The lattice network of Fig. 18 has in each branch a one-point impedance obtained by means of a duplicate of the given network N and an ideal transformer. The two lattice branch impedances are $Z_q+Z_r\pm 2Z_{qr}$ where the three impedances Z_q, Z_r, Z_{qr} are the effective self and mutual impedances of the network N regarded as a transformer. This lattice network has identically the same propaga-

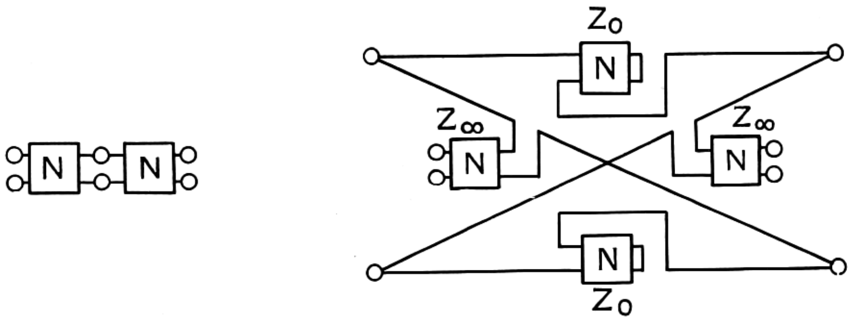

Fig. 17—Lattice Unit Equivalent to Two Sections of Fig. 1 Assumed to be Symmetrical

tion constant as the single network N shown on the left. Since the lattice cannot have different iterative impedances in the two directions, it actually compromises by assuming the sum of the two iterative impedances presented by N. A physical theory of the equival-

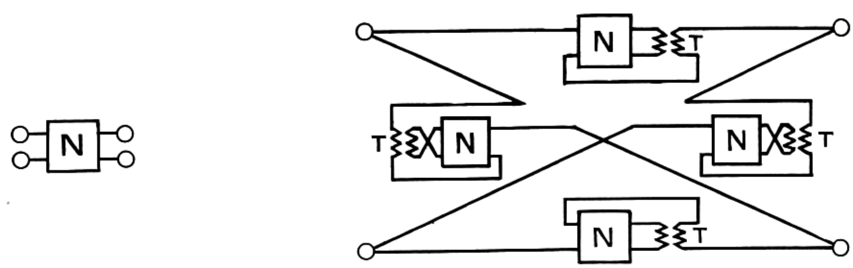

Fig. 18—Lattice Network Having the Same Propagation Constant as N and an Iterative Impedance Equal to the Sum of the Two Iterative Impedances of N

ences shown in Figs. 17 and 18 has not been worked up; the analytical proofs were made by applying the formulas given in the appendix under lattice networks.

Without going to more complex networks it is, of course, not possible to get a symmetrical iterative impedance, but that is not necessary for our present purposes where we are concerned primarily with the propagation constant. It has now been shown with complete generality that:

The lattice artificial line, with physically realizable branch impedances, is identically equivalent in propagation constant and mean iterative impedance to the chain of identical physically realizable networks connected together in sequence through two pairs of terminals.

To complete this simplification of the generalized artificial line it is necessary to know the simplest possible form of the one-point impedances employed in the branches of the lattice network. The discussion of the most general one-point impedance obtainable by means of any network of resistances, self and mutual inductances, leakages and capacities will find its natural place, together with allied theorems, in a paper on the subject of impedances. For the present purpose it is sufficient to state:

The most general branch impedance of the lattice network may be constructed by combining, in parallel, resonant circuits having impedances of the form $R+iLp+(G+iCp)^{-1}$*; or they may equally well be constructed by combining, in series, anti-resonant circuits having impedances of the form* $\left[G+iCp+(R+iLp)^{-1}\right]^{-1}$.

Summary of Physical Theory

The wave-filter under discussion approximates to a resistanceless artificial line, and such an ideal artificial line is capable of two, and only two, fundamentally distinct states of motion. In one state the disturbance is attenuated along the line, and there is no flow of energy other than a back and forth surging of energy, the intensity of which rapidly dies out along the line. In the other state there is a free flow of energy, without loss, from section to section along the line, with no surge of energy between symmetrical sections. Each state holds for one or more continuous bands of frequencies; these bands have been distinguished as stop bands and pass bands.

A high degree of discrimination, between different frequencies, may be obtained, even if each section, taken alone, gives only a moderate difference in attenuation, by the use of a sufficient number of sections in the wave-filter, since the attenuation factors vary in geometrical progression with the number of sections.

Any number of arbitrarily located pass bands may be realized by means of the lattice artificial line; furthermore, the propagation constant at one frequency, and the iterative impedance at one frequency may both be assigned, while the location of zero phase constant and zero iterative impedance at the lower or upper end of each pass band and stop band, respectively, is also optional. This completely determines the lattice artificial line. No additional condition, other than iterative impedance asymmetry, can be realized by replacing the lattice network by any four terminal network.

APPENDIX

Formulas for the Artificial Line

Formulas for the propagation constant and iterative impedance of the generalized artificial line, expressed in a number of equivalent forms, have already been given in my paper on Cisoidal Oscillations,[6] but it seems worth while to deduce the formulas anew here from the free oscillations of the detached unit circuit of Fig. 6, so as to complete the physical theory by deducing the comprehensive mathematical formulas by the same method of procedure.

Ladder Network Formulas

Notation:

Z_1, Z_2 = series impedance and shunt impedance of the section of Fig. 4, which is equivalent to the general network N of Fig. 1.

$\Gamma = A + iB$ = propagation constant per section.

K_1, K_2 = iterative impedances at mid-series and mid-shunt.

$\gamma = \alpha + i\beta = \sqrt{Z_1/Z_2}$ = propagation constant for uniform distribution of Z_1 and $1/Z_2$, per unit length.

$k = \sqrt{Z_1 Z_2}$ = iterative impedance of this same uniform line.

In Fig. 6, the current is indicated as I and the potentials at the ends of the section as $V, Ve^{-\Gamma}$. In order that the free oscillation may be possible the total impedance of the circuit $(Z_1 + Z' + Z'')$ must vanish; this determines the iterative impedance K_2. In addition to this condition it is sufficient to make use of two other simple relations: the proportionality of the potential drops in the direction of the current across Z' and Z'' to Z' and Z'', since they carry the same current (this determines the propagation constant Γ); and the equality of

[6] "Cisodial Oscillations," Trans. A. I. E. E., vol. 30, pp. 873–909, 1911. In the lowest row of squares of Table I, the iterative impedances and propagation constant of any network are given in five different ways, involving one-point and two-point impedances, equivalent star impedances, equivalent delta impedances, equivalent transformer impedances, or the determinant of the network. The only typographical errors in Table I appear to be the four which occur in the first, third and fifth squares of this row: in the values for K_q replace $(S_q - S_{qr})$ by $(S_q - S_r)$ and place a parenthesis before $U_q - U_r)$; in the first value of K_r replace S_{qr} by S_{qr}^2; in the last value for Γ_{qr} add a minus sign so that it reads $\cosh^{-1}$.

K_1, the mid-series iterative impedance of the artificial line, to the total impedance on the right of the mid-point of the series impedance Z_1. These three relations, which can be written down at once, are:

$$Z_1 + Z' + Z'' = Z_1 - \frac{4Z_2K_2^2}{4Z_2^2 - K_2^2} = 0,$$

$$\frac{Ve^{-\Gamma}}{V} = -\frac{Z''}{Z'} = \frac{2Z_2 - K_2}{2Z_2 + K_2},$$

$$K_1 = \frac{1}{2}Z_1 + Z'' = \frac{1}{2}Z_1 + \frac{2Z_2K_2}{2Z_2 + K_2},$$

from which the formulas for Γ, K_1, and K_2, in terms of Z_1, Z_2, are found to be:

$$\Gamma = 2 \sinh^{-1} \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} = 2 \sinh^{-1} \frac{1}{2}\gamma, \tag{1}$$

$$\left.\begin{matrix} K_1 \\ K_2 \end{matrix}\right\} = \sqrt{Z_1Z_2}\left(1 + \frac{Z_1}{4Z_2}\right)^{\pm\frac{1}{2}} = k\left(1 + \frac{1}{4}\gamma^2\right)^{\pm\frac{1}{2}} \text{at mid} \left\{\begin{matrix} \text{series} \\ \text{shunt}, \end{matrix}\right. \tag{2}$$

and the formulas for Z_1 and Z_2 in terms of Γ and K_1 or K_2 are likewise found to be:

$$Z_1 = 2K_1 \tanh \frac{1}{2}\Gamma = K_2 \sinh \Gamma, \tag{3}$$

$$Z_2 = K_1/\sinh \Gamma = \frac{1}{2}K_2 \coth \frac{1}{2}\Gamma. \tag{4}$$

Formulas (3) and (4) are in the nature of design formulas in that they determine the impedance Z_1 and Z_2, at assigned frequencies, which will ensure the assigned values of Γ and K at these frequencies. In general, however, it would not be evident how best to secure these required values of Z_1 and Z_2; complicated or even impossible networks might be called for, even to approximate values of Z_1 and Z_2 assigned in an arbitrary manner. Fortunately, practical requirements are ordinarily satisfied by meeting maximum and minimum values for the attenuation constant throughout assigned frequency bands. Formulas (8) and (9) may be employed for this purpose as explained below.

It is convenient to have formulas (1) and (2) expressed in a variety of ways, since no one form is well suited for calculation throughout the entire range of the variables. Accordingly, the following analytically equivalent expressions are here collected together for reference:

$$\Gamma = i\,2\,\sin^{-1}\frac{\gamma}{2i} = i\cos^{-1}\left(1+\frac{\gamma^2}{2}\right), \tag{5}$$

$$= 2\sinh^{-1}\frac{\gamma}{2} = 2\tanh^{-1}\frac{\frac{\gamma}{2}}{\sqrt{1+\frac{\gamma^2}{4}}} = \cosh^{-1}\left(1+\frac{\gamma^2}{2}\right)$$

$$= 2\log\left[\frac{\gamma}{2}+\sqrt{1+\frac{\gamma^2}{4}}\right], \tag{5a}$$

$$= i\pi + 2\cosh^{-1}\frac{\gamma}{2i} = i\pi + \cosh^{-1}\left(-1-\frac{\gamma^2}{2}\right)$$

$$= i\pi + 2\log\left[\frac{\gamma}{2i}+\sqrt{-1-\frac{\gamma^2}{4}}\right], \tag{5b}$$

$$= i\frac{\pi}{2} - \sinh^{-1}\left(1+\frac{\gamma^2}{2}\right)i, \tag{5c}$$

$$= \gamma - \frac{1}{24}\gamma^3 + \frac{3}{640}\gamma^5 - \frac{5}{7168}\gamma^7 + \ldots, \text{ if } |\gamma| < 2, \tag{5d}$$

$$= 2\cosh^{-1} g + i\,2\sin^{-1}\frac{\beta}{2g}, \tag{5e}$$

$$\text{where } \gamma = \alpha + i\beta,\ 2g = \sqrt{\left(\frac{\alpha}{2}\right)^2 + \left(1+\frac{\beta}{2}\right)^2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + \left(1-\frac{\beta}{2}\right)^2},$$

$$= \cosh^{-1} h + i\cos^{-1}\frac{x}{h}, \tag{5f}$$

$$\text{where } 1+\frac{1}{2}\gamma^2 = x + iy,\ 2h = \sqrt{(x+1)^2+y^2} + \sqrt{(x-1)^2+y^2},$$

$$\left.\begin{matrix} K_1 \\ K_2 \end{matrix}\right\} = k\left(1+\frac{1}{4}\gamma^2\right)^{\pm\frac{1}{2}} = k\left(\cosh\frac{\Gamma}{2}\right)^{\pm 1} = k\left(\frac{\sinh\Gamma}{\gamma}\right)^{\pm 1}$$

$$= k\left(\frac{1}{2}\gamma\coth\frac{1}{2}\Gamma\right)^{\pm 1} \text{ at mid } \left\{\begin{matrix}\text{series} \\ \text{shunt.}\end{matrix}\right. \tag{6}$$

The formulas leave indeterminate the signs of γ, k, Γ, and K, and also a term $\pm i2\pi n$ in γ and Γ. The signs are to be so chosen that the real parts are positive, or become positive when positive resistance is added to the system. The indeterminate $\pm\, i2\pi n$ can be made determinate only after knowing something of the internal structure of the unit network of which the artificial line is composed; the conditions to be met are—absence of phase differences when all branches of the unit network N of Fig. 1 are assumed to be pure resistances and continuity of phase as reactances are gradually introduced to give the actual network.

Formula (5) is adapted for use in the pass bands, since the expressions are real when γ^2 is real, negative and not less than -4; similarly, formulas (5a) and (5b) are adapted for use in the stop ($\pm$) bands, that is, when γ^2 is positive and less than -4 respectively.

From the theory of impedances we know that any resistanceless one-point impedance is expressible in the form

$$Z = iD \frac{p \quad (p_2^2 - p^2) \ldots (p_{2n-2}^2 - p^2)}{(p_1^2 - p^2)\ (p_3^2 - p^2) \ldots (p_{2n-1}^2 - p^2)} \tag{7}$$

where the factor D and the roots $p_1, p_2, \ldots p_{2n}$ are arbitrary positive, reals subject only to the condition that each root is at least as large as the preceding one. This enables us to write down the forms which the quotient and product of two resistanceless one-point impedances may assume, which are as follows:

$$\frac{Z'}{Z''} = G \left(\frac{p_1^2 - p^2}{p_2^2 - p^2}\right)^{\pm 1} \left(\frac{p_3^2 - p^2}{p_4^2 - p^2}\right)^{\pm 1} \ldots \left(\frac{p_{2n-1}^2 - p^2}{p_{2n}^2 - p^2}\right)^{\pm 1} \tag{8}$$

$$Z'Z'' = -H \left(\frac{p^2}{p_1^2 - p^2}\right)^{\pm 1} \left(\frac{p_2^2 - p^2}{p_3^2 - p^2}\right)^{\pm 1} \ldots \left(\frac{p_{2n-2}^2 - p^2}{p_{2n-1}^2 - p^2}\right)^{\pm 1} (p_{2n}^2 - p^2)^{\pm 1} \tag{9}$$

where G, H and the roots $p_1, p_2, \ldots p_{2n}$ are arbitrary positive reals, subject only to the condition that each root is at least as large as the preceding one, and the $2n+1$ and optional $\pm$ signs are mutually independent. Conversely, if the relations (8) and (9) are prescribed, then the required individual impedances Z' and Z'' are each of the form (7) and thus physically realizable.

If in formulas 1, 2, 5 and 6 we substitute for $Z_1/Z_2 = \gamma^2$ and $Z_1 Z_2 = k^2$ the right-hand side of formulas (8) and (9), respectively, we obtain formulas for the propagation constant and iterative impedance of an artificial resistanceless line in terms of frequencies at which the propagation constant becomes zero or infinite. Ordinarily, however, we are more interested in having expressions in terms of the frequencies which terminate the pass bands. To secure these the substitutions should be $4[8]/(4 - [8])$ and $[9]\ (1 - [8]/4)^{\pm 1}$, where [8] and [9] stand for the entire right-hand sides of formulas (8) and (9). This substitution amounts to obtaining the lattice network giving the required pass bands, and then transforming to the ladder network having the same propagation constant and the same iterative impedance at mid-series or mid-shunt.

Lattice Network Formulas Fig. 14

The impedances of a single section between terminals 1 and 2, with the far end of the section 3 and 4 either short-circuited or open, are readily seen to be

$$Z_0 = \frac{2Z_1Z_2}{\frac{1}{2}Z_1 + 2Z_2}, \quad Z_\infty = \frac{1}{2}\left(\frac{1}{2}Z_1 + 2Z_2\right). \tag{10}$$

Since $\sqrt{Z_0Z_\infty}$ and $\sqrt{Z_0/Z_\infty}$ are the iterative impedance and the hyperbolic tangent of the propagation constant for any symmetrical artificial line, we have the following analytically equivalent formulas for the lattice network where $\gamma = \sqrt{Z_1/Z_2}$, and $k = \sqrt{Z_1Z_2}$ as for the ladder type.

Lattice Formulas

$$\Gamma = 2\tanh^{-1}\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} = 2\tanh^{-1}\frac{1}{2}\gamma, \tag{11}$$

$$K = \sqrt{Z_1Z_2} = k. \tag{12}$$

$$Z_1 = 2K\tanh\frac{1}{2}\Gamma, \tag{13}$$

$$Z_2 = \frac{1}{2}K\coth\frac{1}{2}\Gamma. \tag{14}$$

$$\Gamma = i2\tan^{-1}\frac{\gamma}{2i} = i\cos^{-1}\frac{1+\frac{1}{4}\gamma^2}{1-\frac{1}{4}\gamma^2}, \tag{15}$$

$$= 2\tanh^{-1}\frac{\gamma}{2} = 2\sinh^{-1}\frac{\frac{1}{2}\gamma}{\sqrt{1-\frac{1}{4}\gamma^2}} = \cosh^{-1}\frac{1+\frac{1}{4}\gamma^2}{1-\frac{1}{4}\gamma^2}$$

$$= \log\frac{1+\frac{1}{2}\gamma}{1-\frac{1}{2}\gamma}, \tag{15a}$$

$$= i\pi + 2\coth^{-1}\frac{\gamma}{2} = i\pi + \cosh^{-1}\frac{1+\frac{1}{4}\gamma^2}{-1+\frac{1}{4}\gamma^2}$$

$$= i\pi + \log\frac{1+\frac{1}{2}\gamma}{-1+\frac{1}{2}\gamma}, \tag{15b}$$

$$= i\frac{\pi}{2} - \sinh^{-1}\frac{1+\frac{1}{4}\gamma^2}{1-\frac{1}{4}\gamma^2}i, \tag{15c}$$

$$= \gamma + \frac{1}{12}\gamma^3 + \frac{1}{80}\gamma^5 + \frac{1}{448}\gamma^7 + \ldots, |\gamma| < 2, \tag{15d}$$

$$= \frac{1}{2}\log\frac{\left(1+\frac{\alpha}{2}\right)^2+\left(\frac{\beta}{2}\right)^2}{\left(1-\frac{\alpha}{2}\right)^2+\left(\frac{\beta}{2}\right)^2} + i\tan^{-1}\frac{\beta}{1-\left(\frac{\alpha}{2}\right)^2-\left(\frac{\beta}{2}\right)^2}, \tag{15e}$$

where $\gamma = \alpha + i\beta$.

In these formulas $Z_1/Z_2 = \gamma^2$ and $Z_1 Z_2 = k^2$ might be expressed in terms of the resonant and anti-resonant complex frequencies of Z_1 and Z_2, the frequencies being made complex quantities so as to include the damping. Where there is no damping, that is, where all network impedances are devoid of resistance, the simplified forms of these expressions are given by formulas (8) and (9). The use of these formulas for designing wave filters having assigned pass bands is explained at page 23.