let us consider the free oscillations of Fig. 6; first, with K_2 assumed to be a pure reactance; second, with K_2 assumed to be a pure resistance; and third, in order to show that this third assumption is contrary to fact, with K_2 assumed to be an impedance with both resistance and reactance.

With K_2 a reactance, the circuit contains nothing but reactances, and free oscillations are possible if, and only if, the total impedance of the circuit is zero. The end impedances Z' and Z'' being different, the potentials at the ends of the mesh will be different, and this means that the corresponding wave on the infinite line will be attenuated, since the ratio between these potentials is the rate at which the amplitudes fall off per section.

With K_2 a pure resistance, a free oscillation is possible only if the dissipation in the positive resistance at the right end of the circuit is exactly made up by the hypothetical source of energy existing in the negative resistance $-K_2$ at the left end of the circuit. An exact balance between the energy supplied at one end and that lost at the other end is possible, since the equal positive and negative resistances K_2, $-K_2$ carry equal currents. This continuous transfer of energy from the left of the oscillating circuit of Fig. 6 to the right end is the action which goes on in every section of the infinite artificial line, and serves to pass forward the energy along the infinite line.

If K_2 were complex, $-K_2$ on the left of Fig. 6 and $+K_2$ on the right would not carry the same fraction of the circulating current I, since they are each shunted by a reactance $2Z_2$ which would allow less of the current to flow through $+K_2$ than through $-K_2$, if $2Z_2$ makes the smaller angle with $+K_2$, and vice versa. No balance between absorbed and dissipated energy is possible under these conditions when the equal and opposite resistance components carry unequal currents. A complex K_2, therefore, gives no free oscillation, and cannot occur with a resistanceless artificial line.

It is perhaps more instructive to consider the transmission on the line as a whole, rather than to confine attention exclusively to the oscillations of the simple circuit of Fig. 6 and so, at this point, without following further the conclusions to be drawn directly from this oscillating circuit, the fundamental energy theorem of resistanceless artificial lines will be stated, and then proved as a property of an infinite artificial line.

Energy Flow Theorem

Upon an infinite line of periodic recurrent structure, and devoid of resistance, a sinusoidal e.m.f. produces one of two steady states, viz.:

1. *A to-and-fro surging of energy without any resultant transfer of energy; currents and potential differences each attenuated from section to section, but everywhere in the same or opposite phase and mutually in quadrature, or,*

2. *A continuous, non-attenuated flow of energy along the line to infinity with no energy surging between symmetrical sections; current and potential non-attenuated, but retarded or advanced in phase from section to section, and mutually in phase at mid-shunt and mid-series points.*

The critical frequencies separating the two states of motion are the totality of the resonant frequencies of the series impedance, the anti-resonant frequencies of the shunt impedance, and the resonant frequencies of a single mid-shunt section of the line.

To prove the several statements of this theorem let us consider first the consequences of assuming that the wave motion, in progressing along the line, is attenuated, and next the consequences of assuming that the wave motion changes its phase. If the wave is attenuated, however little, at a sufficient distance it becomes negligible, and the more remote portions of the line may be completely removed without appreciable effect upon the disturbance in the nearer portion of the line. That part of the line which then remains is a finite network of pure reactances, and in any such network all currents are always in the same, or opposite, phase; so, also, are the potential differences; moreover, the two are mutually in quadrature; there is no continuous accumulation of energy anywhere, but only an exchange of energy back and forth between the inductances, the capacities and the generator. Continuously varying the amount of the assumed attenuation will cause a continuous variation in the corresponding frequency. The motion of the assumed character may, therefore, be expected to occur throughout continuous ranges or bands of frequencies and not merely at isolated frequencies.

The question may be asked—How far does the energy surge? Is the surge localized in the individual section, or does the surge carry the energy back and forth over more than one section, or even in and out of the line as a whole? To answer this question, it would be necessary, as we will now proceed to prove, to know something about the actual construction of the individual section. If each section is actually made up as shown in Fig. 6, and this is entirely possible in the present case (since only positive and negative reactances would be called for), then the section is capable of free oscillation, as explained above, and the surging is localized within the section; twice during each cycle the amount of energy increases on the right and decreases on the left. But we do not know that the section is made up like Fig. 6; we only know that it is equivalent to Fig. 6 as regards input and output relations. As far as these external relations go, the actual network may be made exclusively of either inductances or capacities with the connections shown in Fig. 4 or with the cross-connections of Fig. 5, according as the current is to have the same or opposite signs in consecutive sections. In any network made up exclusively of inductances or of capacities, the total energy falls to zero when the current or the potential falls to zero, respectively. Twice, therefore, in every cycle the total energy surges into this line and then it all returns to the generator. With other networks, surgings intermediate between these two extremes will occur. The theorem, therefore, does not limit the extent of the surging.

Under the second assumption, the phase difference between the currents at two given points, separated by a periodic interval, is to be an angle which is neither zero nor a multiple of $\pm\pi$. The assumed difference in phase can only be due to the infinite extension of the artificial line since, as previously noted, no finite sequence of inductances and capacities can produce any difference in phase. That infinite lines do produce phase differences is well-known; in particular, an infinite, uniform, perfectly conducting, metallic pair shows a continuous retardation in phase. If the infinitely remote sections of the artificial line are to have this controlling effect on the wave motion, the wave motion must actually extend to infinity, that is, there can be no attenuation. The wave progressing indefinitely to infinity without attenuation must be supplied continuously with energy; this energy must flow along the entire line with neither loss nor gain in the reactances it encounters on the way. This continuous flow of energy can take place only provided the currents and potentials are not in quadrature; they may be in phase. In considering the free oscillations of Fig. 6 it was shown that K_2 is real if it is not pure reactance. That is, for the mid-shunt section the current and potential are in phase. It is easy to show that they are also in phase at the mid-series point which is also a point of symmetry.

This flow-of-energy state of motion thus necessarily characterizes a phase-retarded wave on a resistanceless artificial line, regardless of the amount of the assumed positive or negative retardation, which may be taken to have any value between zero and exact opposition of phase. Continuously varying the retardation throughout the 180 degrees will, in general, call for a continuous change in the frequency of the wave motion. The second state of motion occurs, therefore, throughout continuous ranges or bands of frequencies.

No other state of motion is possible. With given initial amplitude and phase any possible wave motion is completely defined by its attenuation and phase change. All possible combinations of these two elements have been included in the two states, since the excluded conditions on each assumption have been included as a consequence of the other assumption. Thus, the exclusion of no attenuation in the first assumption was found necessarily to accompany the phase change of the second assumption; currents in phase or opposed, which were excluded from the second assumption, were found to be necessary features accompanying the first assumption. There remains only to consider the critical frequencies separating the two states of motion. At these frequencies there can be no attenuation and lag angles of multiples of $\pm\pi$, including zero, only. At symmetrical points the iterative impedance of the line must be a pure reactance to satisfy the first state of motion, and a pure resistance to satisfy the second state of motion. The only iterative impedances which satisfy these conditions are zero and infinity.

Some details relating to the pass and stop bands and the critical frequencies are brought together in the following table, where "stop ($\pm$)" refers to stop bands, the current being in phase or opposed in successive sections, and where γ and k refer to the line obtained by uniformly distributing $1/Z_2$ with respect to Z_1.

TABLE I.

For Ladder Artificial Line, Fig. 4

Band	Critical Frequency	Ratio $\frac{Z_1}{4Z_2}$	Uniform Line		Artificial Line			
			γ	k	Γ	$e^{-\Gamma}$	K_1	K_2
Stop (+)		>0	+real	imag.	+real	$0<<1$	imag.	imag.
	$Z_1=0$ $Z_2=\infty$	0 0	0 0	0 ∞	0 0	1 1	0 ∞	0 ∞
Pass		$0>>-1$	imag.	+real	imag.	$e^{i\theta}$	+real	+real
	$Z_1+4Z_2=0$	-1	$i2$	$i2Z_2$	$i\pi$	-1	0	∞
Stop (−)		<-1	imag.	+real	$i\pi$ + real	$-1<<0$	imag.	imag.
	$Z_1=\infty$ $Z_2=0$	$-\infty$ $-\infty$	∞ ∞	∞ 0	∞ ∞	0 0	∞ $\frac{1}{2}Z_1$	$2Z_2$ 0

It is not necessary to check the table item by item, many of which have already been proven, but it will be instructive to check some of the items by assuming that $Z_1/4Z_2$, called the ratio for brevity, is positive to begin with, and that a continuous increase in frequency reduces the ratio to zero and back through $\mp\infty$ to its original positive value. This cycle starts with a stop (+) band since the artificial line is in effect a network of reactances, all of which have the same sign; there is attenuation and the iterative impedances are imaginary. When the ratio decreases to zero, there must be either resonance which makes $Z_1 = 0$, or anti-resonance which makes $Z_2 = \infty$; in either case the artificial line has degenerated into a much simpler circuit; it is a shunt made up of all Z_2's combined in parallel, or a simple series circuit made up of all Z_1's, respectively; the iterative impedances are 0 and ∞, respectively; there is no attenuation in either case.

With a somewhat further increase of the frequency the ratio will assume a small negative value with the result that the artificial line will have both kinetic and potential energy. An analogy now exists between the artificial line and an ordinary uniform transmission line, which possesses both kinetic and potential energy, and is ordinarily visualized as being equivalent to many small positive reactances, in series, bridged, to the return conductor, by large negative reactances. The fact that uniform lines do freely transmit waves is a well-known physical principle, and it is not necessary to repeat here the physical theory of such transmission merely to show that the same phenomenon occurs with the identical structure when it is called an artificial line or wave-filter.

In order to determine just how far the ratio may depart from zero, on the negative side, without losing the property of free transmission, we look for any change in the action of the individual section of the artificial line which is fundamental; nothing less than a fundamental change in the behavior of the individual section can produce such a radical change in the line as an abrupt transition from the free transmission of a pass band to the to-and-fro surging of energy in a stop band. Now as the ratio is made more and more negative by the assumed increase of frequency, the value -1 is reached, at which frequency the symmetrical section (Fig. 6) of the artificial line is capable of free oscillation by itself. This is well recognized as a most fundamental change in the properties of any network, and it affords grounds for expecting a complete change in the character of the propagation over the artificial line. The change must be to a stop band with currents in opposite phase, since at resonance the potentials at the two ends of a section are in opposite phase.

Further increase in the frequency cannot make any change in the absolute difference in phase between the two ends of the other section, since opposition is the greatest possible difference in phase; the wave now adapts itself to increasing frequency by altering its attenuation.

Upon continuing the increase of frequency, so as to reduce the ratio to $-\infty$, we arrive at either anti-resonance corresponding to $Z_1 = \infty$ or resonance corresponding to $Z_2 = 0$; the artificial line has now degenerated into a row of isolated impedances Z_2, or into a series of impedances Z_1 short-circuited to the return wire; in either case the attenuation is infinite since no wave is transmitted. Passing beyond this critical frequency the ratio becomes positive, according to our assumption, and we are again in a stop (+) band.

While in this rapid survey of what happens during this frequency cycle little has been actually proven, it should have been made physically clear why abrupt changes in the character of the transmission occur at the frequencies making the ratio equal to 0, -1 or ∞, since the line degenerates into a simpler structure, or the phase change reaches its absolute maximum, on account of resonance, at these particular frequencies.

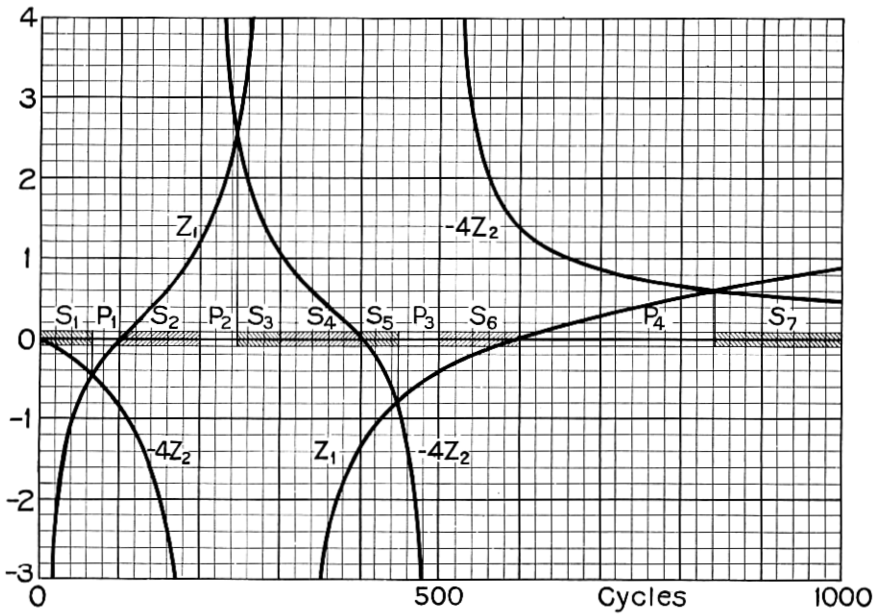

Fig. 7—Graph for Locating the Pass Bands and Stop Bands of Fig. 4
$Z_1 = -i(1^2 - x^2)(6^2 - x^2)/8x(3^2 - x^2)$,
$-4Z_2 = -i4x(4^2 - x^2)/(2^2 - x^2)(5^2 - x^2)$ and x = cycles/100

Information as to the location of the bands is often obtained most readily by plotting both Z_1 and $-4Z_2$, as illustrated in Fig. 7, and determining the critical frequencies by noting where the curves cross each other and the abscissa axis, as well as where they become infinite. Any particular band is then a pass band, a stop (+) band or a stop (−) band, according as Z_1, the abscissa axis, or $-4Z_2$ lies between the other two of the three lines. In Fig. 7 the pass bands are P_1, P_2, P_3, P_4; the stop (+) bands are S_2, S_4, S_6; and the stop (−) bands are S_1, S_3, S_5, S_7, and they illustrate quite a variety of sequences. By altering the curves the bands may be shifted, may be made to coalesce, or may be made to vanish.

Wave-Filter Curves

The pass band and stop band characteristics of wave-filters are concretely illustrated for a few typical cases by the curves of Figs. 8–13, which show the attenuation constant A, the phase constant B, and both the resistance R and reactance X components of the iterative impedance for a range of frequencies which include all of the critical frequencies, except infinity. The heavy curves apply to the ideal resistanceless case, while the dotted curves assume a power factor equal to $1/(20\pi)$ for each inductance which is a value readily obtained in practice. This value is, however, not sufficiently large to make these small scale curves entirely clear, since considerable portions of the dotted curves appear to be coincident with the heavy line curves; but this, as far as it goes, proves the value of the present discussion which rests upon a close approximation of actual wave-filters to the ideal resistanceless case.

The low pass resistanceless wave-filter, as shown by Fig. 8, presents no attenuation below 1,000 cycles; above this frequency the attenuation constant increases rapidly, in fact, the full line attenuation curve increases at the start with maximum rapidity, since it is there at right angles to the axis. The dotted attenuation curve, which includes the effective resistance in the inductance coils, follows the ideal attenuation curve closely, except in the neighborhood of 1,000 cycles, where resistance rounds off the abrupt corner which is present in the ideal A curve. The phase constant B is, at the start, proportional to the frequency, as for an ordinary uniform transmission line; its slope becomes steeper as the critical frequency 1,000 is approached where the curve reaches the ordinate π, at which value it remains constant for all higher frequencies. As shown by the dotted B curve, resistance rounds off the corner at the critical frequency, but

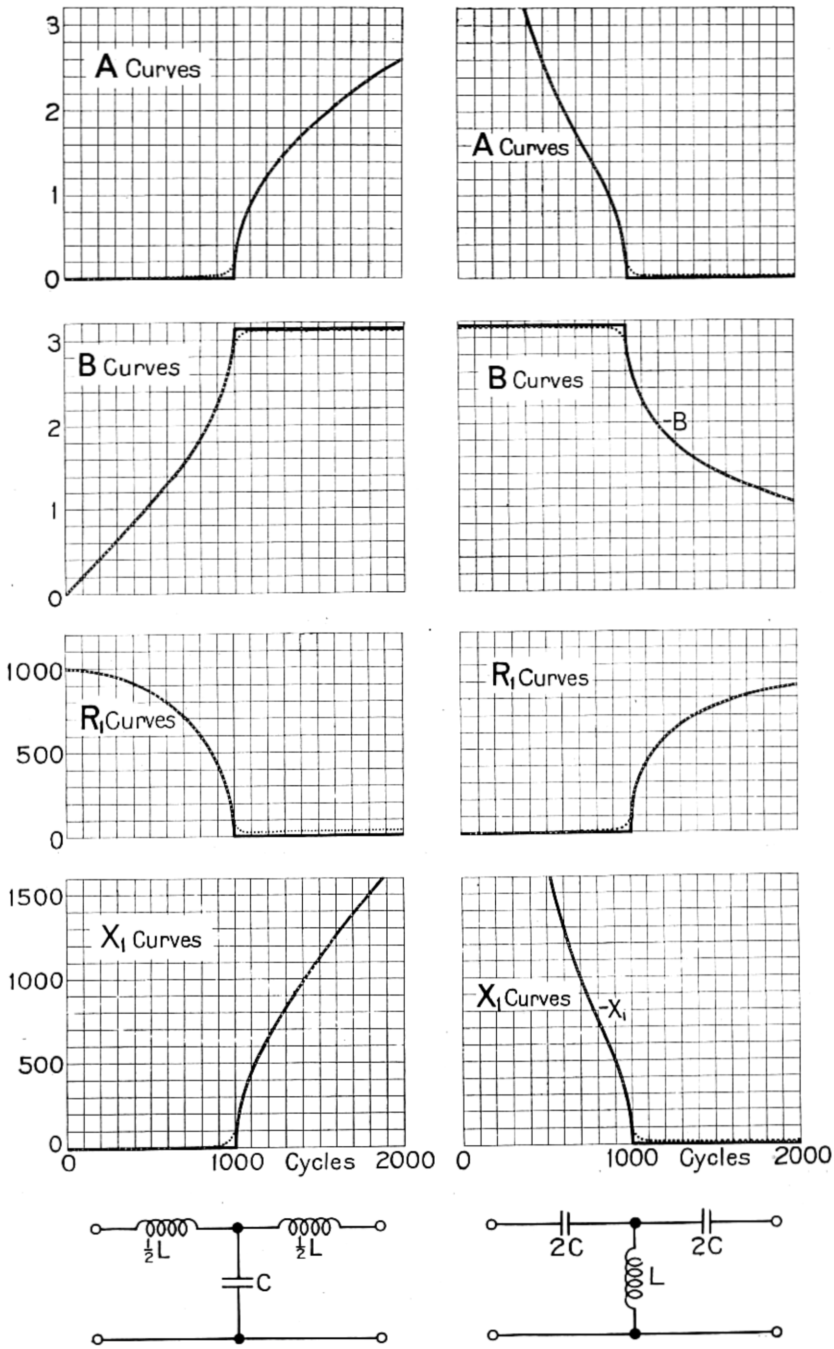

Fig. 8—Low Pass Wave-Filter: $L = 1/\pi$ Henry, $C = 1/\pi$ Microfarad
Fig. 9—Complementary High Pass Wave-Filter: $L = 1/4\pi$, $C = 1/4\pi$