgoing Green function and the bound and resonant states of the problem is relevant.

Some years ago, Berggren did interesting work in relation to eigenfunction expansions involving resonant states. He considered a situation where a set of resonant states, the so-called proper resonant states [6])†, form part of a complete set of states which also includes the bound states of the system and a set of continuum states. It is relevant at this point to indicate that the continuous contribution arises from an integration along a line in the complex momentum plane that passes through the origin at an angle of 45° and eventually returns to the real axis. The eigenfunction expansion considered by Berggren does not include in a discrete way any contribution from resonance poles situated in the third quadrant of the complex k-plane. Actually such a contribution is incorporated in the integral term. It is not evident that the resonant poles situated in the third quadrant yield a slowly energy varying and as usually expected, negligible contribution. Therefore we believe that it is worthwhile investigating eigenfunction expansions involving complex poles in both the third and fourth quadrants of the complex k-plane.

† Proper resonant states are those associated with poles of the scattering matrix situated in the fourth quadrant of the complex momentum plane and such that the real part of the complex momentum is greater than the corresponding magnitude of the imaginary part.

In this paper we would like to present such an eigenfunction expansion, namely, one including the whole set of eigenfunctions associated with the poles of the complete outgoing Green function for the case of the continuum solutions of the wave equation. Our development is based on a Cauchy expansion of the complete outgoing Green function in the complex momentum plane. This procedure requires the investigation of the asymptotic behaviour of that function as the momentum tends to infinity in all directions of the momentum plane.

2. The wave function in the interior region and the outgoing Green function

The case of scattering by a potential is particularly convenient to illustrate our formalism.

We shall restrict ourselves to the case of S-wave scattering (though the same method is applicable for any other fixed angular momentum). We are thus dealing with the radial Schrödinger equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\phi(r)+[k^2-V(r)]\phi(r) = 0, \tag{2.1}$$

where k^2 is the energy in appropriate units. The wave function $\phi(r)$ obeys the boundary condition

$$\phi(r = 0) = 0. \tag{2.2}$$

The interaction potential $V(r)$ is assumed to be spherically symmetric and of finite range. That is

$$V(r) \equiv 0, \quad r > a. \tag{2.3}$$

As is well known, the solution of the wave equation for distances $r > a$, may be written as

$$\phi(r) = \tfrac{1}{2}i[\mathrm{e}^{-ikr}-S(k)\mathrm{e}^{ikr}], \quad r \geqq a, \tag{2.4}$$

where, of course, $S(k)$ is the scattering matrix. It is also well known that the poles of $S(k)$ lie either in the lower half of the complex momentum plane or on the positive imaginary axis. The latter represents exponentially decreasing functions and belong to bound states, while the former are associated with exponentially increasing functions, and are called antibound states when the poles are situated on the negative imaginary axis and resonant states otherwise.

A particularly important role in our development is played by the outgoing Green function $G^+(r, r'; k)$ of the complete problem, which obeys the inhomogeneous equation

$$\frac{\partial^2}{\partial r^2}G^+(r, r'; k)+[k^2-V(r)]G^+(r, r'; k) = \delta(r-r'), \tag{2.5}$$

with the boundary conditions at $r = 0$ and $r = a_+$[†] respectively given by

$$G^+(0, r'; k) = 0, \tag{2.6}$$

$$\left[\frac{\partial}{\partial r} G^+(r, r'; k)\right]_{r=a_+} = [ikG^+(r, r'; k)]_{r=a_+}, \qquad r' \leqq a. \tag{2.7}$$

Let us derive an expression connecting the wave function in the region $r \leqq a$ and the outgoing Green function. This relation follows immediately by using the Green theorem between eqs. (2.1) and (2.5) together with its respective boundary conditions, i.e. eqs. (2.2), (2.6) and (2.7), and the symmetry property of the Green function. We obtain

$$\phi(r) = G^+(r, a_+; k)\,[ik\phi(a_+) - \phi'(a_+)], \quad r \leqq a, \tag{2.8}$$

where the dummy variable r' is changed by the variable r. The expression inside the brackets on the right-hand side of eq. (2.8) depends only on the value at $r = a$. Due to the continuity of the solution $\phi(r)$ at the point $r = a$, we may utilize eq. (2.4) in order to write eq. (2.8) as

$$\phi(r) = -G^+(r, a; k)\, k e^{-ika}, \quad r \leqq a, \tag{2.9}$$

this expression relates, for a given value of the real momentum k, the interior wave function with the outgoing Green function multiplied by a constant factor.

It follows from expression (2.9) that an eigenfunction expansion for the wave function in the region $r \leqq a$ may be obtained by expanding the outgoing Green function in terms of its poles in the complex energy or momentum planes. The relevant point is that the residues at the poles of the outgoing Green function are precisely the bound, antibound or resonant states of the problem.

The outgoing Green function $G^+(r, r'; k)$ is defined for all $r, r' \geqq 0$, and is considered as a function of the complex momentum or energy variables; it possesses, except in very special cases, the same distribution of poles as the scattering matrix $S(k)$. In the complex momentum plane, the outgoing Green function is a meromorphic function. In general all its poles are simple. Poles of second or higher order occur only in very special circumstances. We assume that we are dealing with problems in which these difficulties are not present. Let us consider the contour integral

$$I = \frac{1}{2\pi i}\int_\Gamma \frac{G^+(r, a; k')}{k' - k}\, dk', \tag{2.10}$$

where Γ represents a large closed contour in the complex momentum k-plane about the origin in the clockwise direction which excludes all simple poles, k_n, and the value $k' = k$. Hence by using Cauchy's theorem it follows that

$$I = \frac{1}{2\pi i}\left[-\int_C \frac{G^+(r, a; k')}{k' - k}\, dk' + \sum_n^N \int_{C_n} \frac{G^+(r, a; k')}{k' - k}\, dk' + \int_{C_k} \frac{G^+(a, r'; k')}{k' - k}\, dk'\right] = 0, \tag{2.11}$$

[†] We define $a_\pm = \lim_{\varepsilon \to 0} a \pm \varepsilon$.

where in the above expression, C represents a large circle of radius R centered at the origin; the contours C_n, encircle each of the simple poles k_n that are enclosed by the contour C, and the contour C_k encloses the momentum $k' = k$. The contours C, C_n and C_k, are all in the counterclockwise direction.

By using the residue theorem, we may write expression (2.11) as

$$G^+(r, a; k) = \sum_n^N \frac{\rho_n}{k-k_n} + \frac{1}{2\pi i}\int_C \frac{G^+(r, a; k')}{k'-k}\,dk'. \tag{2.12}$$

The expression for the residues has been derived elsewhere [11, 12, 14]) and will not be repeated here. The residues are given by

$$\rho_n = U_n(a)U_n(r)/2k_n \left[\int_0^a U_n^2(r)dr + iU_n^2(a)/2k_n\right]. \tag{2.13}$$

The functions U_n that appear in the above relation represent bound, antibound or resonant states of the problem. These eigenfunctions obey the radial equation

$$\frac{d^2}{dr^2} U_n(r) + [k_n^2 - V(r)]U_n(r) = 0, \tag{2.14}$$

with the boundary condition at the origin

$$U_n(r = 0) = 0, \tag{2.15}$$

and the outgoing boundary condition

$$\left[\frac{d}{dr} U_n(r)\right]_{r=a} = [ik_n U_n(r)]_{r=a}. \tag{2.16}$$

The quantity inside the brackets appearing in the denominator of eq. (2.13) gives the normalization of these eigenfunctions

$$\int_0^a U_n^2(r)dr + iU_n^2(a)/2k_n = 1. \tag{2.17}$$

This normalization procedure is consistent with bound state eigenfunctions. In fact it is easy to convince oneself that the above normalization reduces to the usual procedure in the case of bound states.

Therefore, by using eqs. (2.13) and (2.17) in eq. (2.12) we obtain

$$G^+(r, a; k) = \sum_n^N \frac{U_n(a)U_n(r)}{2k_n(k-k_n)} + \frac{1}{2\pi i}\int_C \frac{G^+(a, r; k')dk'}{k'-k}. \tag{2.18}$$

Substitution of eq. (2.18) into eq. (2.9) gives the following relation for the wave function in the region $r \leqq a$:

$$\phi(r) = \sum_n C_n U_n(r) + \frac{ke^{-ika}}{2\pi i}\int_C \frac{G^+(r, a; k')}{k'-k}\,dk', \tag{2.19}$$

where the coefficient C_n is given by

$$C_n = k\mathrm{e}^{-ika}U_n(a)/2k_n(k_n-k). \tag{2.20}$$

The expansion of the wave function in the interior region given above contains a finite number of discrete terms and an integral contribution. Our aim is to obtain an eigenfunction expansion involving only discrete terms. Since for finite range potentials the number of complex poles is infinite [13]), one has to consider the limit of an infinite radius for the contour C of the corresponding integral term in order to include all the poles into the sum of eq. (2.19). A sufficient condition for the vanishing of the integral term is

$$G^+(r, a; k) \to 0 \quad \text{as } |k| \to \infty,$$

where obviously the variable k is complex. Therefore in order to see if the contribution of the integral term vanishes, one has to look at the behaviour of the outgoing Green function $G^+(r, a; k)$ as the complex momentum tends to infinity.

Using eq. (2.8) one can get an expression for $G^+(r, a; k)$ of the following form

$$G^+(r, a; k) = G^+(a, a; k)\phi(r)/\phi(a), \quad r < a. \tag{2.21}$$

The above expression is particularly convenient to study the behaviour of the outgoing Green function with k. In fact in that way one can make use of the results obtained by Peierls [5]) in connection with the behaviour of the scattering matrix $S(k)$ as k tends to infinity. This follows immediately by realizing the close connection between the outgoing Green function evaluated at $r = r' = a$ and the scattering matrix. In fact by substituting eq. (2.4) for $r = a$ into eq. (2.8) we obtain a relation that can be written as

$$G^+(a, a; k) = \frac{1}{2ki}[1-S(k)\mathrm{e}^{2ika}]. \tag{2.22}$$

Peierls showed that for finite range potentials†, $S(k)$ behaves for very large k as $4k^2 \exp[-2ika]$ in the lower half k-plane and as $\exp[-2ika]/4k^2$ in the upper half k-plane. Therefore by using the above results into (2.22) it follows that $G^+(a, a; k)$ for very large k is proportional to k in the lower half k-plane and goes as $1/k$ in the upper half k-plane. On the other hand, it is well known [13]) that along the real axis $S(k)$ tends to the unity as k tends to infinity and consequently $G^+(a, a; k)$ goes to zero.

In summary we obtain the result that $G^+(a, a; k)$ diverges along any direction of the lower half k-plane as k tends to infinity.

Let us now consider the behaviour of the second factor on the right-hand side of eq. (2.21), i.e. $\phi(r)/\phi(a)$ as the momentum increases.

† The analysis is made for potentials having a finite different from zero value at the point $r = a$. This includes most of the interesting cases. One should notice that any long range potential can be cut off in a way that satisfies the above condition.

It is well known [13]) that the "physical" solution may be compared with the so called "regular" solution $\psi(k, r)$ through the relation

$$\phi(r) = k\psi(k, r)/J(k), \tag{2.23}$$

where $J(k)$ is the Jost function. In contrast to $\phi(r)$, the "regular" solution is everywhere an analytic function of k. Hence it is convenient to express the factor $\phi(r)/\phi(a)$ in terms of "regular" solutions,

$$\phi(r)/\phi(a) = \psi(k, r)/\psi(k, a). \tag{2.24}$$

The "regular" solution obeys the integral equation [13]),

$$\psi(k, r) = \frac{\sin kr}{k} + \frac{\gamma}{k}\int_0^r \mathrm{d}r' \sin k(r-r')V(r')\psi(k, r'). \tag{2.25}$$

The important point is that for finite range potentials the "regular" function $\psi(k, r)$ behaves as

$$\psi(k, r) = \sin kr/k, \tag{2.26}$$

when $|k| \to \infty$; thus, $\psi\ (k, r)$ approaches its "unperturbed" value as $|k| \to \infty$ in all directions of the complex plane. Hence, substitution of (2.25) into (2.23) leads to

$$\phi(r)/\phi(a) = \sin kr/\sin ka, \quad r < a. \tag{2.27}$$

Since the radius a is always larger than the variable r, it is straightforward to see that in both the upper and the lower halves of the complex k-plane the quantity $\phi(r)/\phi(a)$ vanishes exponentially as k goes to infinity. In a similar way it may be seen that this quantity oscillates indefinitely along the real k-axis.

From the above analysis it follows that the outgoing Green function $G^+(r, a; k)$ vanishes in all directions of the complex plane as the momentum tends to infinity, i.e.

$$G^+(r, a; k) \to 0 \quad \text{as} \quad |k| \to \infty. \tag{2.28}$$

Hence the integral term of eq. (2.19) goes to zero in the limit of infinite radius and we obtain the interesting result that in the region $r < a$ the Green function $G^+(a, r; k)$ may be written as

$$G^+(a, r; k) = \sum_n^{\infty} \frac{U_n(a)U_n(r)}{2k_n(k-k_n)}, \qquad r < a. \tag{2.29}$$

Consequently in the region $r < a$, the wave function $\phi(r)$ can be expanded in terms of the discrete sum

$$\phi(r) = \sum_n^{\infty} C_n U_n(r), \quad r < a, \tag{2.30}$$

where the expansion coefficients C_n are given by eq. (2.20).

We may ask the question as to what happens at the boundary value $r = a$. It is evident that eq. (2.29) cannot be extended to deal with this case. Since $G^+(a, a; k)$

diverges as $|k| \to \infty$, we have to return to eq. (2.18), which for $r = a$ reads

$$G^+(a, a; k) = \sum_n^N \frac{U_n^2(a)}{2k_n(k-k_n)} + \frac{1}{2\pi i}\int_C \frac{G^+(a, a; k')}{k'-k}\,dk'. \tag{2.31}$$

Therefore the expression for the wave function at $r = a$ is given by

$$\phi(a) = \sum_n^N C_n U_n(a) + k\frac{e^{-ika}}{2\pi i}\int_C \frac{G^+(a, a; k')}{k-k'}\,dk', \tag{2.32}$$

where the coefficient C_n is the same as in eq. (2.20). Notice that the discrete sum in eq. (2.32) runs up to a finite number of resonance terms. If we want to consider a sum including an infinite number of resonances, the analysis made above on the asymptotic behaviour of the complete outgoing Green function $G^+(a, a; k)$, namely, that this function diverges linearly in k as k goes to infinity, implies that the integral term in eq. (2.32) diverges as the radius R if the circle C tends to infinity. The preceding result means that we have to consider two subtraction terms in the expansion of eq. (2.32) if we want to obtain a purely discrete expansion. Otherwise we have to deal with a finite sum of terms and an additional integral contribution.

The expression for the wave function $\phi(a)$ involving two subtractions follows directly from Cauchy's expansion and is given by

$$\phi(r) = -ke^{-ika}\left[G^+(a, r; 0) + k\left[\frac{d}{dk'}G^+(a, r; k')\right]_{k'=0} + k^2\sum_n^\infty \frac{U_n(a)U_n(r)}{2k_n^3(k-k_n)}\right]. \tag{2.33}$$

3. Expansion coefficients

In this section we derive an alternative expression of the expansion coefficients C_n given by eq. (2.20). Our aim is to obtain an expression for each coefficient C_n in terms of an integral involving the wave function $\phi(r)$ and the corresponding eigenfunction $U_n(r)$. The derivation of such an expression is well known in the framework of other reaction formalisms. In our case one has to proceed in a different way because the eigenfunction expansion of the wave function is different in the region $r < a$ and at the boundary radius $r = a$.

We are interested in deriving integral expressions for the coefficients C_n because they could be useful in connection with the problem of completeness of the set of eigenfunctions $U_n(r)$.

Consider the equation for the eigenfunction $U_n(r)$ and the complete outgoing Green function $G^+(r, r'; k)$ given respectively by eq. (2.14) and eq. (2.5). Using Green's theorem we obtain

$$\left[U_n(r)\frac{d}{dr}G^+(r, r'; k) - G^+(r, r'; k)\frac{d}{dr}U_n(r)\right]_{r=0}^{r=a+} + (k^2-k_n^2)\int_0^{a+} U_n(r)G^+(r, r'; k)dr = \int_0^{a+} U_n(r)\delta(r-r')dr. \tag{3.1}$$

Applying the boundary conditions associated with $U_n(r)$ and $G^+(r, r'; k)$, i.e. eqs. (2.6), (2.7) and (2.16), to expression (3.1) gives the result

$$U_n(r') = i(k-k_n)U_n(a)G^+(a, r'; k)+(k^2-k_n^2)\int_0^a U_n(r)G^+(r, r'; k)\mathrm{d}r, \qquad r' \leqq a. \tag{3.2}$$

It is now convenient to consider eq. (3.2) at the point $r' = a$ and then make use of eq. (2.9). The resulting expression can be written as

$$\int_0^a \phi(r)U_n(r)\mathrm{d}r + i\frac{\phi(a)U_n(a)}{k+k_n} = \frac{kU_n(a)\mathrm{e}^{-ika}}{k_n^2-k^2}. \tag{3.3}$$

Therefore, by comparing eq. (3.3) with the expression for the coefficients C_n, i.e. eq. (2.20), allows the coefficient C_n to be written as the product

$$C_n = h_n a_n. \tag{3.4}$$

The coefficients h_n and a_n are given respectively by

$$h_n = (k+k_n)/2k_n, \tag{3.5}$$

$$a_n = \int_0^a \phi(r)U_n(r)\mathrm{d}r + i\frac{\phi(a)U_n(a)}{k+k_n}. \tag{3.6}$$

Substitution of eqs. (3.5) and (3.6) into (2.30) gives the following expression for the wave function in the region $r < a$:

$$\phi(r) = \sum_n^\infty \frac{k+k_n}{2k_n}\left[\int_0^a \mathrm{d}r'\,\phi(r')U_n(r') + i\frac{\phi(a)U_n(a)}{k+k_n}\right]U_n(r), \qquad r < a. \tag{3.7}$$

The above equation may be written as

$$\phi(r) = \sum_n^\infty \frac{k+k_n}{2k_n}\left[\int_0^a \mathrm{d}r'\,\phi(r')U_n(r')\right]U_n(r) + \phi(a)\sum_n^\infty i\frac{U_n(a)U_n(r)}{2k_n}, \qquad r < a. \tag{3.8}$$

Actually the last term of eq. (3.8) vanishes exactly. In fact let us multiply and divide that term by $k-k_n$ and then make use of the outgoing boundary condition for the eigenfunctions $U_n(r)$ [eq. (2.16)] in order to write the last term of eq. (3.8) as

$$\phi(a)\left[ik\sum_n^\infty \frac{U_n(a)U_n(r)}{2k_n(k-k_n)} - \sum_n \frac{U_n'(a)U_n(r)}{2k_n(k-k_n)}\right], \tag{3.9}$$

where $U_n'(a)$ means the derivative of $U_n(r)$ with respect to r calculated at $r = a$.

Using eq. (2.29) allows eq. (3.9) to be written as

$$\phi(a)[ikG^+(r, a; k) - G'^+(r, a; k)]. \tag{3.10}$$

We can immediately see from the boundary condition obeyed by the Green function $G^+(r, a; k)$, i.e. eq. (2.7), that eq. (3.7) is identically zero. Therefore the eigenfunction

expansion for the wave function in the region $r < a$ reads

$$\phi(r) = \sum_n^{\infty} h_n \left[\int_0^a \phi(r') U_n(r') \mathrm{d}r' \right] U_n(r), \qquad r < a, \tag{3.11}$$

where the coefficients h_n are given by eq. (3.5).

We may interprete the coefficient h_n as a weight factor. This is suggested by the fact that the integral term in eq. (3.11), i.e.

$$b_n = \int_0^a \phi(r') U_n(r') \mathrm{d}r', \tag{3.12}$$

corresponds to the familiar inner product between the wave function $\phi(r)$ and the eigenfunction $U_n(r)$.

Let us now consider the situation at the boundary radius $r = a$. Substitution of eqs. (3.5) and (3.6) into eqn. (2.32) leads to the expression

$$\phi(a) = \sum_n^N h_n \left[\int_0^a \mathrm{d}r' \, \phi(r') U_n(r') + i \frac{\phi(a) U_n(a)}{k + k_n} \right] U_n(a) + \frac{k \mathrm{e}^{-ika}}{2\pi i} \int_C \mathrm{d}k' \frac{G^+(a, a; k')}{k - k'} . \tag{3.13}$$

Since the sum in eq. (3.13) includes a finite number of terms it is evident that the surface term of the corresponding expansion coefficient does not vanish. Clearly, this situation is in contrast to the result obtained previously for the region $r < a$.

It is of interest to mention that in a recent paper, Romo [15]) asserts that bound, antibound and resonant states form a complete set of states in the internal region of space $r \leqq a$. The analysis made by Romo is only for the delta shell potential. Our results agree with those of Romo only in the region $r < a$. We have shown that at the boundary radius $r = a$, a purely discrete expansion is not valid unless we introduce subtraction terms†. Otherwise in addition to a finite sum of discrete terms there is an integral contribution. We would like to point out that Romo's analysis is made in terms of integrals along the internal region in such a way that the features of the eigenfunction expansion at $r = a$ are not exhibited.

4. Conclusions

In this paper we have shown, for the case of finite range potentials, that the poles and residues of the complete outgoing Green function as a function of a complex variable, may be used to obtain a purely discrete eigenfunction expansion of the wave function $\phi(r)$ in the region of space $r < a$. Such an expansion possesses an infinite number of terms involving the bound, antibound and resonant eigenfunctions of the

† Actually the delta shell potential possess the peculiarity that at $r = a$ it has an infinite value. The exact expression of $G^+(a, a; k)$ associated with the delta shell potential $V(r) = \lambda\delta(r-a)$ is $G^+(a, a; k) = 1/ik - k \cot ka - \lambda$. It is straightforward to show that $G^+(a, a; k)$ behaves respectively as $1/k$ in the upper k-plane and as a constant in the lower k-plane. Therefore a purely discrete expansion requires only one subtraction term.

problem. On the other hand we have obtained that at the boundary radius $r = a$, a purely discrete eigenfunction expansion of the wave function $\phi(a)$ is not valid unless we introduce into the expansion two subtraction terms. Otherwise we have to consider, in addition to a finite number of discrete terms, an integral contribution. Since the resonance contribution is included into the discrete part we may expect, if the discrete sum runs over sufficiently larger number of resonances, that the integral term corresponds to a slowly energy-varying quantity.

Therefore we reach the conclusion that for finite range potentials the residues and poles of the complete outgoing Green function do not give a complete representation of the wave function in the interior region $r \leqq a$. Since the scattering matrix is constructed from the condition of the continuity of the interior and exterior wave solutions at the boundary $r = a$, we conclude also that a purely discrete expansion for the scattering matrix is not valid unless we introduce two subtraction terms.

It is a pleasure to thank Prof. R. E. Peierls for an illuminating discussion and Dr. M. Berrondo for useful discussions.

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