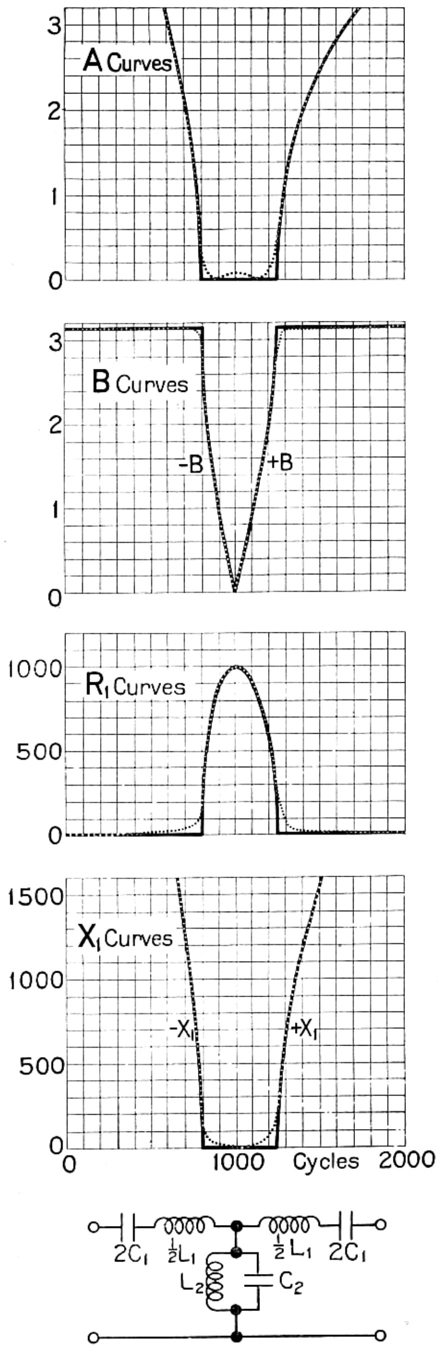
otherwise leaves the curve approximately unchanged. The full line curves for R_1 and X_1 show that in the ideal case the iterative impedance is pure resistance and pure reactance in the pass and stop bands respectively, and that resistance smooths the abrupt transition at the critical frequency.

The high pass wave-filter shown by Fig. 9 passes the band which is stopped by the low pass wave-filter of Fig. 8, and vice versa. For this reason the two wave-filters are said to be complementary.

Another set of two complementary wave-filters is shown by Figs. 10 and 11, one of which passes only a single band of frequencies, not extending to either zero or infinity, while the other passes the remaining frequencies only. The single pass band of Fig. 10, embracing a total phase change 2π on the B curve, is actually a case of confluent pass bands, each of which embraces the normal angle π. The tendency of the two simple pass bands to separate, and leave a stop band between them, is shown by the hump in the dotted attenuation constant curve at 1,000 cycles. If, instead of the two simple bands having been brought together, one of them had been relegated to zero or infinity, the single remaining pass band would have exhibited the normal angular range π in the B curve, and there would have been no hump in the dotted A curve. The stop band of Fig. 11 also illustrates peculiarities which are not necessary features of a wave-filter with a single stop band in this position. This wave-filter is obtained from Fig. 7 by making all bands vanish except P_2, S_3, S_5 and P_3,—by extending P_2 to zero, P_3 to infinity, and making S_3 and S_5 coalesce, so that the attenuation becomes infinite in the stop band without passing from a stop $(-)$ to a stop $(+)$ band. The coalescing stop bands are responsible for the rapid changes in the B, R_1, and X_1 curves of Fig. 11 which would not have appeared if, in Fig. 7, the same pass band had been obtained by retaining P_1, S_2 and P_2 and making all other bands vanish.

An extreme case of complementary wave-filters is shown by Figs. 12 and 13, where no frequencies and all frequencies are passed respectively. The first result is obtained by combining inductances alone, which, as has been pointed out above, can give only an attenuated disturbance devoid of wave characteristics. The wave-filter shown for passing all frequencies has inductance coils in the line, and capacities diagonally bridged across the line. This wave-filter combines a constant iterative impedance with a progressive change in phase which is sometimes useful.[4] An outstanding char-

[4] A theoretical use of the phase shifting afforded by the lattice artificial line was made at page 253 of "Maximum Output Networks for Telephone Substation and Repeater Circuits," Trans. A. I. E. E., vol. 39, pp. 231–280, 1920.



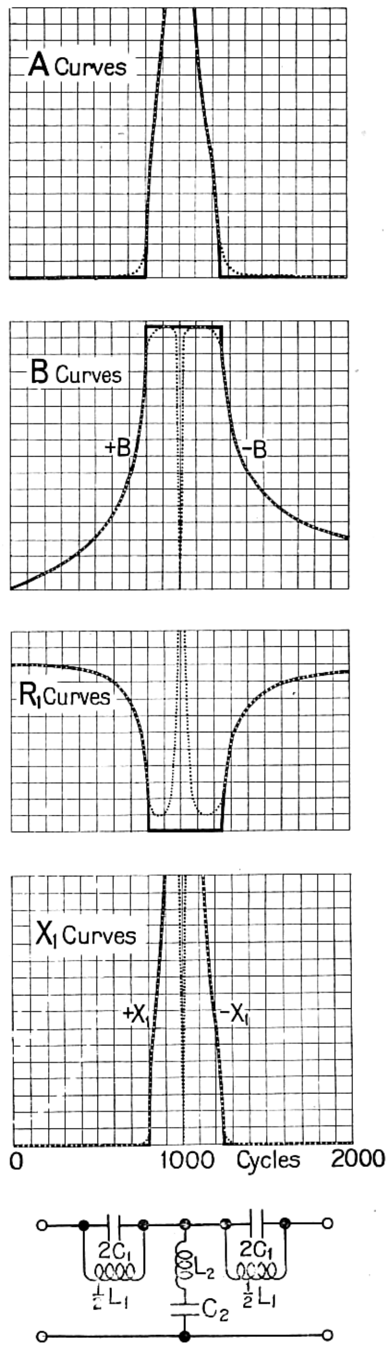
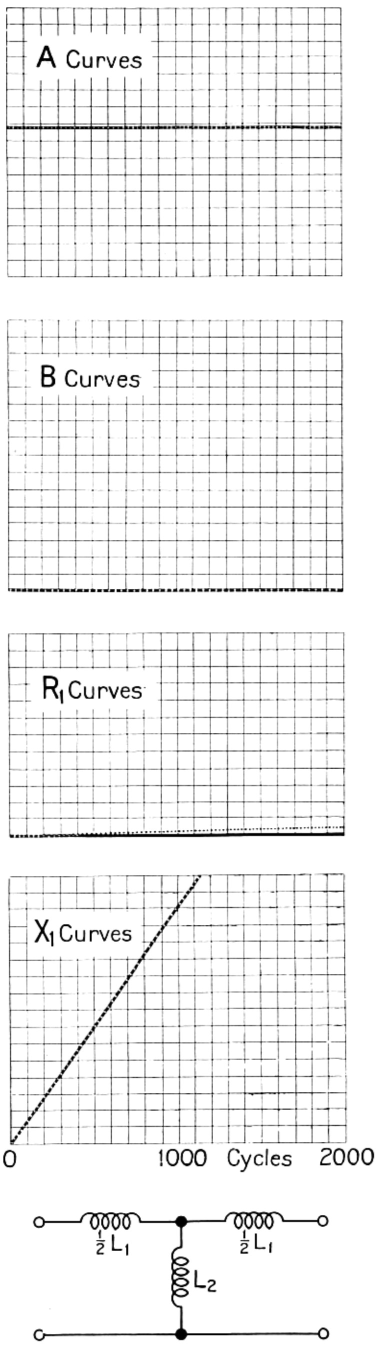


Fig. 10—Single Band Pass Wave-Filter: $L_1 = 20/9\pi$, $L_2 = 9/80\pi$, $C_1 = 9/80\pi$, $C_2 = 20/9\pi$

Fig. 11—Complementary High and Low Pass Wave-Filter: $L_1 = 9/20\pi$, $L_2 = 5/9\pi$, $C_1 = 5/9\pi$, $C_2 = 9/20\pi$



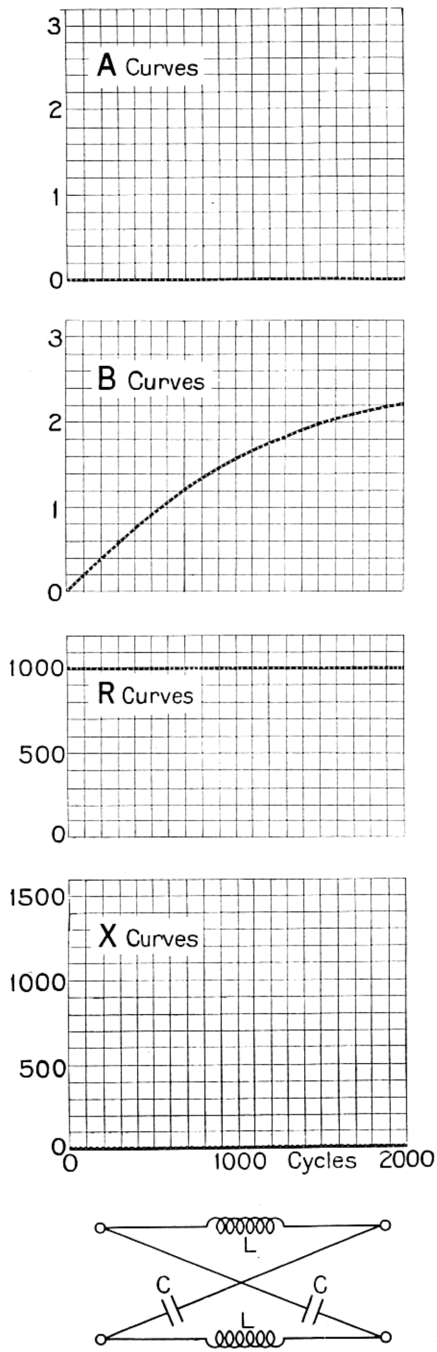


Fig. 12—No Pass Wave-Filter: $L_1 = 1/\pi$, $L_2 = 1/4\pi$
Fig. 13—Complementary All Pass Wave-Filter: $L = 1/2\pi$, $C = 1/2\pi$

acteristic of this type of artificial line is that it has, for all frequencies, the same iterative impedance as a uniform line with the same total series and shunt impedances. This artificial line will be considered in more detail in the next section of this paper.

Lattice Artificial Lines

Up to this point we have considered the properties of artificial line networks which were supposed to be given. In practice the problem is ordinarily reversed, and we ask the questions: May the locations of the bands be arbitrarily assigned? May additional conditions be imposed? How may the corresponding network be determined, and what is its attenuation in terms of the assigned critical frequencies? These questions might be answered by a study of Fig. 7, in

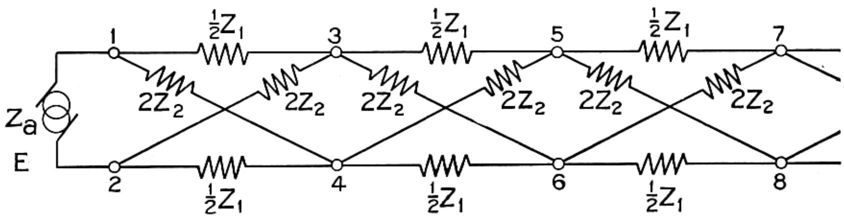


Fig. 14—Lattice Artificial Line

all its generality, but it seems simpler to base the discussion upon the artificial line shown in Fig. 14, which is to be a generalization of Fig. 13 to the extent of making the two impedances Z_1 and Z_2 any possible actual driving-point impedances. It is sometimes illuminating to regard this artificial line as a nest of bridges, one within another, as shown by Fig. 15.

On interchanging terminals 3 with 4 and 7 with 8 in Fig. 14 the network of lines remains unchanged; thus, Z_1 and $4Z_2$ may be interchanged in the formulas for the artificial line with no change in the result, except, possibly, one corresponding to a reversal of the current at alternate junction points. Another elementary feature of this artificial line is that it degenerates into a simple shunt or a simple series circuit at the resonant or anti-resonant frequencies, respectively, of either Z_1 or Z_2, and these are the critical frequencies, terminating the pass bands. At other frequencies, a positive ratio $Z_1/4Z_2$ must give a stop band, since the reactances are all of one sign. If a small negative value of this ratio gives free transmission, as we naturally expect, there will be identical transmission, except for a reversal of

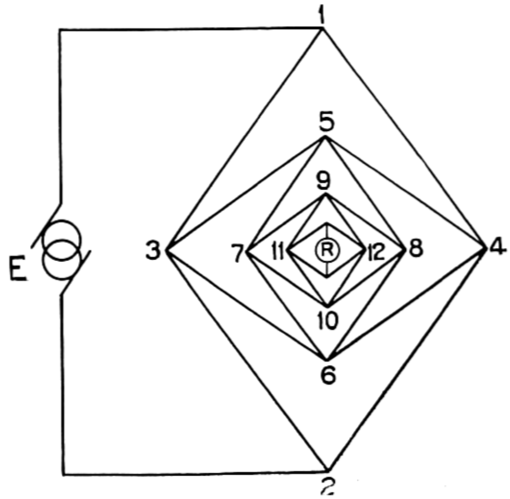


Fig. 15—Lattice Artificial Line Drawn to Show the Chain of Bridge Circuits

sign, when the ratio has the reciprocal value, which will be a large negative quantity, since we may always interchange Z_1 and $4Z_2$. The consequences of this and of other elementary properties of this artificial line are brought together in the following table:

TABLE II

For Lattice Artificial Line, Fig. 14

Band	Critical Frequency	Ratio $\frac{Z_1}{4Z_2}$	Uniform Line		Artificial Line		
			γ	k	Γ	$e^{-\Gamma}$	K
Stop (+)		$1>>0$	$2>>0$	imag.	+ real	$0<<1$	imag.
	$Z_1=0$ $Z_2=\infty$	0 0	0 0	0 ∞	0 0	1 1	0 ∞
Pass		<0	imag.	+ real	imag.	$e^{i\theta}$	+ real
	$Z_1=\infty$ $Z_2=0$	∞ ∞	∞ ∞	∞ 0	$i\pi$ $i\pi$	-1 -1	∞ 0
Stop (−)		<1	<2	imag.	$i\pi$ + real	$-1<<0$	imag.
	$Z_1=4Z_2$	1	2	$2Z_2$	∞	0	$2Z_2$

The cycle of bands: stop (+), pass, stop (−), adopted for the table, carries the attenuation factor $e^{-\Gamma}$ around the periphery of a unit semi-circle; in the stop (+) band it traverses the radius from 0 to 1, in the pass band it travels along the unit circle through 180 degrees to the value -1, completing the cycle from -1 to 0 in the stop (−) band. In this cycle there are four points of special interest, corresponding to ratio values 1, 0, -1 and ∞, for which the wave is infinitely attenuated, unattenuated with an angular change of 0, of 90, and of 180 degrees, respectively. It is at the 90 degree angle that resonance of the individual section occurs; the iterative impedance is then equal to $2|Z_2|$.

Graph of the Ratio $Z_1/4Z_2$ for Fig. 14

If we plot Z_1 and $4Z_2$ the pass bands are shown by the points where the curves become zero or infinite, and the intersections of the two

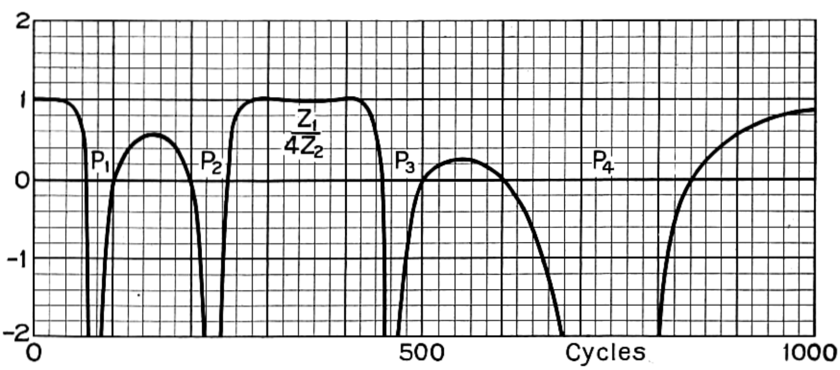


Fig. 16—Graph for Locating the Pass and Stop Bands of the Lattice Artificial Line, where $Z_1/4Z_2 = \left[(x_1^2 - x^2)\,(x_2^2 - x^2)^{-2}\,(x_3^2 - x^2)\right]\ldots$, x = cycles/100, and the resonant roots $x_1, x_3, \ldots$ are 0.650, 1, 2, 2.452, 4.442, 5, 6, 8.476 and the double anti-resonant roots $x_2, x_5, \ldots$ are 0.766, 2.301, 4.585, 7.423

curves show the frequencies at which the attenuation becomes infinite. These intersections must be at an acute angle since each branch of the two curves has a positive slope throughout its entire length; for this reason it may be desirable to plot the ratio rather than the individual curves; this is especially desirable in cases where the two curves do not intersect, but are tangent. Fig. 16 is for a lattice network equivalent to two sections of the ladder type illustrated by Fig. 7, and so cannot include a stop (−) band. Accordingly, the ratio does not go above unity, although it reaches unity at the two frequencies 300 and 400, corresponding to the infinite attenuation where stop (−) and stop (+) bands meet in Fig. 7. It is also unity at the extreme frequencies zero and infinity. The four pass bands have, of course, the same locations as in Fig. 7.

Multiplying the ratio by a constant greater than unity introduces stop (−) bands along with the stop (+) bands; multiplying it by a constant less than unity removes all infinite attenuations; these changes within the stop bands are made without altering the locations of the four pass bands.

Wave-Filter Having Assigned Pass Bands

In connection with practical applications we especially desire to know what latitude is permitted in the preassignment of properties for a wave-filter. If we consider first the ideal lattice wave-filter, its limitations are those inherent in the form which its two independent resistanceless one-point impedances[5] Z_1 and Z_2 may assume. The mathematical form of this impedance is shown by formula (7) of the appendix, which may be expressed in words as follows:

Within a constant factor the most general one-point reactance obtainable by means of a finite, pure reactance network is an odd rational function of the frequency which is completely determined by assigning the resonant and anti-resonant frequencies, subject to the condition that they alternate and include both zero and infinity.

The corresponding general expressions for the quotient and product of the impedances Z_1 and Z_2 are shown by formulas (8) and (9). Definite, realizable values for all of the $2n+2$ parameters and $2n+1$ optional signs occurring in these formulas may be determined in the following manner:

(a) Assign the location of all n pass bands, which must be treated as distinct bands even though two or more are confluent; this fixes the values of the $2n$ roots $p_1 \ldots p_{2n}$ which correspond to the successive frequencies at the two ends of the bands.

(b) Assign to the lower or upper end of each pass band propagation without phase change from section to section; this fixes the corresponding optional sign in formula (8) as + or −, respectively.

(c) Assign a value to the propagation constant at any one non-critical frequency (that is, assign the attenuation constant in a

[5] A one-point impedance of a network is the ratio of an impressed electromotive force at a point to the resulting current at the same point—in contradistinction to two-point impedances, where the ratio applies to an electromotive force and the resulting current at two different points.

stop band or the phase constant in a pass band); this fixes the value of the constant G and thus completely determines formula (8) on which the propagation constant depends.

(d) Assign to the lower or upper end of each stop band the iterative impedance zero; this fixes the corresponding optional sign in formula (9) as $+$ or $-$, respectively.

(e) Assign the iterative impedance at any one non-critical frequency (subject to the condition that it must be a positive resistance in a pass band and a reactance in a stop band); this fixes the constant H and thereby the entire expression (9) upon which the iterative impedance depends.

The quotient and product of the impedances Z_1 and Z_2 are now fully determined; the values of Z_1 and Z_2 are easily deduced and also the propagation constant and iterative impedance by formulas (11) and (12); Z_1 and Z_2 are physically realizable except for the necessary resistance in all networks.

These important results may be summarized as follows:

A lattice wave-filter having any assigned pass bands is physically realizable; the location of the pass bands fully determines the propagation constant and iterative impedance at all frequencies when their values are assigned at one non-critical frequency, and zero phase constant and zero iterative impedance are assigned to the lower or upper end of each pass band and stop band, respectively.

Lattice Artificial Line Equivalent to the Generalized Artificial Line of Fig. 1

Since any number of arbitrarily preassigned pass bands may be realized by means of the lattice network, it is natural to inquire whether this network does not present a generality which is essentially as comprehensive as that obtainable by means of any network N in Fig. 1, provided the generalized line is so terminated as to equalize its iterative impedances in the two directions. This proves to be the case.

If network N has identical iterative impedances in both directions, the lattice network equivalent to two sections of N is shown by Fig. 17; each lattice impedance is secured by using an N network; the N's placed in the two series branches of the lattice have their far terminals short-circuited so that they each give the impedance denoted by Z_0; the N's in the two diagonal branches have their far ends open and they each give the impedance denoted by Z_∞.