# GRAVITATION, GAUGE THEORIES AND DIFFERENTIAL GEOMETRY

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## 1. Introduction

Advances in mathematics and physics have often occurred together. The development of Newton's theory of mechanics and the simultaneous development of the techniques of calculus constitute a classic example of this phenomenon. However, as mathematics and physics have become increasingly specialized over the last several decades, a formidable language barrier has grown up between the two. It is thus remarkable that several recent developments in theoretical physics have made use of the ideas and results of modern mathematics and, in fact, have elicited the direct participation of a number of mathematicians. The time therefore seems ripe to attempt to break down the language barriers between physics and certain branches of mathematics and to re-establish interdisciplinary communication (see, for example, Robinson [1977]; Mayer [1977]).

The purpose of this article is to outline various mathematical ideas, methods, and results, primarily from differential geometry and topology, and to show where they can be applied to Yang–Mills gauge theories and Einstein's theory of gravitation.

We have several goals in mind. The first is to convey to physicists the bases for many mathematical concepts by using intuitive arguments while avoiding the detailed formality of most textbooks. Although a variety of mathematical theorems will be stated, we will generally give simple examples motivating the results instead of presenting abstract proofs.

Another goal is to list a wide variety of mathematical terminology and results in a format which allows easy reference. The reader then has the option of supplementing the descriptions given here by consulting standard mathematical references and articles such as those listed in the bibliography.

Finally, we intend this article to serve the dual purpose of acquainting mathematicians with some basic physical concepts which have mathematical ramifications; physical problems have often stimulated new directions in mathematical thought.

### 1.1. Gauge theories

By way of introduction to the main text, let us give a brief survey of how mathematicians and physicists noticed and began to work on certain problems of mutual interest. One crucial step was taken by Yang and Mills [1954] when they introduced the concept of a non-abelian gauge theory as a generalization of Maxwell's theory of electromagnetism. The Yang–Mills theory involves a self-interaction among gauge fields, which gives it a certain similarity to Einstein's theory of gravity (Utiyama [1956]). At about the same time, the mathematical theory of fiber bundles had reached the advanced stage described in Steenrod's book (Steenrod [1953]) but was generally unknown to the physics community. The fact that Yang–Mills theories and the affine geometry of principal fiber bundles are one and the same thing was eventually recognized by various authors as early as 1963 (Lubkin [1963]; Hermann [1970]; Trautman [1970]), but few of the implications were explored. The potential utility of the differential geometric methods of fiber bundles in gauge theories was pointed out to the bulk of the physics community by the paper of Wu and Yang [1975]. For example, Wu and Yang showed how the long-standing problem of the Dirac string for magnetic monopoles (Dirac [1931]) could be resolved by using overlapping coordinate patches with gauge potentials differing by a gauge transformation; for mathematicians, the necessity of using coordinate patches is a trivial consequence of the fact that non-trivial fiber bundles cannot be described by a single gauge potential defined over the whole coordinate space.

Almost simultaneously with the Wu–Yang paper, Belavin, Polyakov, Schwarz and Tyupkin [1975] discovered a remarkable finite-action solution of the Euclidean SU(2) Yang–Mills gauge theory, now generally known as the "instanton" or, sometimes, the "pseudoparticle". The instanton has self-dual or anti-dual field strength and carries a non-vanishing topological quantum number; from the mathematical point of view, this number is the integral of the second Chern class, which is an integer characterizing the topology of an SU(2) principal fiber bundle. 't Hooft [1976a, 1977] recognized that the instanton provided a mechanism for breaking the chiral $U(1)$ symmetry and solving the long-standing problem of the ninth axial current, together with a possible mechanism for the violation of $CP$ symmetry and fermion number.

Another important consequence of the instanton is that it revealed the existence of a periodic structure of the Yang–Mills vacua (Jackiw and Rebbi [1976b]; Callan, Dashen and Gross [1976]). The instanton action gives the lowest order approximation to the quantum mechanical tunneling amplitude between these states. The true ground state of the theory becomes the coherent mixture of all such vacuum states.

Following the BPST instanton, which had topological index $\pm 1$ for self-dual or anti-dual field strength, Witten [1977], Corrigan and Fairlie [1977], Wilczek [1977], 't Hooft [1976b] and Jackiw, Nohl and Rebbi [1977] found ways of constructing "multiple instanton" solutions characterized by (anti)-self-dual field strength and arbitrary integer topological index $\pm k$. At this point, the question was whether or not the parameter space of the $k$-instanton solution was exhausted by the $(5k+4)$ parameters of the Jackiw–Nohl–Rebbi solution (for $k=1$ and $k=2$, the number of parameters reduces to 5 and 13, respectively). The answer was provided both by mathematicians and physicists. Schwarz [1977] and Atiyah, Hitchin and Singer [1977] used the Atiyah–Singer index theorem [1968] to show that the parameter space was $(8k-3)$-dimensional. The same result was found by Jackiw and Rebbi [1977] and Brown, Carlitz and Lee [1977] using physicists' methods. It was also noted that the Dirac equation in the presence of the $k$-(anti)-instanton field would have $k$ zero frequency modes of chirality $\pm 1$. Physicists' arguments leading to this result were found by Coleman [1976], who integrated the local equation for the Adler–Bell–Jackiw anomaly (Adler [1969]; Bell and Jackiw [1969]). The number of parameters for self-dual Yang–Mills solutions for general Lie groups was worked out by Bernard, Christ, Guth and Weinberg [1977] and by Atiyah, Hitchin and Singer [1978]. It became apparent that the same class of problems was being attacked simultaneously by mathematicians and physicists and that a new basis existed for mutual discourse.

The attention of the mathematicians was now drawn to the problem of constructing Yang–Mills solutions with index $k$ which exhausted the available free parameters for a given gauge group. The first concrete steps in this direction were taken by Ward [1977] and by Atiyah and Ward [1977] who adapted Penrose's twistor formalism to Yang–Mills theory to show how the problem could be solved. Atiyah, Hitchin, Drinfeld and Manin [1978] then used a somewhat different approach to give a construction of the most general solutions with self-dual field strength. The remarkable fact about this construction is that powerful tools of algebraic geometry made it possible to reduce the non-linear Yang–Mills differential equations to *linear algebraic equations*. The final link in the chain was provided by Bourguignon, Lawson and Simons [1979], who showed that, for compactified Euclidean space-time, *all* stable finite action solutions of the Euclidean Yang–Mills equations have self-dual field strength. Thus all stable finite action solutions of the Euclidean Yang–Mills equations are, in principle, known.

Finally, we note an interesting parallel development concerning the choice of gauge in a Yang–Mills theory. Gribov [1977, 1978] and Mandelstam [1977] noticed that the traditional Coulomb gauge choice does not determine a unique gauge potential; there exist an infinite number of gauge-equivalent fields all obeying the Coulomb gauge condition. The gauge-choice ambiguity can be avoided if the underlying space-time is a flat space (see, e.g., Coleman [1977]). However, Singer [1978a] showed that the Gribov ambiguity was incurable if he assumed a compactified Euclidean space-time manifold. Singer's calculation introduced powerful methods for examining the functional space of the path-integral using the differential geometry of infinite-dimensional fiber bundles; the exploitation of such techniques may eventually lead to a more satisfactory understanding of the path integral approach to the quantization of gauge theories.

### *1.2. Gravitation*

The methods of differential geometry have always been essential in Einstein's theory of gravity (see, e.g., Trautman [1964]; Misner, Thorne and Wheeler [1973]). However, the discovery of the Yang–Mills instanton and its relevance to the path integral quantization procedure led to the hope that similar new approaches might be used in quantum gravity. The groundwork for the path integral approach to quantum gravity was laid by De Witt [1967a,b,c]. Prescriptions were subsequently developed for giving an appropriate boundary correction to the action (Gibbons and Hawking [1977]) and for avoiding the problem of negative gravitational action (Gibbons, Hawking and Perry [1978]).

The problem was then to determine which classical Euclidean Einstein solutions might be important in the gravitational path integral and which, if any, might play a physical role similar to that of the Yang–Mills instanton. The Euler–Poincaré characteristic $\chi$ and the signature $\tau$ were identified by Belavin and Burlankov [1976] and by Eguchi and Freund [1976] as gravitational analogs of the Yang–Mills topological index $k$. Eguchi and Freund went on to suggest the Fubini–Study metric on two-dimensional complex projective space as a possible gravitational instanton, but the absence of well-defined spinors on this manifold lessens its appeal. Hawking [1977] then proposed a Euclidean Taub–NUT metric with self-dual curvature as a gravitational instanton, and furthermore presented a new multiple-center solution reminiscent of the $k > 1$ Yang–Mills solutions. However, Hawking's metrics had a distorted asymptotic behavior at infinity and, in fact, resembled magnetic monopoles more than instantons. It was also noted by Eguchi, Gilkey and Hanson [1978], by Römer and Schroer [1977] and by Pope [1978] that special care was required to compute the topological invariants for manifolds with boundary, such as those Hawking considered; here, the Atiyah–Patodi–Singer index theorem [1973, 1975a,b, 1976] with boundary corrections was applied to the study of physical questions arising in quantum gravity.

Starting from the idea that since the Yang–Mills instanton potential is asymptotically a pure gauge, a gravitational instanton should have an asymptotically flat metric, Eguchi and Hanson [1978] derived a new Euclidean Einstein metric with self-dual curvature which seems to be the closest gravitational analog of the Yang–Mills instanton. Although this metric is asymptotically flat, the manifold's boundary at infinity is not the three-sphere of ordinary Euclidean space, but is a three-sphere with opposite points identified (Belinskii et al. [1978]). Essentially this same metric was found independently by Calabi [1979] as the solution to an abstract mathematical problem. Gibbons and Hawking [1978] subsequently realized that this metric was the first of a class of metrics found by making a simple modification to Hawking's original multicenter metric (Hawking [1977]). The metrics in this new class are all *asymptotically locally Euclidean*: they are asymptotically flat, but the boundaries are three-spheres with points identified under the action of some discrete group. The manifolds described by these metrics are distinguished by the signature $\tau$, which takes on all integer values and plays the role of the Yang–Mills index $k$. An explicit construction by Hawking and Pope [1978b] and an index theory calculation by Hanson and Römer [1978] show that the metrics with signature $\tau$ give a spin 3/2 anomaly $2\tau$, but do not contribute at all to the spin 1/2 axial anomaly as did the Yang–Mills index $k$. This distinction appears to have its origins in the existence of supersymmetry. Hitchin [1979] has now discussed further generalizations of these metrics and pointed out the existence of complex algebraic manifolds whose asymptotic boundaries are three-spheres identified under the action of all possible groups. He has also suggested that these manifolds may admit metrics with self-dual curvatures. These manifolds appear to exhaust the class of asymptotically locally Euclidean Einstein solutions with self-dual curvature, and thus provide a complete classification of this type of gravitational instanton. In principle, the Penrose construction can be used to find the self-dual metrics on each of these manifolds, so that the gravitational problem is nearing the same degree of completeness that exists for the Yang–Mills theory.

### *1.3. Outline*

In the main body of this article, we will attempt to provide a physicist with the mathematical ideas underlying the sequence of discoveries just described. In addition, we wish to provide a mathematician with a feeling for some of the physical problems to which mathematical methods might apply. In section 2, we introduce the basic concepts of manifolds and differential forms, and then discuss the elements of de Rham cohomology. In section 3, we consider Riemannian geometry and explain the relationship between classical tensor analysis and modern differential geometric notation. Section 4 is devoted to an exposition of the geometry of fiber bundles. We introduce the concepts of connections and curvatures on fiber bundles in section 5 and give some physical examples. In section 6, we develop the theory of characteristic classes, which are the topological invariants used to classify fiber bundles. The Atiyah–Singer index theorem for manifolds without boundary is discussed in section 7. The generalization of the index theorem to manifolds with boundary is presented in section 8. Section 9 contains a brief discussion of Yang–Mills instantons and a list of mathematical results relevant to Yang–Mills theories, while section 10 treats gravitational instantons and gives a list of mathematical results associated with gravitation.

A number of basic mathematical formulas are collected in the appendices, while the bibliography contains suggestions for further reading.

Due to limitations of time and space, we have not been able to provide detailed treatments of a number of interesting mathematical and physical topics; brief discussions of some such topics are given in sections 9 and 10. We also note that many of the "mathematical" results we present have also been discovered by physicists using different methods of calculation; we have made no attempt to treat in detail these alternative derivations, but refer the reader instead to the bibliography for appropriate review articles elaborating on the conventional physical approaches.

## 2. Manifolds and differential forms

Manifolds are generalizations of the familiar ideas of lines, planes and their higher dimensional analogs. In this section, we introduce the basic concepts of manifolds, differential forms and de Rham cohomology (see, for instance, Flanders [1963]). Various examples are given to show how these tools can be used in physical problems.

### *2.1. Definition of a manifold*

A real (complex) $n$-dimensional *manifold* $M$ is a space which looks like a Euclidean space $\mathbb{R}^n(\mathbb{C}^n)$ around *each point*. More precisely, a manifold is defined by introducing a set of neighborhoods $U_i$ covering $M$, where each $U_i$ is a subspace of $\mathbb{R}^n(\mathbb{C}^n)$. Thus, a manifold is constructed by pasting together many pieces of $\mathbb{R}^n(\mathbb{C}^n)$.

In fig. 2.1, we show some examples of manifolds in one dimension: fig. 2.1a is a line segment of $\mathbb{R}^1$, the simplest possible manifold. Figure 2.1b shows the circle $S^1$; this is a non-trivial manifold which requires at least two neighborhoods for its construction. Figure 2.2 shows some spaces which are *not* manifolds: no neighborhood of a multiple junction looks like $\mathbb{R}^n$.

*Examples 2.1*

Let us discuss some of the typical $n$-dimensional manifolds which we will encounter.

1. $\mathbb{R}^n$ itself and $\mathbb{C}^n$ itself are the most trivial examples. These are noncompact manifolds.
2. The $n$-sphere $S^n$ defined by the equation

$$\sum_{i=1}^{n+1} x_i^2 = c^2, \qquad c = \text{constant}. \tag{2.1}$$

The "zero-sphere" $S^0$ is just the two points $x = \pm c$. $S^1$ is a circle or ring and $S^2$ is a sphere like a balloon.

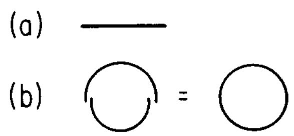


Fig. 2.1. One-dimensional manifolds: (a) is a line segment of $\mathbb{R}^1$. (b) shows the construction of $S^1$ using two neighborhoods.

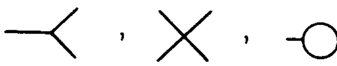

Fig. 2.2. One-dimensional spaces which are not manifolds. The condition that the space looks locally like $\mathbb{R}^1$ is violated at the junctions.

3. *Projective spaces.* Complex projective space, $P_n(\mathbb{C})$, is the set of lines in $\mathbb{C}^{n+1}$ passing through the origin. If $z = (z_0, \ldots, z_n) \neq 0$, then $z$ determines a complex line through the origin. Two points $z$, $z'$ determine the same line if $z = cz'$ for some $c \neq 0$. We introduce the equivalence relation $z \simeq z'$ if there is a non-zero constant such that $z = cz'$; $P_n(\mathbb{C})$ is $\mathbb{C}^{n+1} - \{0\}$ modulo this identification.

We define neighborhoods $U_k$ in $P_n(\mathbb{C})$ as the set of lines for which $z_k \neq 0$ (this condition is unchanged by replacing $z$ by a scalar multiple). The ratio $z_i/z_k = cz_i/cz_k$ is well-defined on $U_k$. Let

$$\zeta_i^{(k)} = z_i/z_k \qquad \text{on } U_k$$

and $\zeta^{(k)} = (\zeta_0^{(k)}, \ldots, \zeta_n^{(k)})$ where we omit $\zeta_k^{(k)} = 1$. This gives a map from $U_k$ to $\mathbb{C}^n$ and defines complex coordinates on $U_k$. We see that

$$\zeta_i^{(j)} = \frac{z_i}{z_k}\frac{z_k}{z_j} = \zeta_i^{(k)}(\zeta_j^{(k)})^{-1}$$

is well-defined on $U_j \cap U_k$. The $(n+1)$ $z_i$'s are "homogeneous coordinates" on $P_n(\mathbb{C})$. Later we will show that the $z_i$'s can be regarded as sections to a line bundle over $P_n(\mathbb{C})$. The $n$ $\zeta_i^{(k)}$'s defined in each $U_k$ are local "inhomogeneous coordinates".

Real projective space, $P_n(\mathbb{R})$, is the set of lines in $\mathbb{R}^{n+1}$ passing through the origin. It may also be regarded as the sphere $S^n$ in $\mathbb{R}^{n+1}$ where we identify antipodal points. (Two unit vectors $x$, $x'$ determine the same line in $\mathbb{R}^{n+1}$ if $x = \pm x'$.)

*Remark*: $P_1(\mathbb{C}) = S^2$ and $P_3(\mathbb{R}) = \mathrm{SO}(3)$.

4. *Group manifolds* are defined by the space of free parameters in the defining representation of a group. Several group manifolds are easily identifiable with simple topological manifolds:

(a) $\mathbb{Z}_2$ is the group of addition modulo 2, with elements (0, 1); $\mathbb{Z}_2$ may also be thought of as the group generated by multiplication by $(-1)$, and thus has elements $\pm 1$. This latter representation shows its equivalence to the zero-sphere,

$$\mathbb{Z}_2 = S^0.$$

(b) $U(1)$ is the group of multiplication by unimodular complex numbers, with elements $e^{i\theta}$. Since $\theta$, $0 \leq \theta < 2\pi$ parametrizes a circle, we see that

$$U(1) = S^1.$$

(c) SU(2). A general SU(2) matrix can be written as

$$u = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix},$$

where $a = x_1 + ix_2$, $b = x_3 + ix_4$, bar denotes complex conjugation and

$$\det u = |a|^2 + |b|^2 = \sum_{i=1}^{4} x_i^2 = 1.$$

Hence we can identify the parameter space of SU(2) with the manifold of the three-sphere $S^3$

$$\mathrm{SU}(2) = S^3.$$

(d) SO(3). It is well-known that SU(2) is the double-covering of SO(3), so that SO(3) can be written as the manifold

$$\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2 = P_3(\mathbb{R})$$

where $P_3(\mathbb{R})$ is three-dimensional real projective space.

*Boundary of a manifold.* The boundary of a line segment is the two end points; the boundary of a disc is a circle. Thus we may, in general, determine another manifold of dimension $(n-1)$ by taking the boundary of an $n$-manifold. We denote the boundary of a manifold $M$ as $\partial M$.
*Note*: The boundary of a boundary is always empty, $\partial\partial M = \emptyset$.

*Coordinate systems.* One of the important themes in manifold theory is the idea of coordinate transformations relating adjacent neighborhoods. Suppose we have a covering $\{U_i\}$ of a manifold $M$ and some coordinate system $\phi_i$ in each neighborhood $U_i$. $\phi_i$ is a mapping from $U_i$ to $\mathbb{R}^n$. Then we need to know how to relate two coordinate systems $\phi_i$ and $\phi_j$ in the overlapping region $U_i \cap U_j$, the shaded area in fig. 2.3. The answer is the following: we take $\phi_i^{-1}$ to be the mapping *back* from $\mathbb{R}^n$, so the transformation from the coordinate system $\phi_i$ to the coordinate system $\phi_j$ is given by the transition function

$$\phi_{ji} = \phi_j \cdot \phi_i^{-1}.$$

This map is required to be $C^\infty$ (have continuous partial derivatives of all orders). If the $\phi_{ji}$ are real analytic, then $M$ is said to be a real analytic manifold. If the $\phi_{ji}$ are holomorphic (i.e., complex valued functions with complex power series), then $M$ is said to be a complex manifold.

*Examples 2.1 (Continued)*

5. *Two sphere.* On $S^2$ we may choose just two neighborhoods, $U_1$ and $U_2$, which cover the northern and southern hemisphere, respectively, and one transition function $\phi_{12}$, where

$$\phi_{12}(x, y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

in the intersection $U_1 \cap U_2$ of the neighborhoods. In terms of complex coordinates, $z = x + \mathrm{i}y$,

$$\phi_{12}(z) = 1/z.$$

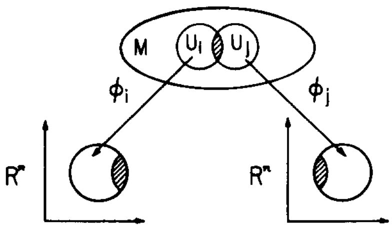


Fig. 2.3. Overlapping neighborhoods of a manifold $M$ and their coordinate systems. $\phi_i$ is a map from $U_i$ to an open subspace of $\mathbb{R}^n$.

Since this transition function is not only smooth but holomorphic, $S^2$ has the structure of a complex manifold (namely $P_1(\mathbb{C})$).

6. *Projective space.* $P_n(\mathbb{C})$ is also a complex manifold because its transition functions are holomorphic

$$\phi_{ji}(z_0, \dots, z_n) = \left(\frac{z_i}{z_j} z_0, \dots, \frac{z_i}{z_j} z_n\right)$$

on $U_i \cap U_j$ (where we recall $z_i \neq 0$, $z_j \neq 0$).

7. *Lie groups in general.* If $A$ is a matrix, then $\exp(A) = I + A + \cdots + A^n/n! + \cdots$ converges to an invertible matrix. Let $G$ be one of the groups: $\mathrm{GL}(k, \mathbb{C})$, $\mathrm{GL}(k, \mathbb{R})$, $U(k)$, $\mathrm{SU}(k)$, $O(k)$, $\mathrm{SO}(k)$ and let $\mathfrak{g}$ be the Lie-algebra of $G$. $\mathfrak{g}$ is a linear set of matrices and $\exp: \mathfrak{g} \to G$ is a diffeomorphism from a neighborhood of the origin in $\mathfrak{g}$ to the identity $I$ in $G$. This defines a coordinate system near $I \in G$; we can define a coordinate system near *any* $g_0 \in G$ by mapping $\mathfrak{g}$ into $g_0 \exp \mathfrak{g}$. The transition functions are thus given by left multiplication in the group. $G$ is a real analytic manifold.

## 2.2. Tangent space and cotangent space

One of the most important concepts used to study the properties of a manifold $M$ is the tangent space $T_p(M)$ at a point $p \in M$. To develop the idea of the tangent space, let us first consider a curve $y = f(x)$ in a plane as shown in fig. 2.4. Consider a point $x = p + v$ very close to $p$; then we may expand $f(x)$ in a Taylor series, yielding

$$f(x = p + v) = f(p) + v \, \mathrm{d}f(x)/\mathrm{d}x\,|_{x=p} + \cdots . \tag{2.2}$$

The slope of the curve, $\mathrm{d}f/\mathrm{d}x$ at $x = p$, is represented in fig. 2.4. If we had an $n$-dimensional surface with coordinates $x^i$, there would be $n$ different directions, so the second term in (2.2) would become

$$v^i \, \partial f(\boldsymbol{x})/\partial x^i \,|_{\boldsymbol{x}=\boldsymbol{p}} .$$

(Here we introduce the convention of implied summation on repeated indices.) We can thus begin to see that, regardless of the particular details of the manifold considered, the directional derivative

$$v^i \, \partial/\partial x^i \,|_{\boldsymbol{x}=\boldsymbol{p}} \tag{2.3}$$

has an intrinsic meaning. $\{\partial/\partial x^i\}$ at $x = p$ defines a basis for the tangent space of $M$ at $p$. A collection of these directional derivatives at each point in $M$ with smoothly varying coefficients $v^i(x)$ is called a *vector field.*

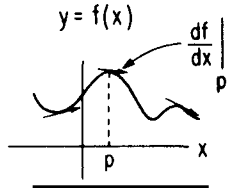


Fig. 2.4. Tangent to a curve $y = f(x)$.

*The tangent space* $T_p(M)$ is thus defined as the vector space spanned by the tangents at $p$ to all curves passing through $p$ in the manifold (see fig. 2.5). No matter how curved the manifold may be, $T_p(M)$ is always an $n$-dimensional vector space at each point $p$.

The tangent space occurs naturally in classical mechanics. We consider a Lagrangian $L(q^i(t), \dot{q}^i(t))$ and recall that $t$-derivatives can be defined using the implicit function rule

$$\mathrm{d}/\mathrm{d}t = \partial/\partial t + \dot{q}^i\, \partial/\partial q^i. \tag{2.4}$$

Comparison with eq. (2.3) shows that the second term in the above equation has the structure of a vector field. *Velocity space* in Lagrangian classical mechanics corresponds exactly to the tangent space of the configuration space: if $M$ has coordinates $\{q^i\}$, then $T_q(M)$ has coordinates $\{\dot{q}^i\}$. Equation (2.4) shows that the operators $\{\partial/\partial q^i\}$ form a basis for $T_q(M)$.

*The cotangent space* $T_p^*(M)$ of a manifold at $p \in M$ is defined as the dual vector space to the tangent space $T_p(M)$. A *dual vector space* is defined as follows: given an $n$-dimensional vector space $V$ with basis $E_i$, $i = 1, \ldots, n$, the basis $e^i$ of the dual space $V^*$ is determined by the inner product

$$\langle E_i, e^j \rangle = \delta^j_i.$$

When we take the basis vectors $E_i = \partial/\partial x^i$ for $T_p(M)$, we write the basis vectors for $T_p^*(M)$ as the differential line elements

$$e^i = \mathrm{d}x^i.$$

Thus the inner product is given by

$$\langle \partial/\partial x^i, \mathrm{d}x^j \rangle = \delta^j_i.$$

Now consider the vector field

$$V = v^i\, \partial/\partial x^i$$

and the covector field

$$U = u_i\, \mathrm{d}x^i.$$

Under general coordinate transformations $x \to x'(x)$, $V$ and $U$ are invariant, but since

$$\mathrm{d}x'^i = \frac{\partial x'^i}{\partial x^j}\mathrm{d}x^j \qquad \frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j},$$

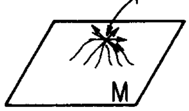


Fig. 2.5. Curves through a point $p$ of $M$. The tangents to these curves span the tangent space $T_p(M)$.

the components $v^i$ and $u_i$ change according to

$$v'^i = v^j \, \partial x'^i / \partial x^j$$
$$u'_i = u_j \, \partial x^j / \partial x'^i .$$

(The invariance of $V$ and $U$ in fact is the origin of the transformation law for contravariant and covariant vectors, respectively.) Thus the inner product

$$\langle V, U \rangle = v^i u_i = v'^i u'_i$$

is invariant under general coordinate transformations.

The idea of the cotangent space also occurs in classical mechanics. Whereas tangent space corresponds to velocity space, cotangent space corresponds to *momentum space*. Here the basis vectors are given by the differential line elements $\mathrm{d}q^i$, so the cotangent vector fields are expressed as

$$p_i \, \mathrm{d}q^i$$

where we identify

$$p_i = \partial L(q^i, \dot{q}^i) / \partial \dot{q}^i .$$

Using the basis elements of $T_p(M)$ and $T_p^*(M)$, we may now extend the concept of a field to include tensor fields over $M$ with $l$ covariant and $k$ contravariant indices, which we write

$$w^{(k)}_{(l)} = w^{i_1 i_2 \dots i_k}_{j_1 j_2 \dots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_l} .$$

The tensor product symbol $\otimes$ implies no symmetrization or antisymmetrization of indices – each basis element is taken to act independently of the others.

## *2.3. Differential forms*

A special class of tensor fields, the totally *antisymmetric covariant tensor fields* are useful for many practical calculations.

We begin by defining Cartan's *wedge product*, also known as the exterior product, as the antisymmetric tensor product of cotangent space basis elements

$$\mathrm{d}x \wedge \mathrm{d}y = \tfrac{1}{2}(\mathrm{d}x \otimes \mathrm{d}y - \mathrm{d}y \otimes \mathrm{d}x)$$
$$= -\mathrm{d}y \wedge \mathrm{d}x .$$

Note that, by definition,

$$\mathrm{d}x \wedge \mathrm{d}x = 0 .$$

The differential line elements $dx$ and $dy$ are called *differential 1-forms* or 1-forms; thus the wedge product is a rule for constructing *2-forms* out of pairs of 1-forms. It is easy to show that the 2-form made in this way has the properties we expect of a differential *area* element. Suppose we change variables to $x'(x, y)$, $y'(x, y)$; then we find

$$\begin{aligned} dx' \wedge dy' &= \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}\right) dx \wedge dy \\ &= \text{Jacobian } (x', y'; x, y)\, dx \wedge dy. \end{aligned}$$

Cartan's wedge product thus is designed to produce the required *signed* Jacobian every time we change variables. Let $\Lambda^p(x)$ be the set of anti-symmetric $p$-tensors at a point $x$. This is a vector space of dimension $n!/p!(n-p)!$. The $\Lambda^p(x)$ patch together to define a bundle over $M$ as we shall discuss later. $C^\infty(\Lambda^p)$ is the space of smooth $p$-forms, represented by anti-symmetric tensors $f_{ij}\ldots(x)$ having $p$ indices contracted with the wedge products of $p$ differentials. The elements of $C^\infty(\Lambda^p)$ may then be written explicitly as follows:

$$\begin{array}{ll} C^\infty(\Lambda^0) = \{f(x)\} & \text{dimension} = 1 \\ C^\infty(\Lambda^1) = \{f_i(x)\, dx^i\} & \dim = n \\ C^\infty(\Lambda^2) = \{f_{ij}(x)\, dx^i \wedge dx^j\} & \dim = n(n-1)/2! \\ C^\infty(\Lambda^3) = \{f_{ijk}(x)\, dx^i \wedge dx^j \wedge dx^k\} & \dim = n(n-1)(n-2)/3! \\ \quad\vdots & \quad\vdots \\ C^\infty(\Lambda^{n-1}) = \{f_{i_1\ldots i_{n-1}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_{n-1}}\} & \dim = n \\ C^\infty(\Lambda^n) = \{f_{i_1\ldots i_n}\, dx^{i_1} \wedge \cdots \wedge dx^{i_n}\} & \dim = 1. \end{array} \tag{2.5}$$

Several important properties emerge: First, we see that $\Lambda^p$ and $\Lambda^{n-p}$ have the same dimension as vector spaces. In particular, $C^\infty(\Lambda^n)$ is representable by a single function times the $n$-volume element. Furthermore, we deduce that $\Lambda^p = 0$ for $p > n$, since some differential would appear twice and be annihilated.

Now it is clear that the wedge product may be used to make $(p+q)$-forms out of a given $p$-form and a given $q$-form. But since one gets zero for $p+q > n$, the resulting forms always belong to the original set of spaces, which we write

$$\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^n.$$

The space $\Lambda^*$ of all possible antisymmetric covariant tensors therefore reproduces itself under the wedge product operation: $\Lambda^*$ is a graded algebra called Cartan's *exterior algebra* of differential forms.
*Remark*: Let $\alpha_p$ be an element of $\Lambda^p$, $\beta_q$ an element of $\Lambda^q$. Then

$$\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p.$$

Hence odd forms anticommute and the wedge product of identical 1-forms will always vanish.

*Exterior derivative*: Another useful tool for manipulating differential forms is the exterior derivative operation, which takes $p$-forms into $(p+1)$-forms according to the rule

$$C^\infty(\Lambda^0)\xrightarrow{\mathrm{d}}C^\infty(\Lambda^1);\qquad \mathrm{d}(f(x))=\frac{\partial f}{\partial x^i}\,\mathrm{d}x^i$$

$$C^\infty(\Lambda^1)\xrightarrow{\mathrm{d}}C^\infty(\Lambda^2);\qquad \mathrm{d}(f_j(x)\,\mathrm{d}x^j)=\frac{\partial f_j}{\partial x^i}\,\mathrm{d}x^i\wedge\mathrm{d}x^j$$

$$C^\infty(\Lambda^2)\xrightarrow{\mathrm{d}}C^\infty(\Lambda^3);\qquad \mathrm{d}(f_{jk}(x)\,\mathrm{d}x^j\wedge\mathrm{d}x^k)=\frac{\partial f_{jk}}{\partial x^i}\,\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k$$

etc.

Here we have taken the convention that the new differential line element is always inserted *before* any previously existing wedge products. Note also that, to be precise, only the totally antisymmetric parts of the partial derivatives contribute.

An important property of the exterior derivative is that it gives *zero* when applied twice:

$$\mathrm{d}\mathrm{d}\omega_p=0.$$

This identity follows from the equality of mixed partial derivatives, as we can see from the following simple example:

$$C^\infty(\Lambda^0)\xrightarrow{\mathrm{d}}C^\infty(\Lambda^1)\xrightarrow{\mathrm{d}}C^\infty(\Lambda^2)$$

$$\mathrm{d}f=\partial_j f\,\mathrm{d}x^j$$

$$\mathrm{d}\mathrm{d}f=\partial_i\,\partial_j f\,\mathrm{d}x^i\wedge\mathrm{d}x^j=\tfrac{1}{2}(\partial_i\,\partial_j f-\partial_j\,\partial_i f)\,\mathrm{d}x^i\wedge\mathrm{d}x^j=0.$$

In vector notation, $\mathrm{d}\mathrm{d}\omega_p=0$ is equivalent to the familiar statements that

$$\mathrm{curl}\cdot\mathrm{grad}\,f=0$$

$$\mathrm{div}\cdot\mathrm{curl}\,f=0,\qquad \text{etc.}$$

We note also the rule for differentiating the wedge product of a $p$-form $\alpha_p$ and a $q$-form $\beta_q$:

$$\mathrm{d}(\alpha_p\wedge\beta_q)=\mathrm{d}\alpha_p\wedge\beta_q+(-1)^p\,\alpha_p\wedge\mathrm{d}\beta_q.$$

*Note*: The exterior derivative anticommutes with 1-forms.

*Examples 2.3*

1. Possible $p$-forms $\alpha_p$ in two-dimensional space are

$$\alpha_0=f(x,y)$$

$$\alpha_1=u(x,y)\,\mathrm{d}x+v(x,y)\,\mathrm{d}y$$

$$\alpha_2=\phi(x,y)\,\mathrm{d}x\wedge\mathrm{d}y.$$

The exterior derivative of a line element gives the two-dimensional curl times the area:

$$d(u(x, y)\,dx + v(x, y)\,dy) = (\partial_x v - \partial_y u)\,dx \wedge dy.$$

2. The three-space $p$-forms $\alpha_p$ are

$$\alpha_0 = f(x)$$
$$\alpha_1 = v_1\,dx^1 + v_2\,dx^2 + v_3\,dx^3$$
$$\alpha_2 = w_1\,dx^2 \wedge dx^3 + w_2\,dx^3 \wedge dx^1 + w_3\,dx^1 \wedge dx^2$$
$$\alpha_3 = \phi(x)\,dx^1 \wedge dx^2 \wedge dx^3.$$

We see that

$$\alpha_1 \wedge \alpha_2 = (v_1 w_1 + v_2 w_2 + v_3 w_3)\,dx^1 \wedge dx^2 \wedge dx^3$$
$$d\alpha_1 = (\varepsilon_{ijk}\,\partial_j v_k)\tfrac{1}{2}\varepsilon_{ilm}\,dx^l \wedge dx^m$$
$$d\alpha_2 = (\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3)\,dx^1 \wedge dx^2 \wedge dx^3.$$

We thus recognize the usual operations of three-dimensional vector calculus.

### *2.4. Hodge star and the Laplacian*

As we have seen from eq. (2.5) and the examples, the number of independent functions in $C^\infty(\Lambda^p)$ is the same as that in $C^\infty(\Lambda^{n-p})$: there exists a *duality* between the two spaces. We are thus motivated to introduce an operator, the *Hodge* $*$ or *duality transformation*, which transforms $p$-forms into $(n-p)$-forms; in a flat Euclidean space the operator is defined by

$$*(dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}) = \frac{1}{(n-p)!}\varepsilon_{i_1 i_2 \ldots i_p i_{p+1} \ldots i_n}\,dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \cdots \wedge dx^{i_n}.$$

Here $\varepsilon_{ijk\ldots}$ is the totally antisymmetric tensor in $n$-dimensions.

*Note*: Later, when we introduce a metric, we will have to be careful about raising and lowering indices and multiplying by $g^{1/2}$. For now, this point is inessential and will be postponed.

Repeating the $*$ operator on a $p$-form $\omega_p$ gives

$$**\,\omega_p = (-1)^{p(n-p)}\omega_p.$$

We note that for $p = n$,

$$dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n} = \varepsilon_{i_1 i_2 \ldots i_n}\,dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n. \tag{2.6}$$

*Inner product*: Letting $\alpha_p$ and $\beta_p$ be $p$-forms, we define the inner product as the integral

$$(\alpha_p, \beta_p) = \int_M \alpha_p \wedge * \beta_p .$$

For general $p$-forms $\alpha_p$, $\beta_p$ with coefficient functions $f_{ijk\ldots}$ and $g_{ijk\ldots}$, it is easy to show that

$$(\alpha_p, \beta_p) = p! \int_M f_{ijk\ldots} g_{ijk\ldots} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n .$$

The inner product has the further property that

$$(\alpha_p, \beta_p) = (\beta_p, \alpha_p)$$

because of the identity

$$\alpha_p \wedge * \beta_p = \beta_p \wedge * \alpha_p$$

which follows from (2.6).

*Adjoint of exterior derivative*: Examining the inner product $(\alpha_p, \mathrm{d}\beta_{p-1})$ and integrating by parts, we find

$$(\alpha_p, \mathrm{d}\beta_{p-1}) = (\delta\alpha_p, \beta_{p-1}),$$

where the adjoint of d is

$$\delta = (-1)^{np+n+1} * \mathrm{d} * .$$

Note that for $n$ *even* and all $p$,

$$\delta = - * \mathrm{d} * ,$$

while for $n$ odd,

$$\delta = (-1)^p * \mathrm{d} * .$$

(*Remark*: Additional factors of $(-1)$ occur for spaces with negative signature.) $\delta$ *reduces* the degree of a differential form by one unit, whereas d increases the degree:

$$\mathrm{d}: C^\infty(\Lambda^p) \to C^\infty(\Lambda^{p+1})$$
$$\delta: C^\infty(\Lambda^p) \to C^\infty(\Lambda^{p-1}).$$

Like d, $\delta$ acting on forms produces conventional tensor calculus operations – for example, with $n = 3$ and $p = 1$, we find

$$\delta(\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}) = -*(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = -\boldsymbol{\nabla} \cdot \boldsymbol{v} .$$

We note that, like d, $\delta$ gives *zero* when repeated:

$$\delta\delta\omega_p = 0.$$

*Laplacian*: The Laplacian on a manifold can be constructed once d and $\delta$ are known (this would, in general, require knowledge of a metric, but we will continue to use a flat metric for the time being). The Laplacian is

$$\Delta = (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\mathrm{d}. \tag{2.7}$$

We sometimes add a subscript to d and $\delta$ to remind ourselves what kind of form we are acting on. Thus we may write the Laplacian on $p$-forms as

$$\Delta\omega_p = \mathrm{d}_{p-1}\,\delta_p\omega_p + \delta_{p+1}\,\mathrm{d}_p\,\omega_p.$$

The Laplacian clearly takes $p$-forms back into $p$-forms,

$$\Delta\colon C^\infty(\Lambda^p) \to C^\infty(\Lambda^p).$$

For example, on 1-forms, we find

$$\Delta(\boldsymbol{v}\cdot \mathrm{d}\boldsymbol{x}) = -\frac{\partial^2 \boldsymbol{v}}{\partial x^k\,\partial x^k}\cdot \mathrm{d}\boldsymbol{x}.$$

Thus $\Delta$ is called a *positive* operator because its Fourier transform introduces a factor of $\mathrm{i}^2$ which cancels the minus sign. An elegant way of proving the positivity of the Laplacian follows from taking the inner product of the two $p$-forms $\omega_p$ and $\Delta\omega_p$. Using (2.7) we find that, provided there are no boundary terms,

$$(\omega_p, \Delta\omega_p) = (\omega_p, \mathrm{d}\delta\omega_p) + (\omega_p, \delta\,\mathrm{d}\omega_p)$$
$$= (\delta\omega_p, \delta\omega_p) + (\mathrm{d}\omega_p, \mathrm{d}\omega_p),$$

which is necessarily $\geq 0$. As a corollary, we see that for sufficiently well-behaved forms, $\omega_p$ is *harmonic*, that is

$$\Delta\omega_p = 0,$$

*if and only if $\omega_p$ is closed,*

$$\mathrm{d}\omega_p = 0$$

and *co-closed,*

$$\delta\omega_p = 0.$$

A $p$-form $\omega_p$ which can be written globally as the exterior derivative of some $(p-1)$-form $\alpha_{p-1}$,

$$\omega_p = \mathrm{d}\alpha_{p-1},$$

is called an *exact* $p$-form. Similarly, a $p$-form $\omega_p$ which can be expressed globally as

$$\omega_p = \delta\alpha_{p+1}$$

is called a *co-exact* $p$-form.

*Hodge's theorem*: Hodge [1952] has shown that if $M$ is a compact manifold without boundary, any $p$-form $\omega_p$ can be uniquely decomposed as a sum of exact, co-exact and harmonic forms,

$$\omega_p = \mathrm{d}\alpha_{p-1} + \delta\beta_{p+1} + \gamma_p$$

where $\gamma_p$ is a *harmonic* $p$-form. For many applications, the essential properties of $\omega_p$ lie entirely in the harmonic piece $\gamma_p$.

*Stokes' theorem*: If $M$ is a $p$-dimensional manifold with a non-empty boundary $\partial M$, then Stokes' theorem says that for any $(p-1)$-form $\omega_{p-1}$,

$$\int_M \mathrm{d}\omega_{p-1} = \int_{\partial M} \omega_{p-1}.$$

If $\partial M$ has several parts, the right-hand side is an *oriented* sum. For $p=1$, where $M$ is a line segment from $a$ to $b$, we find the fundamental theorem of calculus,

$$\int_a^b \mathrm{d}f(x) = f(b) - f(a).$$

For $p=2$, we find

$$\int_{\text{surface}} \mathrm{d}(\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}) = \oint_{\text{line}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}.$$

In 3 dimensions, where we may make the identification

$$\mathrm{d}(\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}) = \tfrac{1}{2}(\partial_i A_j - \partial_j A_i)\,\mathrm{d}x^i \wedge \mathrm{d}x^j \equiv \tfrac{1}{2}\varepsilon_{ijk}B_k\,\mathrm{d}x^i \wedge \mathrm{d}x^j,$$

we recognize the formula for the magnetic flux going through a surface,

$$\int \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}.$$

For $p = 3$, we examine the 2-form

$$\omega = \tfrac{1}{2}\varepsilon_{ijk} E_k \, dx^i \wedge dx^j$$

obeying

$$d\omega = \nabla \cdot \boldsymbol{E} \, dx^1 \wedge dx^2 \wedge dx^3.$$

Then Stokes' theorem becomes

$$\int \nabla \cdot \boldsymbol{E} \, d^3x = \int_{\text{volume}} d\omega = \int_{\text{surface}} \omega = \int \boldsymbol{E} \cdot d\boldsymbol{S}$$

and we recognize Gauss' law.

*Examples 2.4*

1. *Two-dimensions* ($n = 2$):
   Basis of $\Lambda^*$: $(1, dx, dy, dx \wedge dy)$
   Hodge $*$: $*(1, dx, dy, dx \wedge dy) = (dx \wedge dy, dy, -dx, 1)$
   $\delta$ operation:

   $$\delta f(x, y) = 0$$

   $$\delta(u \, dx + v \, dy) = -(\partial_x u + \partial_y v)$$

   $$\delta\phi \, dx \wedge dy = -\partial_x\phi \, dy + \partial_y \phi \, dx$$

   Laplacian: acting on, for instance, 0-forms,

   $$\Delta f = -(\partial_x^2 f + \partial_y^2 f).$$

2. *Euclidean Maxwell's equation* ($\mu = 1, 2, 3, 4$; $i = 1, 2, 3$)

   Gauge potential: $A = A_\mu(x) \, dx^\mu$
   Gauge transform: $A' = A + d\Lambda(x)$
   Field strength: $F = dA = dA'$
   (gauge invariant $= \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) \, dx^\mu \wedge dx^\nu$
   due to $dd\Lambda = 0$) $= \tfrac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$

   $E$ and $B$: $F = E_i \, dx^i \wedge dx^4 + \tfrac{1}{2} B_i \varepsilon_{ijk} \, dx^j \wedge dx^k$

   $$*F = \tfrac{1}{2} E_i \varepsilon_{ijk} \, dx^j \wedge dx^k + B_i \, dx^i \wedge dx^4$$

   duality: $F \leftrightarrow *F$, $E \leftrightarrow B$

Euler eqn. = inhomogeneous eqns: $\delta F = j$

$$\delta F = -\nabla \cdot \boldsymbol{E}\, \mathrm{d}x^4 + (\partial_4 \boldsymbol{E} + \nabla \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{x}$$

$$j = j_\mu\, \mathrm{d}x^\mu = \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{x} + j_4\, \mathrm{d}x^4$$

Bianchi identity = homogeneous eqns: $\mathrm{d}F = \mathrm{dd}A = 0$

$$\mathrm{d}F = \nabla \cdot \boldsymbol{B}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \tfrac{1}{2}(\partial_4 \boldsymbol{B} + \nabla \times \boldsymbol{E})_i \varepsilon_{ijk}\, \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^4 = 0.$$

*Note*: If $j = 0$, then $\mathrm{d}F = \delta F = 0$, so $F$ is *harmonic*, $\Delta F = 0$.

3. *Dirac magnetic monopole* (Dirac [1931]). In order to describe a magnetic charge, we introduce two coordinate patches $U_\pm$ covering the $z > -\varepsilon$ and the $z < +\varepsilon$ regions of $\mathbf{R}^3 - \{0\}$, with overlap region $U_+ \cap U_-$ effectively equal to the $x$–$y$ plane at $z = 0$ minus the origin. The gauge potentials which are well-defined in these respective regions are taken as

$$A_\pm = \frac{1}{2r}\frac{1}{z \pm r}(x\, \mathrm{d}y - y\, \mathrm{d}x) = \tfrac{1}{2}(\pm 1 - \cos\theta)\, \mathrm{d}\phi$$

where $r^2 = x^2 + y^2 + z^2$. $A_+$ and $A_-$ have the Dirac string singularity at $\theta = \pi$ and $\theta = 0$, respectively. Note that $A_+$ and $A_-$ are related by a gauge transformation:

$$A_+ = A_- + \mathrm{d}\tan^{-1}(y/x) = A_- + \mathrm{d}\phi.$$

In the overlap region $\theta = \pi/2$, $r > 0$, both potentials are regular. The field is given by $F = \mathrm{d}A_\pm$ in $U_\pm$, so

$$F = \frac{1}{2r^3}(x\, \mathrm{d}y \wedge \mathrm{d}z + y\, \mathrm{d}z \wedge \mathrm{d}x + z\, \mathrm{d}x \wedge \mathrm{d}y)$$

or

$$\boldsymbol{B} = \boldsymbol{x}/2r^3.$$

*Remark*: *Dirac strings.* In the modern approach to the magnetic monopole, $A_\pm$ are defined only in their respective coordinate patches $U_\pm$. In Dirac's formulation of the monopole, coordinate patches were not used and $A_\pm$ were used over all of $\mathbf{R}^3$. This led to the appearance of fictitious "string singularities" on the $\pm z$ axis.

### *2.5. Introduction to homology and cohomology*

We conclude this section with a brief treatment of the concepts of homology and de Rham cohomology, which form a crucial link between the topological aspects of manifolds and their differentiable structure.

*Homology*: Homology is used to distinguish topologically inequivalent manifolds. For a treatment more mathematically precise than the one given here, see Greenberg [1967] or Spanier [1966].

Let $M$ be a smooth connected manifold. A *p-chain* $a_p$ is a formal sum of the form $a_p = \Sigma_i c_i N_i$ where the $N_i$ are smooth $p$-dimensional oriented submanifolds of $M$. If the coefficients $c_i$ are real (complex), then $a_p$ is a real (complex) chain; if the coefficients $c_i$ are integers, $a_p$ is an integral chain; if the coefficients $c_i \in \mathbb{Z}_2 = \{0,1\}$, then $a_p$ is a $\mathbb{Z}_2$ chain. There are other coefficients which could be considered, but these are the only ones we shall be interested in.

Let $\partial$ denote the operation of taking the oriented boundary. We define $\partial a_p = \Sigma_i c_i \, \partial N_i$ to be a $(p-1)$-chain. Let $Z_p = \{a_p\colon \partial a_p = \emptyset\}$ be the set of *cycles* (i.e., $p$-chains with no boundaries) and let $B_p = \{\partial a_{p+1}\}$ be the set of *boundaries* (i.e., those chains which can be written as $a_p = \partial a_{p+1}$ for some $a_{p+1}$). Since the boundary of a boundary is always empty, $\partial\partial a_p = \emptyset$, $B_p$ is a subset of $Z_p$.

We define the *simplicial homology* of $M$ by

$$H_p = Z_p/B_p.$$

$H_p$ is the set of equivalence classes of cycles $z_p \in Z_p$ which differ only by boundaries; that is $z'_p \simeq z_p$ provided that $z'_p = z_p + \partial a_{p+1}$. We can think of representative cycles in $H_p$ as manifolds patched together to "surround" a hole; we ignore cycles which can be "filled in".

We may choose different coefficient groups to define $H_p(M;\mathbb{R})$, $H_p(M;\mathbb{C})$, $H_p(M;\mathbb{Z})$, or $H_p(M;\mathbb{Z}_2)$. There are simple relations $H_p(M;\mathbb{R}) = H_p(M;\mathbb{Z})\otimes\mathbb{R}$ and $H_p(M;\mathbb{C}) = H_p(M;\mathbb{R})\otimes\mathbb{C} = H_p(M;\mathbb{Z})\otimes\mathbb{C}$. In other words, modulo finite groups (i.e., *torsion*), $H_p(M;\mathbb{R})$, $H_p(M;\mathbb{Z})$, and $H_p(M;\mathbb{C})$ are essentially the same.

The integral homology is fundamental. We can regard any integral cycle as real by embedding $\mathbb{Z}$ in $\mathbb{R}$. We can reduce any integral cycle mod 2 to get a $\mathbb{Z}_2$ cycle. The *universal coefficient theorem* gives a formula for the homology with $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_2$ coefficients in terms of the integral homology. In particular, real homology is obtained from integral homology by replacing all the "$\mathbb{Z}$" factors by "$\mathbb{R}$" and by throwing away any torsion subgroups.

It is clear that $H_p(M;G) = 0$ for $p > \dim(M)$. If $M$ is connected, $H_0(M;G) = G$. If $M$ is orientable, then $H_n(M;G) = G$. If $G$ is a field, then we have Poincaré duality, $H_p(M;G) = H_{n-p}(M;G)$, for orientable $M$ ($G = \mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_2$ but not $\mathbb{Z}$).

*Examples*

1. *Torus.* We illustrate the computation of homology for the torus $T^2$. In fig. 2.6, the curves $a$ and $b$ belong to the same homology class because they bound a two-dimensional strip $\sigma$ (shown as a shaded area).

$$\partial\sigma = a - b.$$

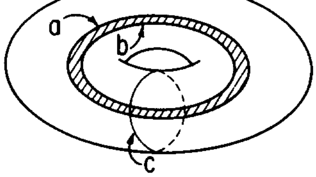


Fig. 2.6. Homology classes of the torus. $a$ and $b$, which bound the shaded area, are homologous. $a$ and $c$ are not.

Curves $a$ and $c$ however do *not* belong to the same homology class. The homology groups of the torus, $M = T^2$, are

$$H_0(M;\mathbb{R}) = \mathbb{R}$$
$$H_1(M;\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$$
$$H_2(M;\mathbb{R}) = \mathbb{R}.$$

The generators of $H_1$ are given by the two curves $a$ and $c$.

2. *Torsion and homology of* $P_3(\mathbb{R}) = \mathrm{SO}(3)$.

The concept of torsion and the effect of different coefficient groups can be illustrated by examining $M = P_3(\mathbb{R}) = \mathrm{SO}(3)$. Let $\rho$ map $S^3$ to $P_3(\mathbb{R})$ by antipodal identification of the points of $S^3$.

Let $S^2$ be the equator of $S^3$, let $S^1$ be the equator of $S^2$, and let $D^k$ be the upper hemisphere of $S^k$. Then $\rho(D^k)$ is a $k$-chain on $P_3(\mathbb{R})$ and

$\partial\rho(D^3) = 0$ (this is a cycle and generates $H_3$ with any coefficients)

$\partial\rho(D^2) = 2\rho(D^1)$ (this is a cycle in $\mathbb{Z}_2$ but not in $\mathbb{R}$ or $\mathbb{Z}$)

$\partial\rho(D^1) = 0$ (this is a cycle. Over $\mathbb{R}$ we have $\rho(D^1) = \partial\frac{1}{2}\rho(D^2)$ so this is a boundary. It is not a boundary over $\mathbb{Z}$ or $\mathbb{Z}_2$ and generates $H_1$ for these groups).

In $\mathbb{Z}_2$ homology $\rho(D^k)$ gives the generators of $H_k(P_3(\mathbb{R}); \mathbb{Z}_2)$. The homology groups of $P_3(\mathbb{R})$ can be shown to be the following:

| | | |
|---|---|---|
| $H_0(M;\mathbb{Z}) = \mathbb{Z}$ | $H_0(M;\mathbb{R}) = \mathbb{R}$ | $H_0(M;\mathbb{Z}_2) = \mathbb{Z}_2$ |
| $H_1(M;\mathbb{Z}) = \mathbb{Z}_2$ | $H_1(M;\mathbb{R}) = 0$ | $H_1(M;\mathbb{Z}_2) = \mathbb{Z}_2$ |
| $H_2(M;\mathbb{Z}) = 0$ | $H_2(M;\mathbb{R}) = 0$ | $H_2(M;\mathbb{Z}_2) = \mathbb{Z}_2$ |
| $H_3(M;\mathbb{Z}) = \mathbb{Z}$ | $H_3(M;\mathbb{R}) = \mathbb{R}$ | $H_3(M;\mathbb{Z}_2) = \mathbb{Z}_2$. |

These groups are different because of the existence of torsion.

*de Rham cohomology*

If $G$ is a field ($G = \mathbb{R}, \mathbb{C}, \mathbb{Z}_2$), the homology group $H_p(M; G)$ is a vector space over $G$. We define the cohomology group $H^p(M; G)$ to be the dual vector space to $H_p(M, G)$. (The definition of $H^p(M;\mathbb{Z})$ is slightly more complicated and we shall omit it.) The remarkable fact is that $H^p(M;\mathbb{R})$ or $H^p(M;\mathbb{C})$ may be understood using differential forms. We define the *de Rham cohomology groups* $H^p_{\mathrm{DR}}(M;\mathbb{R})$ as follows: recall that a $p$-form $\omega_p$ is closed if $\mathrm{d}\omega_p = 0$ and exact if $\omega_p = \mathrm{d}\alpha_{p-1}$. Let

$Z^p_{\mathrm{DR}} = \{\omega_p : \mathrm{d}\omega_p = 0\}$ (the *closed* forms)

$B^p_{\mathrm{DR}} = \{\omega_p : \omega_p = \mathrm{d}\alpha_{p-1}\}$ (the *exact* forms)

$H^p_{\mathrm{DR}}(M;\mathbb{R}) = Z^p/B^p$ (closed modulo exact forms).

The de Rham cohomology is the set of equivalence classes of closed forms which differ only by exact forms; that is

$$\omega_p \simeq \omega'_p$$

if $\omega_p = \omega'_p + d\alpha_{p-1}$ for some $\alpha_{p-1}$.
*Remark*: The space $H^0$ is special because there are no $(-1)$-forms, and thus no 0-forms can be expressed as exterior derivatives. Since the exterior derivative of a constant is zero,

$$H^0 = \{\text{space of constant functions}\}$$

and

$$\dim(H^0) = \text{number of connected pieces of the manifold.}$$

*Poincaré lemma*: The de Rham cohomology of Euclidean space $\mathbb{R}^n$ is trivial,

$$\dim H^p(\mathbb{R}^n) = 0 \qquad p > 0$$
$$(\dim H^0(\mathbb{R}^n) = 1),$$

since any closed form can be expressed as the exterior derivative of a lower form in $\mathbb{R}^n$. For example, in $\mathbf{R}^3$, any closed 1-form can be expressed as the gradient of a scalar function,

$$\nabla \times \boldsymbol{A} = 0 \rightarrow \boldsymbol{A} = \nabla\varphi.$$

Therefore any closed form can be expressed as an exact form in any local $\mathbf{R}^n$ coordinate patch of the manifold. *Non-trivial de Rham cohomology therefore occurs only when the local coordinate neighborhoods are patched together in a globally non-trivial way.*

***de Rham's theorem***: The inner product of a cycle $c_p \in Z_p$ and a closed form $\omega_p \in Z^p_{\mathrm{DR}}$ is defined as

$$\pi(c_p, \omega_p) = \int_{c_p} \omega_p$$

where $\pi(c, \omega) \in \mathbb{R}$ is called a *period*. We note that by Stokes' theorem, when $c_p \in Z_p$ and $\omega_p \in Z^p_{\mathrm{DR}}$, then

$$\int_{c_p} \omega_p + d\alpha_{p-1} = \int_{c_p} \omega_p + \int_{\partial c_p} \alpha_{p-1} = \int_{c_p} \omega_p,$$

and

$$\int_{c_p + \partial a_{p+1}} \omega_p = \int_{c_p} \omega_p + \int_{a_{p+1}} d\omega_p = \int_{c_p} \omega_p.$$

This pairing is thus independent of the choice of the representatives of the equivalence classes and defines a map

$$\pi: H_p(M;\mathbf{R}) \otimes H^p_{\mathrm{DR}}(M;\mathbf{R}) \to \mathbf{R}.$$

de Rham has proven the following fundamental theorems when $M$ is a compact manifold without boundary:

Let $\{c_i\}$, $i = 1, \ldots, \dim H_p(M;\mathbf{R})$, be a set of independent $p$-cycles forming a basis for $H_p(M;\mathbf{R})$.

*First theorem*: Given any set of periods $\nu_i$, $i = 1, \ldots, \dim H_p$, there exists a closed $p$-form $\omega$ for which

$$\nu_i = \pi(c_i, \omega) = \int_{c_i} \omega, \qquad i = 1, \ldots, \dim H_p.$$

*Second theorem*: If all the periods for a $p$-form $\alpha$ vanish,

$$0 = \pi(c_i, \alpha) = \int_{c_i} \alpha, \qquad i = 1, \ldots, \dim H_p$$

then $\alpha$ is *exact.*

In other words, if $\{\omega_j\}$ is a basis for $H^p_{\mathrm{DR}}(M;\mathbf{R})$, then the period matrix

$$\pi_{ij} = \pi(c_i, \omega_j)$$

is invertible. Thus $H^p_{\mathrm{DR}}(M;\mathbf{R})$ is dual to $H_p(M;\mathbf{R})$ with respect to the inner product $\pi$. Therefore de Rham cohomology $H^p_{\mathrm{DR}}$ and simplicial cohomology $H^p$ are *naturally isomorphic*,

$$H^p_{\mathrm{DR}}(M;\mathbf{R}) \simeq H^p(M;\mathbf{R}),$$

and henceforth will be identified.

We define

$$b_p = \dim H_p(M;\mathbf{R}) = \dim H^p(M;\mathbf{R})$$

as the $p$th *Betti number* of $M$. The alternating sum of the Betti numbers is the Euler characteristic

$$\chi(M) = \sum_{p=0}^{n} (-1)^p b_p.$$

The de Rham theorem relates the topological Euler characteristic calculated from $H_p$ to the analytic Euler characteristic calculated from de Rham cohomology. The Gauss–Bonnet theorem gives a formula for $\chi(M)$ in terms of curvature as we shall see later.

We say that a cohomology class is *integral* if $\pi(c, \omega) \in \mathbf{Z}$ for any integral cycle $c$. There is always a natural embedding of $H^p(M;\mathbb{Z})$ in $H^p(M;\mathbb{Z})\otimes\mathbb{R}\simeq H^p(M;\mathbb{R})$. However, $H^p(M;\mathbb{Z})$ is not isomorphic to the set of integral de Rham classes since torsion elements are lost during the embedding; $H^p(M;\mathbb{Z})$ in general has torsion elements while $H^p(M;\mathbb{R})$ (and $H^p(M;\mathbb{C})$) do not.

*Pullback mappings.* If $f\colon M\to N$ and if $\omega_p$ is a $p$-form on $N$, then we can pull back $\omega_p$ to define $f^*\omega_p$ as a $p$-form on $M$. For example, if $x^\mu\in M$, $y^i\in N$, $f^i(x^\mu)=y^i$ and $\omega=g_i(y)\,\mathrm{d}y^i$, then we find $f^*\omega = g_i(f(x))\,\partial_\mu f^i(x)\,\mathrm{d}x^\mu$. Since $\mathrm{d}(f^*\omega_p)=f^*\,\mathrm{d}\omega_p$, $f^*$ pulls back closed forms to closed forms and exact forms to exact forms. This defines a map $f^*\colon H^p(N;\mathbb{R})\to H^p(M;\mathbb{R})$. The dual map $f_*\colon H_p(M;\mathbb{R})\to H_p(N;\mathbb{R})$ goes the other way. $f_*$ is defined on the chain level by using the map $f$ to "push forward" chains on $M$ to chains on $N$. It is easy to check that $f_*$ maps cycles to cycles and boundaries to boundaries. $f^*$ is a zero map if $p>\dim M$ or $\dim N$. We also note that

$$\pi(c,f^*\omega)=\pi(f_*c,\omega).$$

*Ring structure*: The wedge product of two closed forms is again closed; the wedge product of an exact and a closed form is exact. Wedge product preserves the cohomology equivalence relation and induces a map from $H^p(M;\mathbb{R})\otimes H^q(M;\mathbb{R})\to H^{p+q}(M;\mathbb{R})$. This defines a ring structure on $H^*(M;\mathbb{R})=\bigoplus_p H^p(M;\mathbb{R})$. Since

$$f^*(\theta\wedge\omega)=f^*\theta\wedge f^*\omega,$$

pulling back preserves the ring structure. $H^*(M;\mathbb{Z})$ and $H^*(M;\mathbb{Z}_2)$ have ring structures similar to $H^*(M;\mathbb{R})$.

*Poincaré duality*: If $M$ is a compact orientable manifold without boundary, then $H^n(M;\mathbb{R})=\mathbb{R}$ because any $\omega_n\in H^n(M;\mathbb{R})$ may be written up to a total differential as

$$\omega_n=\text{const}\times(\text{volume element in } M).$$

Poincaré duality states that $H^p(M;\mathbb{R})$ is dual to $H^{n-p}(M;\mathbb{R})$ with respect to the inner product

$$(\omega_p,\omega_{n-p})=\int_M \omega_p\wedge\omega_{n-p}.$$

Consequently $H^p$ and $H^{n-p}$ are isomorphic as vector spaces and

$$\dim H^p(M;\mathbb{R})=\dim H^{n-p}(M;\mathbb{R}).$$

Hence the Betti numbers are related by

$$b_p=b_{n-p}.$$

Poincaré duality is valid with $\mathbb{Z}_2$ coefficients regardless of whether or not $M$ is orientable.

*Product formulas*: If $M = M_1 \times M_2$, then

$$H^k(M;\mathbf{R}) \simeq \bigoplus_{p+q=k} H^p(M_1;\mathbf{R}) \otimes H^q(M_2;\mathbf{R}),$$

so $H^*(M;\mathbf{R}) \simeq H^*(M_1;\mathbf{R}) \otimes H^*(M_2;\mathbf{R})$. Furthermore, this is a ring isomorphism. This is the ***Kunneth formula***. This formula is *not* valid with $\mathbf{Z}$ or $\mathbf{Z}_2$ coefficients. Since the Betti numbers are related by

$$b_k(M) = \sum_{p+q=k} b_p(M_1)\, b_q(M_2),$$

we find that the Euler characteristics obey the relation

$$\chi(M = M_1 \times M_2) = \chi(M_1)\,\chi(M_2).$$

*Harmonic forms and de Rham cohomology*

If $M$ is a compact manifold without boundary, we can express each de Rham cohomology class as a harmonic form using the Hodge decomposition theorem,

$$\omega = \mathrm{d}\alpha + \delta\beta + \gamma,$$

where $\gamma$ is harmonic. If $\mathrm{d}\omega = 0$ then $\mathrm{d}\delta\beta = 0$ so $\delta\beta = 0$ and $\omega = \mathrm{d}\alpha + \gamma$. This shows that every cohomology class contains a harmonic representative. If $\omega$ is harmonic, then $\delta\mathrm{d}\alpha = 0$, so $\mathrm{d}\alpha = 0$ and $\omega = \gamma$. This establishes an isomorphism between $H^p(M;\mathbf{R})$ and the set of harmonic $p$-forms $\mathrm{Harm}^p(M;\mathbf{R})$. This is always finite-dimensional, so $H^p(M;\mathbf{R})$ is finite. (If $M$ has a boundary, we must use suitable boundary conditions to obtain this isomorphism.)

If $M = M_1 \times M_2$, $\theta_1$ is harmonic on $M_1$ and $\theta_2$ is harmonic on $M_2$, then $\theta_1 \wedge \theta_2$ is harmonic on $M_1 \times M_2$. This defines the isomorphism

$$\mathrm{Harm}^k(M = M_1 \times M_2;\mathbf{R}) \simeq \bigoplus_{p+q=k} \mathrm{Harm}^p(M_1;\mathbf{R}) \otimes \mathrm{Harm}^q(M_2;\mathbf{R}),$$

which is equivalent to the Kunneth formula defined above.

*Note*: In general the wedge product of two harmonic forms will *not* be harmonic so the ring structure is not given in terms of harmonic forms.

*Note*: If $M$ is oriented, the Hodge operator maps $\Lambda^p \to \Lambda^{n-p}$ with $** = (-1)^{p(n-p)}$. The $*$ operator commutes with the Laplacian and induces an isomorphism

$$*: \mathrm{Harm}^p(M;\mathbf{R}) \simeq \mathrm{Harm}^{n-p}(M;\mathbf{R}).$$

Therefore

$$\dim H^p(M;\mathbf{R}) = \dim H^{n-p}(M;\mathbf{R}).$$

This is another way of looking at Poincaré duality.

*Equivariant cohomology*: An *isometry* of $M$ is a map of $M$ to itself which preserves a given Riemannian metric on $M$. Let $M$ be a manifold on which a finite group $G$ acts by isometries without fixed points and let $N = M/G$. If $\omega_p$ is harmonic on $M$ and $g \in G$, then the pullback $g * \omega_p$ on $M$ is harmonic. If

$$g * \omega_p = \omega_p, \qquad \text{for all } g \in G,$$

then $\omega_p$ is called *G-invariant*. The harmonic $p$-forms on $N = M/G$ can be identified with the $G$-invariant harmonic $p$-forms on $M$.

*Examples 2.5*

1. *de Rham cohomology of* $\mathbb{R}^n$. All closed forms are exact on $\mathbb{R}^n$ except for the scalar functions which belong to $H^0$. If $f$ is a function and $df = 0$, then all the partial derivatives of $f$ vanish so $f$ is constant. $\dim H^0(\mathbb{R}^n; \mathbb{R}) = 1$, $\dim H^k(\mathbb{R}^n; \mathbb{R}) = 0$ for $k \neq 0$.

2. *de Rham cohomology of* $S^n$. Only $H^0$ and $H^n$ are non zero for $S^n$ and both have dimension 1. $H^0$ consists of the constant functions and $H^n$ consists of the constant multiples of the volume element. These are the harmonic forms.

3. *de Rham cohomology of the torus*, $T^2 = S^1 \times S^1$. Let $\theta_1$ and $\theta_2$, $0 \leq \theta_i < 2\pi$, be coordinates on each of the two circles making up the torus. The differential forms $d\theta_i$ are then *closed* but not *exact*, since the $\theta_i$ are defined only modulo $2\pi$ and are therefore not global coordinates. Thus $d\theta_1$ and $d\theta_2$ form a basis for $H^1(T^2; \mathbb{R})$ and $\dim H^1(T^2; \mathbb{R}) = 2$. By the Kunneth formula, $H^2(T^2 = S^1 \times S^1; \mathbb{R}) = H^1(S^1; \mathbb{R}) \otimes H^1(S^1; \mathbb{R})$ and so $H^2(T^2; \mathbb{R})$ is generated by $d\theta_1 \wedge d\theta_2$ where $\dim H^2(T^2; \mathbb{R}) = 1$. Obviously $\dim H^0(T^2; \mathbb{R}) = 1$ also.

It is instructive to work out the Hodge decomposition theorem explicitly for $T^2$ by expanding $C^\infty(\Lambda^p)$ in Fourier series using the coordinates $\theta_i$. We find

$$\omega_0 = \sum a_{nm}\, e^{in\theta_1}\, e^{im\theta_2}$$

$$\omega_1 = \sum b^{(1)}_{nm}\, e^{in\theta_1}\, e^{im\theta_2}\, d\theta_1 + \sum b^{(2)}_{nm}\, e^{in\theta_1}\, e^{im\theta_2}\, d\theta_2$$

$$\omega_2 = \sum c_{nm}\, e^{in\theta_1}\, e^{im\theta_2}\, d\theta_1 \wedge d\theta_2.$$

Now we compute the Laplacians

$$\Delta\omega_0 = \delta d\omega_0 = \sum (n^2 + m^2) a_{nm}\, e^{in\theta_1}\, e^{im\theta_2}$$

$$\Delta\omega_1 = (d\delta + \delta d)\omega_1 = \sum (n^2 + m^2)(b^{(1)}_{nm}\, d\theta_1 + b^{(2)}_{nm}\, d\theta_2)\, e^{in\theta_1}\, e^{im\theta_2}$$

$$\Delta\omega_2 = d\delta\omega_2 = \sum (n^2 + m^2) c_{nm}\, e^{in\theta_1}\, e^{im\theta_2}\, d\theta_1 \wedge d\theta_2$$

and introduce the Green's functions $G_p$ of the form

$$G_0 \cdot \omega_0 = \sum_{(n,m) \neq (0,0)} a_{nm}\, e^{in\theta_1}\, e^{im\theta_2}/(n^2 + m^2), \qquad \text{etc.}$$

Then we may write each element of $C^\infty(\Lambda^p)$ as the sum of a closed, a co-closed, and a harmonic form as follows:

$$\begin{aligned}
\omega_0 &= \Delta G_0\omega_0 + a_{00} = 0 + \delta(\mathrm{d}G_0\omega_0) + a_{00}\\
\omega_1 &= \Delta G_1\omega_1 + b_{00}^{(1)}\,\mathrm{d}\theta_1 + b_{00}^{(2)}\,\mathrm{d}\theta_2\\
&= \mathrm{d}(\delta G_1\omega_1) + \delta(\mathrm{d}G_1\omega_1) + b_{00}^{(1)}\,\mathrm{d}\theta_1 + b_{00}^{(2)}\,\mathrm{d}\theta_2\\
\omega_2 &= \Delta G_2\omega_2 + c_{00}\,\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\\
&= \mathrm{d}(\delta G_2\omega_2) + 0 + c_{00}\,\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2.
\end{aligned}$$

We verify explicitly the dimensions of each cohomology class from the harmonic representatives in the decomposition of $\omega_p$.

4. *de Rham cohomology of* $P_n(\mathbb{C})$. There is an element $x\in H^2(P_n(\mathbb{C});\mathbb{R})$ such that $x^k$ generates $H^{2k}(P_n(\mathbb{C});\mathbb{R})\simeq\mathbb{R}$ for $k=0,\ldots,n$. $H^j(P_n(\mathbb{C});\mathbb{R})=0$ if $j$ is odd or if $j>2n$. $x$ will be the first Chern class of a line bundle as discussed later. It has integral periods as does $x^k$ for $k=0,\ldots,n$. $x^{n+1}=0$ since this would be a $2n+2$ form. There is a natural inclusion of $\mathbb{C}^n$ into $\mathbb{C}^{n+1}$ which induces an inclusion of $P_{n-1}(\mathbb{C})$ into $P_n(\mathbb{C})$ which we denote by $i$. Then $i^*\colon H^k(P_n(\mathbb{C});\mathbb{R})\to H^k(P_{n-1}(\mathbb{C});\mathbb{R})$ is an isomorphism for $k<2n$. Consequently, $x$ is universal; we can view $x$ as belonging to $H^2(P_n(\mathbb{C});\mathbb{R})$ for any $n$. ($x$ is the normalized Kähler form of $P_n(\mathbb{C})$; see example 3.4.3.)

5. *de Rham cohomology of* $U(n)$. Let $g$ be an $n\times n$ unitary matrix $g\in U(n)$. $g^{-1}\,\mathrm{d}g$ is a complex matrix of 1-forms. Let $\omega_k=\mathrm{Tr}(g^{-1}\,\mathrm{d}g)^k$ for $k=1,2,\ldots,2n-1$. Then $\omega_k$ is a complex $k$-form which is closed; $\omega_k=0$ if $k$ is even. The $\{\omega_1,\omega_3,\ldots,\omega_{2n-1}\}$ generate $H^*(U(n);\mathbb{C})$. By adding appropriate factors of $\sqrt{-1}$ to make everything real, we could get corresponding generators for $H^*(U(n);\mathbb{R})$. (If we add appropriate scaling factors, these become integral classes which generate $H^*(U(n);\mathbb{Z})$.) If we then take the mod 2 reduction, we get classes which generate $H^*(U(n);\mathbb{Z}_2)$. $g^{-1}\,\mathrm{d}g$ is the Cartan form which will be discussed later. For example, if $n=2$, then:

$$\begin{aligned}
&H^0(U(2);\mathbb{C})\simeq\mathbb{C}, \qquad H^1(U(2);\mathbb{C})\simeq\mathbb{C}\quad(\text{generator }\omega_1)\\
&H^2(U(2);\mathbb{C})\simeq 0, \qquad H^3(U(2);\mathbb{C})\simeq\mathbb{C}\quad(\text{generator }\omega_3)\\
&H^4(U(2);\mathbb{C})\simeq\mathbb{C}\ (\text{generator }\omega_1\wedge\omega_3), \qquad H^k(U(2);\mathbb{C})=0 \text{ for } k>4.
\end{aligned}$$

Of course, $U(2)=U(1)\times\mathrm{SU}(2)=S^1\times S^3$ topologically (although not as a group). Up to a scaling factor $\omega_1$ is $\mathrm{d}\theta$ on $S^1$ and $\omega_3$ is the volume element on $S^3$. $H^*(S^1\times S^3;\mathbb{C})=H^*(S^1;\mathbb{C})\otimes H^*(S^3;\mathbb{C})$ is just an illustration of the Kunneth formula.

6. *de Rham cohomology of* $\mathrm{SU}(n)$. $\mathrm{SU}(n)$ is a subgroup of $U(n)$; let $i\colon\mathrm{SU}(n)\to U(n)$ be the inclusion map. The $i^*\omega_k\in H^k(\mathrm{SU}(n);\mathbb{R})$ are generators for $k=3,\ldots,2n-1$. ($H^1=0$ since $\mathrm{Tr}(g^{-1}\,\mathrm{d}g)=0$ for $\mathrm{SU}(n)$.) Topologically, $U(n)=S^1\times\mathrm{SU}(n)$ and $H^*(U(n))=H^*(S^1)\otimes H^*(\mathrm{SU}(n))$.

7. *The de Rham cohomology of* $P_n(\mathbb{R})$ is a good example involving torsion.

(a) With real coefficients, we argue that

$$H^k(P_n(\mathbb{R});\mathbb{R})=H_k(P_n(\mathbb{R});\mathbb{R})=\begin{cases}\mathbb{R} & \text{if } k=0 \text{ or } k=n,\ n \text{ odd}\\ 0 & \text{otherwise.}\end{cases}$$

If $k\neq 0$, $n$, then there are no harmonic forms on the universal cover $S^n$ and hence $H^k(P_n(\mathbb{R});\mathbb{R})=0$, for

$k \neq 0, n$. Since $P_n(\mathbb{R})$ is connected, $H^0(P_n(\mathbb{R}); \mathbb{R}) = \mathbb{R}$. Finally, if $n$ is odd, the antipodal map $f(x) = -x$ on $S^n$ preserves the volume element and hence $P_n(\mathbb{R})$ is orientable and $H^n(P_n(\mathbb{R}); \mathbb{R}) = \mathbb{R}$. If $n$ is even, the antipodal map reverses the sign of the volume form so there is no equivariant harmonic $n$-form and $H^n(P_n(\mathbb{R}); \mathbb{R}) = 0$. $P_n(\mathbb{R})$ is not orientable if $n$ is even.

(b) With $\mathbb{Z}_2$ coefficients there is an element $x \in H^1(P_n(\mathbb{R}); \mathbb{Z}_2)$ so that $x^k$ generates $H^k(P_n(\mathbb{R}); \mathbb{Z}_2) \simeq \mathbb{Z}_2$ for $k = 0, \ldots, n$. If $i$: $P_{n-1}(\mathbb{R}) \to P_n(\mathbb{R})$ is the natural inclusion, then $i^*x = x$ so $i^*$: $H^k(P_n(\mathbb{R}); \mathbb{Z}_2) \to H^k(P_{n-1}(\mathbb{R}); \mathbb{Z}_2)$ is an isomorphism for $k = 0, \ldots, n-1$. ($x$ is a Stiefel–Whitney class.)

(c) With integer coefficients,

$$H^k(P_n(\mathbb{R}); \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, \ldots;$$

$$H^n(P_n(\mathbb{R}); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = \text{odd} \\ \mathbb{Z}_2 & \text{if } n = \text{even} \end{cases}$$

$$H_k(P_n(\mathbb{R}); \mathbb{Z}) = \mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \ldots;$$

$$H_n(P_n(\mathbb{R}); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = \text{odd} \\ 0 & \text{if } n = \text{even.} \end{cases}$$

The shift in the relative positions of the $\mathbb{Z}_2$ terms in $H^k$ and $H_k$ is a consequence of the universal coefficient theorem (see, e.g., Spanier [1966]).

## 3. Riemannian manifolds

We now consider manifolds endowed with a metric. We apply the tools of the previous section and present classical Riemannian geometry in a modern notation which is convenient for practical calculations. A still more abstract approach to Riemannian manifolds will be given when we treat connections on fiber bundles.

### 3.1. Cartan structure equations

Suppose we are given a 4-manifold $M$ and a metric $g_{\mu\nu}(x)$ on $M$ in local coordinates $x^\mu$. Then the distance $\mathrm{d}s$ between two infinitesimally nearby points $x^\mu$ and $x^\mu + \mathrm{d}x^\mu$ is given by

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$$

where the $g_{\mu\nu}$ are the components of a symmetric covariant second-rank tensor.

We now decompose the metric into vierbeins (solder forms) or tetrads $e^a{}_\mu(x)$ as follows:

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$$
$$\eta^{ab} = g^{\mu\nu} e^a{}_\mu e^b{}_\nu.$$

Here $\eta_{ab}$ is a flat, usually Cartesian, metric such as the following:

Euclidean space:

$$\eta_{ab} = \delta_{ab}, \qquad a, b = 1, 2, 3, 4;$$

Minkowski space:

$$\eta_{ab} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad a, b = 0, 1, 2, 3.$$

$e^a{}_\mu$ is, in some sense, the *square root* of the metric.

Throughout this section, Greek indices $\mu, \nu, \ldots$ will be raised and lowered with $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$ and Latin indices $a, b, \ldots$ will be raised and lowered by $\eta_{ab}$ and $\eta^{ab}$. We define the inverse of $e^a{}_\mu$ by

$$E_a{}^\mu = \eta_{ab} g^{\mu\nu} e^b{}_\nu$$

which obeys

$$E_a{}^\mu e^b{}_\mu = \delta^b_a$$

$$\eta^{ab} E_a{}^\mu E_b{}^\nu = g^{\mu\nu} \qquad \text{etc.}$$

Thus $e^a{}_\mu$ and $E_a{}^\mu$ are used to interconvert Latin and Greek indices when necessary.

We therefore see that $e^a{}_\mu$ is the matrix which transforms the coordinate basis $\mathrm{d}x^\mu$ of $T^*_x(M)$ to an orthonormal basis of $T^*_x(M)$,

$$e^a = e^a{}_\mu \, \mathrm{d}x^\mu .$$

(Note that while the coordinate basis $\mathrm{d}x^\mu$ is always an exact differential, $e^a$ is not necessarily an exact 1-form.) Similarly, $E_a{}^\mu$ is a transformation from the basis $\partial/\partial x^\mu$ of $T_x(M)$ to the orthonormal basis of $T_x(M)$,

$$E_a = E_a{}^\mu \, \partial/\partial x^\mu .$$

(Note that $E_a$ and $E_b$ do not necessarily commute, while $\partial/\partial x^\mu$ and $\partial/\partial x^\nu$ do commute.)

We now introduce the *affine spin connection one-form* $\omega^a{}_b$ and define

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b \equiv T^a \equiv \tfrac{1}{2} T^a{}_{bc} e^b \wedge e^c . \tag{3.1}$$

This is called the *torsion* 2-form of the manifold. The *curvature* 2-form is defined as

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \tfrac{1}{2} R^a{}_{bcd} e^c \wedge e^d . \tag{3.2}$$

Equations (3.1) and (3.2) are called *Cartan's structure equations*.

*Consistency conditions*: Taking the exterior derivative of (3.1) we find

$$\mathrm{d}T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b . \tag{3.3}$$

Differentiating (3.2), we find the *Bianchi identities*:

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0. \tag{3.4}$$

We define the *covariant derivative* of a differential form $V^a{}_b$ of degree $p$ as

$$DV^a{}_b = dV^a{}_b + \omega^a{}_c \wedge V^c{}_b - (-1)^p V^a{}_c \wedge \omega^c{}_b. \tag{3.5}$$

The consistency condition (3.4) then reads

$$DR^a{}_b = 0.$$

*Gauge transformations*: Consider an orthogonal rotation of the orthonormal frame

$$e^a \to e'^a = \Phi^a{}_b e^b,$$

where

$$\eta_{ab}\Phi^a{}_c\Phi^b{}_d = \eta_{cd}.$$

Note that

$$(d\Phi)^a{}_b(\Phi^{-1})^b{}_c = -\Phi^a{}_b(d\Phi^{-1})^b{}_c.$$

Then we find

$$T'^a = de'^a + \omega'^a{}_b \wedge e'^b$$

where

$$T'^a = \Phi^a{}_b T^b$$

and the new connection is

$$\omega'^a{}_b = \Phi^a{}_c\omega^c{}_d(\Phi^{-1})^d{}_b + \Phi^a{}_c(d\Phi^{-1})^c{}_b.$$

The transformation law for the curvature 2-form is given by

$$R'^a{}_b = d\omega'^a{}_b + \omega'^a{}_c \wedge \omega'^c{}_b = \Phi^a{}_c R^c{}_d(\Phi^{-1})^d{}_b.$$

A similar exercise shows that under a change of frame, the "covariant derivative" (3.5) in fact transforms covariantly,

$$(DV)'^a{}_b = \Phi^a{}_c(DV)^c{}_d(\Phi^{-1})^d{}_b.$$

### *3.2. Relation to classical tensor calculus*

The Cartan differential form approach is, of course, equivalent to the conventional tensor formulation of Riemannian geometry. Here we summarize the relationships among various quantities appearing in the two approaches. Figure 3.1 is a caricature of classical tensor calculus.

*Volume and inner product*: The invariant oriented volume element in $n$ dimensions is

$$\mathrm{d}V = e^1 \wedge e^2 \wedge \cdots \wedge e^n = |g|^{1/2}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \tag{3.6}$$

where $g$ is the determinant of the metric tensor.

In curved space, the Hodge $*$ operation will involve the metric. If

$$\epsilon_{\mu_1\mu_2,\ldots,\mu_n} = \begin{cases} 0 & \text{any two indices repeated} \\ +1 & \text{even permutation} \\ -1 & \text{odd permutation,} \end{cases}$$

then

$$\epsilon_{\mu_1\mu_2,\ldots,\mu_n} = g\epsilon^{\mu_1\mu_2,\ldots,\mu_n}$$

and we define the standard tensor densities

$$E_{\mu_1,\ldots,\mu_n} = |g|^{1/2}\epsilon_{\mu_1,\ldots,\mu_n}$$

$$E^{\mu_1,\ldots,\mu_n} = |g|^{-1/2}\epsilon^{\mu_1,\ldots,\mu_n}.$$

The Hodge $*$ is then defined as the operation which correctly produces the curved space inner product. The inner product for 1-forms is defined using the Hodge $*$ as

$$\alpha \wedge *\beta = g^{\mu\nu}\alpha_\mu\beta_\nu |g|^{1/2}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{3.7}$$

Hodge $*$ is therefore defined as

$$*(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) = \frac{|g|^{1/2}}{(n-p)!}\epsilon^{\mu_1,\ldots,\mu_p}{}_{\mu_{p+1},\ldots,\mu_n}\, \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}.$$

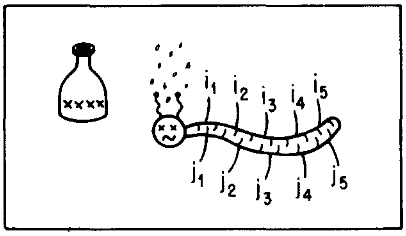


Fig. 3.1. Classical tensor calculus intoxicated by the plethora of indices.

Because of (3.6) we can rewrite this in the form

$$*(e^{a_1} \wedge \cdots \wedge e^{a_p}) = \frac{1}{(n-p)!} \epsilon^{a_1,\ldots,a_p}{}_{a_{p+1},\ldots,a_n} e^{a_{p+1}} \wedge \cdots \wedge e^{a_n}$$

where $\epsilon_{ab\ldots}$ has its indices raised and lowered by the flat metric $\eta_{ab}$. If we convert Greek to Latin indices using the vierbeins, e.g.,

$$\alpha = \alpha_\mu \, dx^\mu = \alpha_a e^a$$

we recover the inner product (3.7):

$$\alpha \wedge * \beta = \eta^{ab} \alpha_a \beta_b (e^1 \wedge e^2 \wedge \cdots \wedge e^n) = g^{\mu\nu} \alpha_\mu \beta_\nu (|g|^{1/2} \, d^n x).$$

The various tensors that we have defined with flat indices $a, b, \ldots$ are, of course, related to the tensor objects with curved indices by multiplication with $e^a{}_\mu$, $E^a{}_\mu$. The curvature two-form is first decomposed as

$$R^a{}_b = \tfrac{1}{2} R^a{}_{bcd} e^c \wedge e^d = \tfrac{1}{2} R^a{}_{b\mu\nu} \, dx^\mu \wedge dx^\nu,$$

and then the Riemann tensor is written

$$\text{Riemann tensor} = R^\alpha{}_{\beta\mu\nu} = E_a{}^\alpha e^b{}_\beta R^a{}_{b\mu\nu}.$$

Similarly, the torsion is

$$T^a = \tfrac{1}{2} T^a{}_{bc} e^b \wedge e^c = \tfrac{1}{2} T^a{}_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

$$T^\alpha{}_{\mu\nu} = E_a{}^\alpha T^a{}_{\mu\nu}.$$

*Levi-Civita connection*: The covariant derivative in the tensor formalism is defined using the Levi-Civita connection $\Gamma^\mu_{\alpha\beta}$, which physicists generally refer to as the Christoffel symbol. The Levi-Civita connection is determined by two conditions, the covariant constancy of the metric and the absence of torsion. In the tensor notation, these conditions are

$$\textit{metricity}: \quad g_{\mu\nu;\alpha} = \partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\alpha\nu} g_{\mu\lambda} = 0 \tag{3.8}$$

$$\textit{no torsion}: \quad T^\mu{}_{\alpha\beta} = \tfrac{1}{2}(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) = 0. \tag{3.9}$$

The Christoffel symbol is then uniquely determined in terms of the metric to be

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}).$$

In Cartan's method, the Levi-Civita *spin connection* is obtained by restricting the affine spin connection

$\omega_{ab}$ in an analogous way. The conditions (3.8) and (3.9) are replaced by

$$\textit{metricity:} \qquad \omega_{ab} = -\omega_{ba} \tag{3.10}$$

$$\textit{no torsion:} \qquad T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0. \tag{3.11}$$

$\omega^a{}_{b\mu}$ is then determined in terms of the vierbeins and inverse vierbeins and is related to $\Gamma^\mu_{\alpha\beta}$ by

$$\begin{aligned}\omega^a{}_{b\mu} &= e^a{}_\nu E_b{}^\nu{}_{;\mu} = e^a{}_\nu(\partial_\mu E_b{}^\nu + \Gamma^\nu_{\mu\lambda} E_b{}^\lambda)\\ &= -E_b{}^\nu e^a{}_{\nu;\mu} = -E_b{}^\nu(\partial_\mu e^a{}_\nu - \Gamma^\lambda_{\mu\nu} e^a{}_\lambda).\end{aligned}$$

From

$$0 = \delta^a{}_{b;\mu} = e^a{}_{\nu;\mu} g^{\nu\lambda} e_{b\lambda} + e^a{}_\nu g^{\nu\lambda}{}_{;\mu} e_{b\lambda} + e^a{}_\nu g^{\nu\lambda} e_{b\lambda;\mu}$$

we see that (3.10) is indeed a consequence of covariant constancy of the metric, (3.8). Similarly, if we write eq. (3.11) as

$$\begin{aligned}0 &= \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + E^{a\alpha} e_{b\alpha;\mu} e^b{}_\nu - E^{a\alpha} e_{b\alpha;\nu} e^b{}_\mu\\ &= \delta^a_b(\Gamma^\lambda_{\nu\mu} e^b{}_\lambda - \Gamma^\lambda_{\mu\nu} e^b{}_\lambda)\end{aligned}$$

we recognize the torsion-free condition (3.9).

The curvature can be extracted from Cartan's equations by computing

$$\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu} = e^a{}_\alpha E_b{}^\beta R^\alpha{}_{\beta\mu\nu}$$

where

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\beta}. \tag{3.12}$$

*Weyl tensor*: A useful object in $n$-dimensional geometry is the *Weyl tensor*, defined as

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} + \frac{\mathcal{R}}{(n-1)(n-2)}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}) - \frac{1}{(n-2)}(g_{\alpha\mu}\mathcal{R}_{\beta\nu} - g_{\alpha\nu}\mathcal{R}_{\beta\mu} - g_{\beta\mu}\mathcal{R}_{\alpha\nu} + g_{\beta\nu}\mathcal{R}_{\alpha\mu}),$$

where $\mathcal{R}_{\mu\nu} = R_{\mu\alpha\nu\beta} g^{\alpha\beta}$ and $\mathcal{R} = \mathcal{R}_{\mu\nu} g^{\mu\nu}$ are the Ricci tensor and the scalar curvature. The Weyl tensor is traceless in all pairs of indices.

*Examples 3.2*

We will for simplicity look only at Levi-Civita connections ($T^a = 0$, $\omega_{ab} = -\omega_{ba}$), so the vierbeins determine $\omega_{ab}$ uniquely.

1. *Coordinate transform of flat Cartesian coordinates to polar coordinates.* The Riemannian curvature remains zero, although the connection may be nontrivial.

a. *Two dimensions*; $\mathbf{R}^2$. $ds^2 = dx^2 + dy^2$; $e^1 = dx$, $e^2 = dy$. If $x = r\cos\theta$, $y = r\sin\theta$, then

$$\begin{pmatrix} e^r = dr \\ e^\theta = r\,d\theta \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Action of Hodge $*$: $\quad *(dx, dy) = (dy, -dx)$

$$*(dr, r\,d\theta) = (r\,d\theta, -dr).$$

Structure equations:

$$de^r - \omega \wedge e^\theta = 0 - \omega \wedge r\,d\theta = 0$$

$$de^\theta + \omega \wedge e^r = dr \wedge d\theta + \omega \wedge dr = 0.$$

Connection and curvature:

$$\omega = d\theta$$

$$R = d\omega = 0.$$

b. *Four dimensions*; $\mathbf{R}^4$. $ds^2 = dx^2 + dy^2 + dz^2 + dt^2$. We define polar coordinates by

$$x + iy = r\cos\frac{\theta}{2}\exp\frac{i}{2}(\psi + \varphi)$$

$$z + it = r\sin\frac{\theta}{2}\exp\frac{i}{2}(\psi - \varphi)$$

$$0 \le \theta < \pi, \qquad 0 \le \varphi < 2\pi, \qquad 0 \le \psi < 4\pi$$

$$\begin{pmatrix} e^0 = dr \\ e^1 = r\sigma_x \\ e^2 = r\sigma_y \\ e^3 = r\sigma_z \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x & y & z & t \\ -t & -z & y & x \\ z & -t & -x & y \\ -y & x & -t & z \end{pmatrix}\begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix}$$

$\sigma_x$, $\sigma_y$ and $\sigma_z$ obey the relation $d\sigma_x = 2\sigma_y \wedge \sigma_z$, cyclic. The connections and curvatures are given by

$$\omega^1{}_0 = \omega^2{}_3 = \sigma_x, \quad \omega^2{}_0 = \omega^3{}_1 = \sigma_y, \quad \omega^3{}_0 = \omega^1{}_2 = \sigma_z$$

$$\begin{aligned} R^0{}_1 &= d\omega^0{}_1 + \omega^0{}_2 \wedge \omega^2{}_1 + \omega^0{}_3 \wedge \omega^3{}_1 \\ &= -2\sigma_y \wedge \sigma_z + (-\sigma_y) \wedge (-\sigma_z) + (-\sigma_z) \wedge \sigma_y = 0, \qquad \text{etc.} \end{aligned}$$

*Remark*: $\sigma_x$, $\sigma_y$ and $\sigma_z$ are the left-invariant 1-forms on the manifold of the group $SU(2) = S^3$ and will appear also in our treatment of the geometry of Lie groups.

2. *Two-sphere.* The metric on $S^2$ is easily found by setting $r$ = constant in the flat $\mathbf{R}^3$ metric:

$$ds^2 = r^2\,d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 = (e^1)^2 + (e^2)^2.$$

We choose

$$e^1 = r\,d\theta, \qquad e^2 = r \sin\theta\, d\varphi$$

so the structure equations

$$de^1 = 0 = -\omega^1{}_2 \wedge e^2$$
$$de^2 = r\cos\theta\, d\theta \wedge d\varphi = -\omega^2{}_1 \wedge e^1$$

give the connection

$$\omega^1{}_2 = -\cos\theta\, d\varphi$$

and the curvature

$$R^1{}_2 = R^1{}_{212} e^1 \wedge e^2$$
$$= d\omega^1{}_2 = \frac{1}{r^2} e^1 \wedge e^2.$$

The Gaussian curvature is thus $K = R_{abab} = 2/r^2$, showing that $S^2$ has constant positive curvature.

3. *4-Sphere with polar coordinates.* The de Sitter metric on $S^4$ with radius $R$ is

$$ds^2 = (dr^2 + r^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2))/(1 + (r/2R)^2)^2.$$

$e^a$ with $a = 0, 1, 2, 3$ is defined by

$$(1 + (r/2R)^2)e^a = \{dr, r\sigma_x, r\sigma_y, r\sigma_z\}.$$

From the structure equations, we find

$$\omega_{i0} = (1 - (r/2R)^2)e^i/r = \sigma_i \frac{1 - (r/2R)^2}{1 + (r/2R)^2}$$

$$\tfrac{1}{2}\epsilon_{ijk}\omega_{jk} = (1 + (r/2R)^2)e^i/r = \sigma_i$$

$$R^{ab} = \frac{1}{R^2} e^a \wedge e^b.$$

The Weyl tensor vanishes identically.

### 3.3. Einstein's equations and self-dual manifolds

Defining the Ricci tensor and scalar curvature in 4 dimensions as

$$\mathcal{R}^{\mu}{}_{\nu} = g^{\alpha\beta}R^{\mu}{}_{\alpha\nu\beta}, \qquad \mathcal{R} = g_{\mu\nu}\mathcal{R}^{\mu\nu}, \tag{3.13}$$

we write Einstein's equations with cosmological term $\Lambda$ as

$$\mathcal{R}^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}\mathcal{R} = T^{\mu\nu} - \Lambda g^{\mu\nu}. \tag{3.14}$$

If the matter energy-momentum tensor $T^{\mu\nu}$ and $\Lambda$ vanish, Einstein's equations imply the vanishing of the Ricci tensor, which we write in the flat vierbein basis as

$$0 = \mathcal{R}^{a}{}_{b} = e^{a}{}_{\mu}E_{b}{}^{\nu}\mathcal{R}^{\mu}{}_{\nu} = R^{a}{}_{cbd}\eta^{cd}. \tag{3.15}$$

We note that in Einstein's theory we always work with the torsion-free Levi–Civita connection, so the consistency condition (3.3) becomes the *cyclic identity*:

$$R^{a}{}_{b} \wedge e^{b} = 0 \rightarrow \epsilon_{abcd}R^{e}{}_{bcd} = 0. \tag{3.16}$$

Now let us define the dual of the Riemann tensor as

$$\tilde{R}_{abcd} = \tfrac{1}{2}\epsilon_{abmn}R_{mncd}. \tag{3.17}$$

Suppose the Riemann tensor is (anti)-self-dual,

$$R_{abcd} = \pm\tilde{R}_{abcd}.$$

Then the cyclic identity implies Einstein's empty space equations,

$$\begin{aligned} 0 = \epsilon_{abcd}R_{ebcd} &= \pm\tfrac{1}{2}\epsilon_{abcd}\epsilon_{ebmn}R_{mncd} \\ &= \pm(\mathcal{R}\delta_{ae} - 2\mathcal{R}_{ae}). \end{aligned}$$

*Remark*: A similar argument can be used to show that Einstein's empty space equations may be written as

$$\tilde{R}^{a}{}_{b} \wedge e^{b} = 0, \qquad (\tilde{R}_{ab} = \tfrac{1}{2}\epsilon_{abcd}R^{cd} = \tfrac{1}{2}\tilde{R}_{abcd}e^{c} \wedge e^{d}).$$

The equivalence of the cyclic identity and Einstein's equations for self-dual $R^{a}{}_{b}$ is then obvious.

From the relation between $R_{ab}$ and $\omega_{ab}$,

$$R_{01} = d\omega_{01} + \omega_{02} \wedge \omega_{21} + \omega_{03} \wedge \omega_{31}$$

$$R_{23} = d\omega_{23} + \omega_{20} \wedge \omega_{03} + \omega_{21} \wedge \omega_{13} \qquad \text{etc.,}$$

we notice that $R_{ab}$ is self-dual, $R_{ab} = \pm\tilde{R}_{ab}$, if $\omega_{ab}$ is self-dual,

$$\omega_{ab} = \pm\tilde{\omega}_{ab}.$$

Therefore one way to generate a solution of Einstein's equations is to find a metric with self-dual connection.

*Remark*: Suppose $R_{ab} = \pm\tilde{R}_{ab}$ but $\omega_{ab} \neq \pm\tilde{\omega}_{ab}$. Then we decompose $\omega_{ab}$ into self-dual and anti-self-dual parts. Using an $O(4)$ gauge transformation one can always remove the piece of $\omega_{ab}$ with the wrong duality. The only change in $R_{ab}$ under the gauge transformation is a rotation by an orthogonal matrix which preserves its duality properties. Thus any self-dual $R_{ab}$ can be considered to come from a self-dual connection $\omega_{ab}$ if we work in an appropriate "self-dual gauge".

*Self-dual and conformally self-dual structures in 4 dimensions*

In the case of four dimensions some simplification occurs since the dual of the curvature 2-form is also a 2-form. Let us define self-dual and anti-self-dual bases for $\Lambda^2$ using the vierbein one-forms $e^a$:

$$\text{basis of } \Lambda^2_\pm = \begin{cases} \lambda^1_\pm = e^0 \wedge e^1 \pm e^2 \wedge e^3 \\ \lambda^2_\pm = e^0 \wedge e^2 \pm e^3 \wedge e^1, \\ \lambda^3_\pm = e^0 \wedge e^3 \pm e^1 \wedge e^2 \end{cases} \qquad *\lambda^i_\pm = \pm\lambda^i_\pm.$$

The curvature tensor can then be viewed as a $6\times 6$ matrix $R$ mapping $\Lambda^2_\pm$ into $\Lambda^2_\pm$ (see, e.g., Atiyah, Hitchin and Singer [1978]),

$$R\Lambda^2 = \left(\begin{array}{c|c} A & C^+ \\ \hline C^- & B \end{array}\right) \begin{pmatrix} \lambda^i_+ \\ \lambda^i_- \end{pmatrix},$$

where $A$ is the $3\times 3$ matrix whose first column is

$$A_{11} = R_{0101} + R_{0123} + R_{2301} + R_{2323}$$
$$A_{21} = R_{0201} + R_{0223} + R_{3101} + R_{3123}$$
$$A_{31} = R_{0301} + R_{0323} + R_{1201} + R_{1223}.$$

That is,

$$A_{ij} = +(R_{0i0j} + \tfrac{1}{2}\epsilon_{jkl}R_{0ikl}) + \tfrac{1}{2}\epsilon_{imn}(R_{mn0j} + \epsilon_{jkl}R_{mnkl})$$

and $B$ and $C^\pm$ are defined by changing the four signs in the definition of $A$ as follows:

$$A \sim (+,+,+,+)$$
$$B \sim (+,-,-,-)$$
$$C^+ \sim (+,-,+,-)$$
$$C^- \sim (+,+,-,+).$$

The Hodge $*$ duality transformation acts on $R$ from the left as the matrix

$$* = \begin{pmatrix} I_3 & 0 \\ 0 & -I_3 \end{pmatrix}.$$

Now if we let

$$S = \operatorname{Tr} A = \operatorname{Tr} B$$

and subtract the trace, we find

$$W = R - \tfrac{5}{6} I_6 = \begin{pmatrix} W_+ & C^+ \\ C^- & W_- \end{pmatrix}$$

where

$C^+ =$ tracefree Ricci tensor
$W_+ + W_- =$ tracefree Weyl tensor.

The interesting spaces can then be categorized as

Einstein: $C^\pm = 0$ $\quad(\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu})$
Ricci flat: $C^\pm = 0, \quad S = 0$ $\quad(\mathcal{R}_{\mu\nu} = 0)$
Conformally flat: $W_\pm = 0$
Self-dual: $W_- = 0, \quad C^\pm = 0$
Anti-self-dual: $W_+ = 0, \quad C^\pm = 0$
Conformally self-dual: $W_- = 0$
Conformally anti-self-dual: $W_+ = 0$.

*Beware*: What physicists refer to as *self-dual metrics* are those which have self-dual Riemann tensor and which mathematicians may call "half-flat". The spaces which a physicist describes as having a *self-dual Weyl tensor* or as *conformally self-dual* may be called simply "self-dual" by mathematicians.

*Examples 3.3*

1. *Schwarzschild metric*. The best-known solution to the empty space Einstein equations is the Schwarzschild "black hole" metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \frac{1}{1 - 2M/R}\,\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$$

$$0 \le \theta < \pi, \quad 0 \le \varphi < 2\pi.$$

Choosing the vierbeins

$$e^0 = \left(1 - \frac{2M}{R}\right)^{1/2}\mathrm{d}t, \quad e^1 = \left(1 - \frac{2M}{R}\right)^{-1/2}\mathrm{d}R, \quad e^2 = R\,\mathrm{d}\theta, \quad e^3 = R\sin\theta\,\mathrm{d}\varphi$$

and raising and lowering Latin indices with $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$, we find the connections

$$\omega^0{}_1 = \frac{M}{R^2}\,\mathrm{d}t \qquad \omega^2{}_3 = -\cos\theta\,\mathrm{d}\varphi$$

$$\omega^0{}_2 = 0 \qquad \omega^3{}_1 = (1-2M/R)^{1/2}\sin\theta\,\mathrm{d}\varphi$$

$$\omega^0{}_3 = 0 \qquad \omega^1{}_2 = -(1-2M/R)^{1/2}\,\mathrm{d}\theta.$$

Then the curvature 2-forms are

$$R^0{}_1 = \frac{2M}{R^3}\,e^0\wedge e^1 \qquad R^2{}_3 = \frac{2M}{R^3}\,e^2\wedge e^3$$

$$R^0{}_2 = \frac{-M}{R^3}\,e^0\wedge e^2 \qquad R^3{}_1 = \frac{-M}{R^3}\,e^3\wedge e^1$$

$$R^0{}_3 = \frac{-M}{R^3}\,e^0\wedge e^3 \qquad R^1{}_2 = -\frac{M}{R^3}\,e^1\wedge e^2,$$

and we easily verify that the Schwarzschild metric satisfies the Einstein equations outside the singularity at $R=0$.

2. *Self-dual Taub–NUT metric.* One example of a metric which satisfies the Euclidean Einstein equations with self-dual Riemann tensor is the self-dual Taub–NUT metric (Hawking [1977]):

$$\mathrm{d}s^2 = \frac{1}{4}\frac{r+m}{r-m}\,\mathrm{d}r^2 + (r^2-m^2)(\sigma_x{}^2+\sigma_y{}^2) + 4m^2\frac{r-m}{r+m}\,\sigma_z{}^2$$

where $\sigma_x$, $\sigma_y$ and $\sigma_z$ are defined in example 3.2.1 and $m$ is an arbitrary constant. We choose

$$e^a = \left\{\frac{1}{2}\left(\frac{r+m}{r-m}\right)^{1/2}\mathrm{d}r,\quad (r^2-m^2)^{1/2}\sigma_x,\quad (r^2-m^2)^{1/2}\sigma_y,\quad 2m\left(\frac{r-m}{r+m}\right)^{1/2}\sigma_z\right\}$$

and find the connections

$$\omega^0{}_1 = \frac{2r}{r+m}\,\sigma_x \qquad \omega^2{}_3 = \frac{2m}{r+m}\,\sigma_x$$

$$\omega^0{}_2 = \frac{2r}{r+m}\,\sigma_y \qquad \omega^3{}_1 = \frac{2m}{r+m}\,\sigma_y$$

$$\omega^0{}_3 = \frac{4m^2}{(r+m)^2}\,\sigma_z \qquad \omega^1{}_2 = \left(2-\frac{4m^2}{(r+m)^2}\right)\sigma_z,$$

and curvatures

$$R^0{}_1 = -R^2{}_3 = \frac{-m}{(r+m)^3}(e^0 \wedge e^1 - e^2 \wedge e^3)$$

$$R^0{}_2 = -R^3{}_1 = \frac{-m}{(r+m)^3}(e^0 \wedge e^2 - e^3 \wedge e^1)$$

$$R^0{}_3 = -R^1{}_2 = \frac{2m}{(r+m)^3}(e^0 \wedge e^3 - e^1 \wedge e^2).$$

3. *Metric of Eguchi and Hanson* [1978]. Another solution of the Euclidean Einstein equations with self-dual curvature is given by

$$ds^2 = \frac{dr^2}{1-(a/r)^4} + r^2(\sigma_x{}^2 + \sigma_y{}^2 + (1-(a/r)^4)\sigma_z{}^2)$$

where $a$ is an arbitrary constant. Choosing the vierbeins

$$e^a = \{(1-(a/r)^4)^{-1/2}\, dr, r\sigma_x, r\sigma_y, r(1-(a/r)^4)^{1/2}\sigma_z\}$$

we find self-dual connections

$$\omega^0{}_1 = -\omega^2{}_3 = -(1-(a/r)^4)^{1/2}\sigma_x$$

$$\omega^0{}_2 = -\omega^3{}_1 = -(1-(a/r)^4)^{1/2}\sigma_y$$

$$\omega^0{}_3 = -\omega^1{}_2 = -(1+(a/r)^4)\sigma_z,$$

and curvatures

$$R^0{}_1 = -R^2{}_3 = \frac{2a^4}{r^6}(-e^0 \wedge e^1 + e^2 \wedge e^3)$$

$$R^0{}_2 = -R^3{}_1 = \frac{2a^4}{r^6}(-e^0 \wedge e^2 + e^3 \wedge e^1)$$

$$R^0{}_3 = -R^1{}_2 = -\frac{4a^4}{r^6}(-e^0 \wedge e^3 + e^1 \wedge e^2).$$

The apparent singularities in the metric at $r = a$ can be removed by choosing the angular coordinate ranges

$$0 \le \theta < \pi, \quad 0 \le \varphi < 2\pi, \quad 0 \le \psi < 2\pi.$$

Thus the boundary at $\infty$ becomes $P_3(\mathbf{R})$. (If $0 \le \psi < 4\pi$, it would have been $S^3$.) See section 10 for further discussion.

### 3.4. Complex manifolds

$M$ is a complex manifold of dimension $n$ if we can find complex coordinates with holomorphic transition functions in a real manifold with real dimension $2n$. Let $z_k = x_k + \mathrm{i}y_k$ be local complex coordinates; the conjugate coordinates are $\bar{z}_k = x_k - \mathrm{i}y_k$. We define:

$$\partial/\partial z_k = \tfrac{1}{2}(\partial/\partial x_k - \mathrm{i}\,\partial/\partial y_k) \qquad \partial/\partial \bar{z}_k = \tfrac{1}{2}(\partial/\partial x_k + \mathrm{i}\,\partial/\partial y_k)$$

$$\mathrm{d}z_k = \mathrm{d}x_k + \mathrm{i}\,\mathrm{d}y_k \qquad \mathrm{d}\bar{z}_k = \mathrm{d}x_k - \mathrm{i}\,\mathrm{d}y_k.$$

Then it is easily checked that

$$\mathrm{d}f = \sum_k (\partial f/\partial z_k)\,\mathrm{d}z_k + \sum_k (\partial f/\partial \bar{z}_k)\,\mathrm{d}\bar{z}_k = \partial f + \bar{\partial} f \tag{3.18}$$

where

$$\partial f = \sum_k (\partial f/\partial z_k)\,\mathrm{d}z_k$$

$$\bar{\partial} f = \sum_k (\partial f/\partial \bar{z}_k)\,\mathrm{d}\bar{z}_k.$$

If $f(z)$ is a holomorphic function of a single variable,

$$\bar{\partial} f = (\partial f/\partial \bar{z})\,\mathrm{d}\bar{z} = 0.$$

In general, a function $f$ on $\mathbb{C}^n$ is holomorphic if $\partial f/\partial \bar{z}_k = 0$ for $k = 1, \ldots, n$ or equivalently if $\bar{\partial} f = 0$.

If $w_k$ is another set of local complex coordinates, then

$$\mathrm{d}w_k = \partial w_k + \bar{\partial} w_k = \partial w_k = \sum_j \frac{\partial w_k}{\partial z_j}\,\mathrm{d}z_j$$

$$\mathrm{d}\bar{w}_k = \sum_j \frac{\partial \bar{w}_k}{\partial \bar{z}_j}\,\mathrm{d}\bar{z}_j.$$

We define the complex tangent and cotangent spaces in terms of their local bases as follows:

$$T_c(M) = \{\partial/\partial z_j\} \qquad \bar{T}_c(M) = \{\partial/\partial \bar{z}_j\}$$

$$T_c^*(M) = \{\mathrm{d}z_j\} \qquad \bar{T}_c^*(M) = \{\mathrm{d}\bar{z}_j\}.$$

In fact, these spaces are invariantly defined independent of the particular local complex coordinates which are chosen. We note that $T(M) \otimes \mathbb{C} = T_c(M) \oplus \bar{T}_c(M)$ and $T^*(M) \otimes \mathbb{C} = T_c^*(M) \oplus \bar{T}_c^*(M)$.

We can define *complex* exterior forms $\Lambda^{p,q}$ which have bases containing $p$ factors of $\mathrm{d}z_k$ and $q$

factors of $d\bar{z}_k$. The operators $\partial$ and $\bar{\partial}$ act as

$$\partial: C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p+1,q}), \qquad \bar{\partial}: C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1}).$$

Clearly we can define $d\omega = \partial\omega + \bar{\partial}\omega$ for any form $\omega \in \Lambda^{p,q}$. These operators satisfy the relations:

$$\partial\partial\omega = 0, \qquad \bar{\partial}\bar{\partial}\omega = 0, \qquad \partial\bar{\partial}\omega = -\bar{\partial}\partial\omega. \tag{3.19}$$

We define the conjugate operators with respect to the inner product by

$$\delta = (-1)^{np+n+1} * d * \equiv d^* = \partial^* + \bar{\partial}^*. \tag{3.20}$$

There are then three kinds of Laplacians:

$$\Delta = (d + \delta)^2$$

$$\Delta' = 2(\partial + \partial^*)^2$$

$$\Delta'' = 2(\bar{\partial} + \bar{\partial}^*)^2.$$

*Almost complex structure*: A manifold $M$ has an almost complex structure if there exists a linear map $J$ from $T(M)$ to $T(M)$ such that $J^2 = -1$. For example, take a Cartesian coordinate system $(x, y)$ on $\mathbf{R}^2$ and define $J$ by the $2 \times 2$ matrix

$$J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$J^2\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}.$$

Clearly $J$ is equivalent to multiplication by $i = \sqrt{-1}$,

$$i(x + iy) = ix - y$$

$$i^2(x + iy) = -(x + iy).$$

As an operator, $J$ has eigenvalues $\pm i$. We note that, obviously, no $J$ can be found on odd-dimensional manifolds.

*Kähler manifolds*: Let us consider a Hermitian metric on $M$ given by

$$ds^2 = g_{a\bar{b}}\, dz^a\, d\bar{z}^b, \tag{3.21}$$

where $g_{a\bar{b}}$ is a Hermitian matrix. We define the Kähler form

$$K = \frac{i}{2} g_{a\bar{b}}\, dz^a \wedge d\bar{z}^b.$$

Then

$$\bar{K} = -\frac{i}{2} \bar{g}_{a\bar{b}}\, d\bar{z}^a \wedge dz^b = \frac{i}{2} g_{b\bar{a}}\, dz^b \wedge d\bar{z}^a = K$$

is a real 2-form.

A metric is said to be a Kähler metric if $dK = 0$, i.e., the Kähler form is closed. $M$ is a Kähler manifold if it admits a Kähler metric. Any Riemann surface (real dimension 2) is automatically Kähler since $dK = 0$ for any 2-form. There are, however, complex manifolds of real dimension 4 which admit no Kähler metric.

If $dK = 0$, then, in fact, $K$ is *harmonic* and

$$dK = \delta K = 0.$$

For a Kähler metric, all the Laplacians are equal; $\Delta = \Delta' = \Delta''$. A Kähler manifold is *Hodge* if there exists a holomorphic line bundle whose first Chern form is the Kähler form of the manifold. Hodge manifolds are given by algebraic equations in $P_n(\mathbb{C})$ for some large $n$.

If a metric is a Kähler metric, then the set of the forms

$$K, K \wedge K, \ldots, \underbrace{K \wedge K \wedge \cdots \wedge K}_{(n \text{ times})}$$

are all non-zero and harmonic. They define cohomology classes in $H^p(M; \mathbb{R})$ for $p = 2, \ldots, 2n$. (If the metric is Hodge, then these are all integral classes.) $P_n(\mathbb{C})$ is a Kähler manifold and all of its cohomology classes are generated by scalar multiples of the set of forms given above.

If $M$ is any complex manifold, it has a natural orientation defined by requiring that

$$\int_M K \wedge \cdots \wedge K > 0.$$

*Examples 3.4*

1. *Flat two space.* Taking $z = x + iy$, we choose the flat metric

$$ds^2 = dx^2 + dy^2 = (dx + i\,dy)(dx - i\,dy) = dz\, d\bar{z}.$$

Hence the Kähler form is

$$K = \frac{i}{2}(dx + i\,dy) \wedge (dx - i\,dy) = dx \wedge dy$$

which is obviously closed and coclosed.

2. *Two sphere,* $S^2 = P_1(\mathbb{C})$. We convert the standard metric on $S^2$ with radius $\frac{1}{2}$ into complex

coordinates:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1 + x^2 + y^2)^2} = \frac{\mathrm{d}z\, \mathrm{d}\bar{z}}{(1 + z\bar{z})^2}.$$

The Kähler form is then

$$K = \frac{\mathrm{i}}{2} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + z\bar{z})^2} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{(1 + x^2 + y^2)^2}.$$

Choosing vierbeins $e^1 = \mathrm{d}x/(1 + x^2 + y^2)$, $e^2 = \mathrm{d}y/(1 + x^2 + y^2)$ we find

$$K = e^1 \wedge e^2$$

$$* K = 1$$

so $K$ is harmonic. We note that

$$K = \frac{\mathrm{i}}{2} \partial \bar{\partial} \ln(1 + z\bar{z}).$$

3. *Fubini–Study metric on* $P_n(\mathbb{C})$. The Fubini–Study metric on $P_n(\mathbb{C})$ is given by the Kähler form

$$K = \frac{\mathrm{i}}{2} \partial \bar{\partial} \ln\Big(1 + \sum_{\alpha=1}^{n} z^\alpha \bar{z}^\alpha\Big)$$

$$= \frac{\mathrm{i}}{2} \frac{\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta}{(1 + \Sigma\, z^\gamma \bar{z}^\gamma)^2} [\delta_{\alpha\beta}(1 + \Sigma\, z^\gamma \bar{z}^\gamma) - \bar{z}^\alpha z^\beta].$$

For $P_2(\mathbb{C})$, we find

$$\mathrm{d}s^2 = \frac{\Sigma\, \mathrm{d}z^\alpha\, \mathrm{d}\bar{z}^\alpha}{1 + \Sigma\, z^\gamma \bar{z}^\gamma} - \frac{\Sigma\, \bar{z}^\alpha\, \mathrm{d}z^\alpha\, \bar{z}^\beta\, \mathrm{d}z^\beta}{(1 + \Sigma\, z^\gamma \bar{z}^\gamma)^2}$$

$$= \frac{\mathrm{d}r^2 + r^2({\sigma_x}^2 + {\sigma_y}^2 + {\sigma_z}^2)}{1 + r^2} - \frac{r^2(\mathrm{d}r^2 + r^2{\sigma_z}^2)}{(1 + r^2)^2}$$

$$= \frac{\mathrm{d}r^2 + r^2{\sigma_z}^2}{(1 + r^2)^2} + \frac{r^2({\sigma_x}^2 + {\sigma_y}^2)}{1 + r^2}.$$

Choosing the vierbein one-forms

$$e^0 = \mathrm{d}r/(1 + r^2), \qquad e^1 = r\sigma_x/(1 + r^2)^{1/2}$$

$$e^2 = r\sigma_y/(1 + r^2)^{1/2}, \qquad e^3 = r\sigma_z/(1 + r^2)$$

we find the connection one-forms

$$\omega^0{}_1 = -\frac{1}{r} e^1 \qquad \omega^2{}_3 = \frac{1}{r} e^1$$

$$\omega^0{}_2 = -\frac{1}{r} e^2 \qquad \omega^3{}_1 = \frac{1}{r} e^2$$

$$\omega^0{}_3 = \frac{r^2-1}{r} e^3 \qquad \omega^1{}_2 = \frac{1+2r^2}{r} e^3.$$

The curvatures are constant:

$$R_{01} = e^0 \wedge e^1 - e^2 \wedge e^3 \qquad R_{23} = -e^0 \wedge e^1 + e^2 \wedge e^3$$

$$R_{02} = e^0 \wedge e^2 - e^3 \wedge e^1 \qquad R_{31} = -e^0 \wedge e^2 + e^3 \wedge e^1$$

$$R_{03} = 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \qquad R_{12} = 2e^0 \wedge e^3 + 4e^1 \wedge e^2.$$

We find that the Ricci tensor is

$$\mathcal{R}_{ab} = 6\delta_{ab}$$

so Einstein's equation

$$\mathcal{R}_{ab} - \tfrac{1}{2}\delta_{ab}\mathcal{R} = -\Lambda\delta_{ab}$$

is solved with the cosmological constant,

$$\Lambda = +6.$$

The Weyl tensor for the Fubini–Study metric is

$$W_{abcd} = R_{abcd} - 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}),$$

so the two-forms $W_{ab} = \frac{1}{2}W_{abcd}\, e^c \wedge e^d$ are *self-dual*:

$$W_{01} = W_{23} = -e^0 \wedge e^1 - e^2 \wedge e^3$$

$$W_{02} = W_{31} = -e^0 \wedge e^2 - e^3 \wedge e^1$$

$$W_{03} = W_{12} = 2e^0 \wedge e^3 + 2e^1 \wedge e^2.$$

More geometrical properties of $P_2(\mathbb{C})$ will be explored later.

4. $S^1 \times S^{2n-1}$. Let $c$ be a complex number with $|c|>1$. On $\mathbb{C}^n - \{0\}$ we introduce the equivalence relation $z \simeq z'$ if $z = c^k z'$ for some integer $k$. The resulting quotient manifold will be a complex manifold and will be topologically equivalent to $S^1 \times S^{2n-1}$. We suppose $n \geq 2$, then $H^*(S^1; \mathbb{R}) \otimes H^*(S^{2n-1}; \mathbb{R}) = H^*(S^1 \times S^{2n-1}; \mathbb{R})$ implies that $H^2(S^1 \times S^{2n-1}; \mathbb{R}) = 0$, so this complex manifold does not admit any Kähler metric. It is worth noting that different values of the constant $c$ yield inequivalent complex manifolds (although the underlying topological type is unchanged).

5. *Metrics on the group manifolds of* $U(n)$, $\mathrm{SU}(n)$, $O(n)$, $\mathrm{SO}(n)$. Let $g(t)$, $h(t)$: $[0, \varepsilon) \to G$ be two curves with $g(0) = h(0) = g_0$. We define a metric on $G$ by defining the inner product of the two tangent vectors $(g'(0), h'(0)) = -\mathrm{Tr}(g_0^{-1} g'(0) g_0^{-1} h'(0))$. It is easily verified that this is a positive definite metric which is both left and right invariant on these groups; i.e., multiplication on either the right or the left is an isometry which preserves this metric. Up to a scaling factor, this is the *Killing metric*.

## 4. Geometry of fiber bundles

Many important concepts in physics can be interpreted in terms of the geometry of fiber bundles. Maxwell's theory of electromagnetism and Yang–Mills theories are essentially theories of connections on principal bundles with a given gauge group $G$ as the fiber. Einstein's theory of gravitation deals with the Levi–Civita connection on the frame bundle of the space-time manifold.

In this section, we shall define the notion of a fiber bundle and study the geometrical properties of a variety of interesting bundles. We begin for simplicity with vector bundles and then go on to treat principal bundles.

### *4.1. Fiber bundles*

We begin our treatment of fiber bundles with an informal discussion of the basic concepts. We shall then outline a more formal mathematical approach. Suppose we are given some manifold $M$ which we shall call the *base manifold* as well as another manifold $F$ which we shall call the *fiber*. A *fiber bundle E* over $M$ with fiber $F$ is a manifold which is locally a direct product of $M$ and $F$. That is, if $M$ is covered by a set of local coordinate neighborhoods $\{U_i\}$, then the bundle $E$ is topologically described in each neighborhood $U_i$ by the product manifold

$$U_i \times F$$

as shown in fig. 4.1.

A little thought shows that the local direct-product structure still leaves a great deal of information about the *global* topology of $E$ undetermined. To completely specify the bundle $E$, we must provide a set of *transition functions* $\{\Phi_{ij}\}$ which tell how the fiber manifolds match up in the overlap between two neighborhoods, $U_i \cap U_j$. We write $\Phi_{ij}$ as a mapping

$$\Phi_{ij}: F|_{U_i} \to F|_{U_j} \text{ in } U_i \cap U_j, \tag{4.1}$$

as illustrated in fig. 4.2. Thus, although the local topology of the bundle is trivial, the global topology

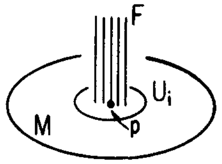


Fig. 4.1. Local direct-product structure of a fiber bundle. A vertical line represents a fiber associated to each point, such as $p$, in $U_i$.

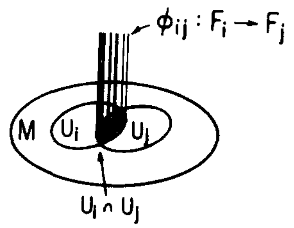


Fig. 4.2. The transition function $\Phi_{ij}$ defines the mapping of the coordinates of the fibers over $U_i$ to those over $U_j$ in the overlap region $U_i \cap U_j$.

determined by the transition functions may be quite complicated due to the relative twisting of neighboring fibers. For this reason, fiber bundles are sometimes called *twisted products* in the mathematical literature.

Example: *The Möbius strip.* A simple non-trivial fiber bundle is the Möbius strip, which we may construct as follows: Let the base manifold $M$ be the circle $S^1$ parametrized by the angle $\theta$. We cover $S^1$ by two semicircular neighborhoods $U_\pm$ as shown in fig. 4.3a,

$$U_+ = \{\theta: -\epsilon < \theta < \pi + \epsilon\}, \qquad U_- = \{\theta: \pi - \epsilon < \theta < 2\pi + \epsilon = 0 + \epsilon\}.$$

We take the fiber $F$ to be an interval in the real line with coordinates $t \in [-1, 1]$. The bundle then consists of the two local pieces shown in fig. 4.3b,

$$U_+ \times F \text{ with coordinates } (\theta, t_+), \qquad U_- \times F \text{ with coordinates } (\theta, t_-),$$

and the transition functions relating $t_+$ to $t_-$ in $U_+ \cap U_-$. This overlap consists of two regions I and II illustrated in fig. 4.3c. We choose the transition functions to be:

$$t_+ = t_- \text{ in region I} = \{\theta: -\epsilon < \theta < \epsilon\}$$

$$t_+ = -t_- \text{ in region II} = \{\theta: \pi - \epsilon < \theta < \pi + \epsilon\}.$$

Identifying $t$ with $-t$ in region II twists the bundle and gives it the non-trivial global topology of the Möbius strip, as shown in fig. 4.3d.

*Trivial bundles*: If all the transition functions can be taken to be the identity, the global topology of the bundle is that of the direct product

$$E = M \times F.$$

Such bundles are called *trivial fiber bundles* or sometimes simply trivial bundles. For example, if we had set $t_+ = t_-$ in both regions I and II in the example above, we would have found a trivial bundle equal to the cylinder $S^1 \times [-1, 1]$.

It is a theorem that *any fiber bundle over a contractible base space is trivial.* Thus, for example, all

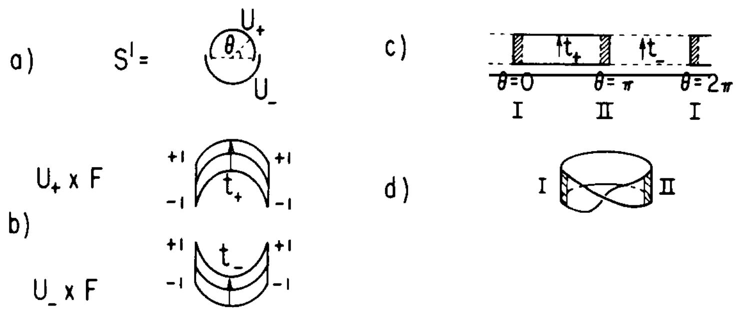


Fig. 4.3. Möbius strip. (a) The base space $S^1$ is covered by two neighborhoods $U_\pm$ which overlap at $\theta \sim 0$ and $\theta \sim \pi$. (b) Pieces of the bundle formed by taking the direct product of $U_\pm$ with the fiber $[-1, +1]$ having coordinates $t_\pm$. (c) The overlapping regions I and II of $U_+ \times F$ and $U_- \times F$. (d) A non-trivial bundle, the Möbius strip, is obtained by setting $t_+ = t_-$ in region I and $t_+ = -t_-$ in region II.

fiber bundles over a coordinate ball in $\mathbb{R}^n$ or over the sphere $S^n$ minus a single point are necessarily trivial. Non-trivial fiber bundles can only be constructed when the global topology of the base space is non-trivial.

*Sections*: A *cross section* or simply a *section* $s$ of a fiber bundle $E$ is a rule which assigns a preferred point $s(x)$ on each fiber to each point $x$ of the base manifold $M$, as illustrated in fig. 4.4. A *local section* is a section which is only defined over a subset of $M$. We can always define local sections in the local patches $U_i \times F$ from which the bundle is constructed. These sections are simply functions from $U_i$ into $F$. The existence of *global* sections depends on the global geometry of the bundle $E$. There exist fiber bundles which have no global sections.

*Formal approach to fiber bundles*

A more sophisticated description of fiber bundles requires us to define a *projection* $\pi$ which maps the fiber bundle $E$ onto the base space $M$ by shrinking each fiber to a point. If $x \in M$, $\pi^{-1}(x)$ is the *fiber over* $x$; $\pi^{-1}(x)$ acts like a flashlight shining through a hole at $x$ to produce a "light ray" equal to the fiber. We sometimes denote the fiber $F$ over $x$ as $F_x$.

We let $\pi^{-1}(U_i)$ denote the subset of $E$ which projects down to the neighborhood $U_i$ in $M$. By assumption, there exists an isomorphism which maps $U_i \times F$ to $\pi^{-1}(U_i)$. This amounts to an assignment of local coordinates in the bundle often referred to as a *trivialization*. It is important to observe that this isomorphism is not *canonical*; we cannot simply identify $U_i \times F$ with $\pi^{-1}(U_i)$. We are now ready to give our formal description:

*Formal definition of a fiber bundle*: A fiber bundle $E$ with fiber $F$ over the base manifold $M$ consists of a

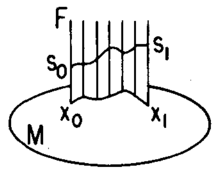


Fig. 4.4. A local cross-section or section of a bundle is a mapping which assigns a point $s$ of the fiber to each point $x$ of the base.

topological space $E$ together with a projection $\pi$: $E \to M$ which satisfies the local triviality condition: For each point $x \in M$, there exists a neighborhood $U_i$ of $x$ and an isomorphism $\Phi_i$ which maps $U_i \times F$ to the subset $\pi^{-1}(U_i)$ of the bundle $E$. Letting $(x, f)$ denote a point of $U_i \times F$, we require that $\pi(\Phi_i(x, f)) = x$ as a consistency condition. When we ignore the action of $\Phi_i(x, f)$ on the argument $x$, we may regard it as an $x$-dependent map $\Phi_{ix}(f)$ taking $F$ into $F_x$.

The *transition functions* are defined as

$$\Phi_{ij} = \Phi_i^{-1}\Phi_j \tag{4.2}$$

in the overlap of the neighborhoods $U_i$ and $U_j$. For each fixed $x \in U_i \cap U_j$, this is a map from $F$ onto $F$; $\Phi_{ij}$ relates the local product structure over $U_i$ to that over $U_j$. We require that these transition functions belong to a group $G$ of transformations of the fiber space $F$. $G$ is called the *structure group* of the fiber bundle.

The transition functions satisfy the cocycle conditions:

$$\Phi_{ii} = \text{identity}$$

$$\Phi_{ij}\Phi_{jk} = \Phi_{ik} \text{ for } x \in U_i \cap U_j \cap U_k.$$

A set of transition functions can be used to define a consistent procedure for gluing together local pieces of a bundle if and only if the cocycle conditions are satisfied. A bundle is completely determined by its transition functions.

*Pullback bundles*: Let $E$ be a fiber bundle over the base manifold $M$ with fiber $F$ and suppose that $h$: $M' \to M$ is a map from some other manifold $M'$ to $M$. The *pullback bundle* $E'$ denoted by $h^*E$, is defined by copying the fiber of $E$ over each point $x = h(x')$ in $M$ over the point $x'$ in $M'$. If we denote a point of $M' \times E$ by the pair $(x', e)$, then

$$E' = h^*E = \{(x', e) \in M' \times E \text{ such that } \pi(e) = h(x')\}. \tag{4.3}$$

Thus $E'$ is a *subset* of $M' \times E$ obtained by restricting oneself to the curve $\pi(e) = h(x')$. [Example: let $h$ be the identity map and let $E = M = M' = \mathbb{R}$; then $x = x'$ is a line in $\mathbb{R}^2 = M' \times M$ and $E' = \mathbb{R}$.] If $\{U_i\}$ is a covering of $M$ such that $E$ is locally trivial over $U_i$ and if $\Phi_{ij}(x)$ are the transition functions of $E$, then $\{h^{-1}(U_i)\}$ is a covering of $M'$ such that $E' = h^*E$ is locally trivial. The corresponding transition functions of the pullback bundle are:

$$\Phi'_{ij}(x') = (h^*\Phi_{ij})(x') = \Phi_{ij}(h(x')). \tag{4.4}$$

It is clear that if $M = M'$ and if $h(x) = x$ is the identity map, then $h^*(E)$ can be naturally identified with the original bundle $E$.

*Homotopy axiom*: If $h$ and $g$ are two maps from $M'$ to $M$, we say that they are *homotopic* if there exists a map $H$: $M' \times [0, 1] \to M$ such that $H(x', 0) = h(x')$ and $H(x', 1) = g(x')$. If we let $h_t(x') = h(x', t)$, then we are simply smoothly pushing the map $h = h_0$ to the map $g = h_1$. It is a theorem that if $h$ and $g$ are homotopic then $h^*E$ is isomorphic to $g^*E$. For example, if $M$ is contractible, we can let $h(x) = x$ be the identity map and let $g(x) = x_0$ be the map which collapses all of $M$ to a point. These maps are homotopic so $E = h^*E$ is isomorphic to $g^*E = M \times F$; this proves that $E$ is trivial if $M$ is contractible.

## 4.2. Vector bundles

Let us consider a bundle $E$ with a $k$-dimensional real fiber $F = \mathbb{R}^k$ over an $n$-dimensional base space $M$; $k$ is commonly called the *bundle dimension* and we shall write $\dim(E) = k$ even though this is in reality the dimension of the fiber alone. (The total dimension of $E$ is of course $(n+k)$.) $E$ is called a *vector bundle* if is transition functions belong to $GL(k, \mathbb{R})$ rather than to the full group of diffeomorphisms (differentiable transformations which are 1–1 and onto) of $\mathbb{R}^k$. Since $GL(k, \mathbb{R})$ preserves the usual operations of addition and scalar multiplication on a vector space, the fibers of $E$ inherit the structure of a vector space. We can think of a vector bundle as being a family of vector spaces (the fibers) which are parametrized by the base space $M$. Clearly there is a similar notion of a complex vector bundle if we replace $\mathbb{R}^k$ by $\mathbb{C}^k$ and $GL(k, \mathbb{R})$ by $GL(k, \mathbb{C})$.

*Vector space structure on the set of sections*: We can use the vector space structure on the fibers of a vector bundle to define the pointwise addition or scalar multiplication of sections. We write sections of a vector bundle in the form $s(x)$ to emphasize their vector-valued nature. Thus if $s(x)$ and $s'(x)$ are two local sections to $E$, we can define the local section $(s+s')(x) = s(x) + s'(x)$ by adding the values in the fibers. If $f(x)$ is a smooth continuous function on $M$, we can define the new section $[fs](x) = s(x)f(x)$ by pointwise scalar multiplication in the fibers.

*Zero section*: The origin $\{0\}$ of $\mathbb{C}^k$ or $\mathbb{R}^k$ is preserved by the general linear group and represents a distinguished element of the fiber of a vector bundle. Let $s(x) = 0$; this defines a global section called the *zero-section* of the vector bundle. We can always choose a non-zero section in any single neighborhood $U_i$. If we assume that this section is zero near the boundary of $U_i$, we can extend this section continuously to be zero outside of $U_i$. Therefore, any vector bundle has many global sections, although there may be no global sections which are everywhere non-zero.

*Moving frames*: At each point $x$ of some neighborhood $U$ of $M$, we can choose a basis $\{e_1(x), \ldots, e_k(x)\}$ for the $k$-dimensional fiber over $x$. We assume that the basis varies continuously with $x$ if it varies at all; such a collection of bases defined for all $x$ in $U$ is called a *frame*. If we have chosen a local trivialization of $U \times \mathbb{C}^k \to \pi^{-1}(U)$, then we can regard the $e_i(x)$ as vector-valued functions from $U$ into $\mathbb{C}^k$ and the entire frame as a matrix-valued function from $U$ into $GL(k, \mathbb{C})$. The coordinate frame is then the set of constant sections:

$$
\begin{aligned}
e_1(x) &= (1, 0, \ldots, 0) \\
e_2(x) &= (0, 1, \ldots, 0) \\
&\vdots
\end{aligned}
$$

We remark that one may still discuss the notion of a frame without necessarily having chosen a local trivialization.

A choice of frame $\{e_j(x)\}$ may in fact be used to specify the isomorphism $\Phi$ mapping $U \times \mathbb{C}^k \rightarrow \pi^{-1}(U)$. If $x \in U$ in $M$ and if $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$, we define

$$\Phi(x, z) = \sum_{j=1}^{k} e_j(x)\, z^j(x). \tag{4.5}$$

This introduces a local trivialization. Clearly $\Phi(x; 0, \ldots, 1, 0, \ldots, 0) = e_j(x)$ is just the vector in the fiber $\pi^{-1}(x)$ associated with the section $e_j(x)$.

*Change of frames*: Let $U$ and $U'$ be two neighborhoods in $M$ and suppose that we have frames $\{e_i\}$ and $\{e'_i\}$ over $U$ and $U'$. Let $\{z^i\}$ and $\{z''\}$ be the respective fiber coordinates, and let $\Phi = \Phi_{UU'}$ be the $\mathrm{GL}(k, \mathbb{C})$-valued transition function on $U \cap U'$. Then the frames, coordinates, and transition functions are related as follows:

$$\begin{aligned} e'_j(x) &= e_i(x)\, \Phi^{-1}_{ij}(x) \\ z''(x) &= \Phi_{ij}(x)\, z^j(x) \\ \Phi_{ij} &= (i, j) \text{ element of the matrix } \Phi_{U'U}. \end{aligned} \tag{4.6}$$

Hence

$$e_i z^i = e'_i z''$$

as required. *Note*: reversing the order in which the transition matrix acts would interchange the roles of right and left multiplication and would change the sign convention in the curvature from $R = \mathrm{d}\omega + \omega \wedge \omega$ to $R = \mathrm{d}\omega - \omega \wedge \omega$.

*Line bundles*: A line bundle is a vector bundle with a one-dimensional vector space as fiber. It is a family of lines parametrized by the base space $M$. If we replace the interval $[-1, 1]$ by the real line $\mathbb{R}$ in the Möbius strip example, we find a non-trivial real line bundle over the circle. If we replace $[-1, 1]$ by the complex numbers $\mathbb{C}$, the resulting line bundle is isomorphic to $S^1 \times \mathbb{C}$ and is therefore trivial. Note that $\mathrm{GL}(1, \mathbb{C})$ and $\mathrm{GL}(1, \mathbb{R})$ are *Abelian* groups so right and left multiplication commute; consequently, for line bundles, it does not matter whether we write the transition function on the left or on the right.

*Tangent and cotangent bundles*: We let the *tangent bundle* $T(M)$ and the *cotangent bundle* $T^*(M)$ be the real vector bundles whose fibers at a point $x \in M$ are given by the tangent space $T_x(M)$ or the cotangent space $T^*_x(M)$. These spaces were discussed earlier; we observe that if $x = (x_1, \ldots, x_n)$ is a local coordinate system defined on some neighborhood $U$ in $M$, then we can choose the following standard bases for the local frames:

$$\begin{aligned} &\{\partial/\partial x_1, \ldots, \partial/\partial x_n\} \quad \text{for the tangent bundle } T(M) \\ &\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\} \qquad\quad \text{for the cotangent bundle } T^*(M). \end{aligned} \tag{4.7}$$

If $U'$ is another neighborhood in $M$ with local coordinates $x'$, the transition functions in $U \cap U'$ are given by:

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x'_j} \cdot \frac{\partial x'_j}{\partial x_i} \qquad \text{on } T(M)$$
$$dx_i = dx'_j \cdot \frac{\partial x_i}{\partial x'_j} \qquad \text{on } T^*(M). \tag{4.8}$$

The *complexified* tangent and cotangent bundles $T(M) \otimes \mathbb{C}$ and $T^*(M) \otimes \mathbb{C}$ of a real manifold $M$ are defined by permitting the coefficients of the frames $\{\partial/\partial x_i\}$ and $\{dx_i\}$ to be complex.

If $M$ is a complex manifold with local complex coordinates $z_j$, we define the *complex tangent bundle* $T_c(M)$ to be the sub-bundle of $T(M) \otimes \mathbb{C}$ which is spanned by the holomorphic tangent vectors $\partial/\partial z_j$. The (complex) dimension of $T_c(M)$ is half the real dimension of $T(M)$. If we forget the complex structure on $T_c(M)$ and consider $T_c(M)$ as a real bundle, then $T_c(M)$ is isomorphic to $T(M)$.

*Constructions on vector bundles*

If $V$ is a vector space, we define the dual space $V^*$ to be the set of linear functionals. If $V$ and $W$ are a pair of vector spaces, we can define the Whitney sum $V \oplus W$ and the tensor product $V \otimes W$. These and other constructions can be carried over to the vector bundle case as we describe in what follows.

*Digression on dual vector spaces*: We first recall some facts concerning the dual space $V^*$ of linear functionals. An element $v^* \in V^*$ is just a linear map $v^*: V \to \mathbb{R}$. The sum and scalar multiple of linear maps are again linear maps so $V^*$ is a vector space. If $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k\}$ is a basis for $V$ and $v^* \in V^*$, then $v^*(\boldsymbol{e}_j z^j) = z^j v^*(\boldsymbol{e}_j)$, so the action of $v^*$ on a section is determined by the value of the linear map on the basis. We define the *dual basis* $\{\boldsymbol{e}^{*1}, \ldots, \boldsymbol{e}^{*k}\}$ of the dual space $V^*$ of linear functionals by

$$\boldsymbol{e}^{*i}(\boldsymbol{e}_j) = \delta^i_j \qquad \text{i.e.} \quad \boldsymbol{e}^{*i}(\boldsymbol{e}_j z^j) = z^i.$$

These equations show that we can regard the $\boldsymbol{e}^{*i}$ themselves as defining coordinates on $V$. Similarly, the $\boldsymbol{e}_j$ define coordinates on $V^*$. We see that

$$\dim(V) = \dim(V^*) = k.$$

If we change bases and set $\boldsymbol{e}_i = \boldsymbol{e}'_j \Phi_{ji}$, then the new dual basis is given by

$$\boldsymbol{e}^{*i} = \Phi^{-1}_{ij} \boldsymbol{e}^{*\prime j} = \boldsymbol{e}^{*\prime j} (\Phi^{t})^{-1}_{ji}. \tag{4.9}$$

The dual basis transforms just as a set of coordinates on $V$ transforms.

Dual vector spaces arise naturally whenever we have two vector spaces $V$ and $W$ together with a non-singular inner product $(v, w) \in \mathbb{R}$ or $\mathbb{C}$ where $v \in V$, $w \in W$. Since $(v, w)$ is a linear functional on $v$, we can regard $w$ as an element of the dual space $V^*$ whose action is defined by

$$w(v) = (v, w).$$

Since the inner product is non-singular, we may identify $W$ with $V^*$. Conversely, $V$ and $V^*$ possess a natural inner product defined by the action of $v^*$ on $v$:

$$(v, v^*) = v^*(v).$$

We may regard $V$ itself as a space of linear functionals dual to $V^*$ if we define the action of $v \in V$ by

$$v(v^*) = (v, v^*) = v^*(v).$$

If $V$ is finite dimensional, we find that $V^{**} = V$; this conclusion is false if $V$ is infinite dimensional.

*A simple example*: Let $V$ be the vector space of all polynomials of degree 1 or 0. Let $V = W$ and define an inner product by $(v, w) = \int_0^1 v(x)\, w(x)\, dx$. If $\{1, x\}$ is a basis for $V$, the corresponding dual basis for $W \simeq V^*$ relative to this pairing is $\{4 - 6x, -6 + 12x\}$.

*Dual bundles*: Let $E$ be a vector bundle with fiber $F_x$; let $E^*$ be the *dual vector bundle* with pointwise fiber $F_x^*$. If $\{\boldsymbol{e}_i\}$ is a local frame for $E$, we have the dual frame $\{\boldsymbol{e}^{*j}\}$ for $E^*$ defined by $\boldsymbol{e}^{*i}(\boldsymbol{e}_j) \equiv (\boldsymbol{e}_j, \boldsymbol{e}^{*i}) = \delta_{ij}$. If the transition functions of $E$ are given by $k \times k$ matrices $\Phi$, then the transition functions of $E^*$ are given by the $k \times k$ matrices $(\Phi^t)^{-1}$.

If $E = T(M)$ is the tangent bundle, then $E^* = T^*(M)$ is the cotangent bundle. The $\{\partial/\partial x_i\}$ and the $\{dx_i\}$ are dual bases in the usual sense and the transition matrices given earlier satisfy all the required properties.

*Whitney sum bundle*: The Whitney sum $V \oplus W$ of two vector spaces $V$ and $W$ is defined to be the set of all pairs $(v, w)$. The vector space structure of $(v, w)$ is

$$(v, w) + (v', w') = (v + v', w + w') \quad \text{and} \quad \lambda(v, w) = (\lambda v, \lambda w).$$

If we identify $v$ with $(v, 0)$ and $w$ with $(0, w)$, then $V$ and $W$ are subspaces of $V \oplus W$. If $\{\boldsymbol{e}_i\}$ and $\{f_j\}$ form bases for $V$ and $W$, respectively, then $\{\boldsymbol{e}_i, f_j\}$ is a basis for $V \oplus W$ so $\dim(V \oplus W) = \dim(V) + \dim(W)$.

If $E$ and $F$ are vector bundles over $M$, the fiber of the *Whitney sum bundle* $E \oplus F$ is obtained by taking the Whitney direct sum of the fibers of $E$ and $F$ at each point $x \in M$. If $\dim(E) = j$ and $\dim(F) = k$ and if the transition functions of $E$ and $F$ are the $j \times j$ matrices $\Phi$ and the $k \times k$ matrices $\Psi$, respectively, then the transition matrices of $E \oplus F$ are just the $(j+k) \times (j+k)$ matrices $\Phi \oplus \Psi$ given by:

$$\begin{pmatrix} \Phi & 0 \\ 0 & \Psi \end{pmatrix} = \Phi \oplus \Psi. \tag{4.10}$$

If $\{\boldsymbol{e}_i\}$, $\{f_l\}$ are local frames for $E$ and $F$, then $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_j, f_1, \ldots, f_k\}$ is a local frame for $E \oplus F$. Clearly, $\dim(E \oplus F) = \dim(E) + \dim(F) = j + k$.

*Tensor product bundle*: The tensor product bundle $E \otimes F$ of $E$ and $F$ is obtained by taking the tensor product of the fibers of $E$ and of $F$ at each point $x \in M$. The transition matrices for $E \otimes F$ are obtained by taking the tensor product of the transition functions of $E$ and the transition functions of $F$. A local frame for $E \otimes F$ is given by $\{e_i \otimes f_l\}$ so $\dim(E \otimes F) = \dim(E)\dim(F)$.

*Bundles of linear maps*: If $V$ and $W$ are vector spaces, we define $\mathrm{Hom}(V, W)$ to be the *space of all linear maps* from $V$ into $W$. For example, $\mathrm{Hom}(V, \mathbb{R}) = V^*$ since $V^*$ is by definition the space of all linear maps from $V$ to $\mathbb{R}$. If $\dim(V) = j$ and $\dim(W) = k$, then $\mathrm{Hom}(V, W)$ can be identified with the set of all $k \times j$ matrices and is a vector space in its own right. If $E$ and $F$ are vector bundles, we define $\mathrm{Hom}(E, F)$ to be the vector bundle whose fiber is Hom of the fibers of $E$ and $F$. There is a natural isomorphism $\mathrm{Hom}(V, W) = V^* \otimes W$ and similarly $\mathrm{Hom}(E, F) = E^* \otimes F$. Since $E^{**} = E$, the isomorphism $\mathrm{Hom}(E^*, F) = E \otimes F$ can be used to give an alternative definition of the tensor product.

*Other constructions*: Let $\otimes^p(E) = E \otimes \cdots \otimes E$ be the bundle of $p$-tensors. $\Lambda^p(E)$ is the bundle of antisymmetric $p$-tensors and $S^p(E)$ is the bundle of symmetric $p$-tensors; these are both sub-bundles of $\otimes^p(E)$. If $\dim(E) = k$, then

$$\dim(\otimes^p(E)) = k^p, \qquad \dim(\Lambda^p(E)) = \binom{k}{p}, \qquad \dim(S^p(E)) = \binom{k+p-1}{p}.$$

The transition functions of $\Lambda^p(E)$ and $S^p(E)$ are $p$-fold tensor products of the transition functions of $E$ with the appropriate symmetry properties. Note that $C^\infty(\Lambda^p(T^*(M)))$ is just the space of $p$-forms on $M$.

*Complementary bundles* (*normal bundles*): If $E$ is a real or complex vector bundle over $M$ with fiber $V$ of dimension $k$, we can always construct a (nonunique) complementary bundle $\bar{E}$ such that the Whitney sum $E \oplus \bar{E} \simeq M \times \mathbb{C}^l$ is a trivial bundle with fiber $\mathbb{C}^l$ for some $l > k$. A frequent application of this fact occurs in the construction of the *tangent* and *normal bundles* of a manifold. If $M$ is an $n$-dimensional complex manifold embedded in $\mathbb{C}^m$, the bundle of tangent vectors $T_c(M)$ (dimension $= n$) and the bundle of normal vectors $N_c(M)$ (dimension $= m - n$) are both non-trivial in general. However, the Whitney sum is the trivial $n + (m - n) = m$-dimensional bundle $I_m$:

$$T_c(M) \oplus N_c(M) = I_m = M \times \mathbb{C}^m. \tag{4.11}$$

*Fiber metrics* (*inner products*): *A fiber metric* is a pointwise inner product between two sections of a vector bundle which allows us to define the *length* of a section at a point $x$ of the base. In local coordinates, a fiber metric is a positive definite symmetric matrix $h_{ij}(x)$. The inner product of two sections is then

$$(s, s') = h_{ij}(x)\, z^i(x)\, \bar{z}'^j(x), \tag{4.12}$$

where $\bar{z}$ denotes complex conjugation if the fiber is complex. Under a change of frame, we obviously find

$$h \to (\Phi^{\mathrm{t}})^{-1} h \bar{\Phi}^{-1}.$$

A fiber metric defines a (conjugate) linear isomorphism between $E$ and $E^*$.

If $E$ is a real vector bundle with a fiber metric, the fiber metric defines a pairing of $E$ with itself and gives an isomorphism between $E$ and $E^*$. If $E = T(M)$, the fiber metric is simply a Riemannian metric on $M$; thus $T(M)$ *is always isomorphic to* $T^*(M)$.

If $E$ is a complex vector bundle, the fiber metric is conjugate linear in the second factor. This defines a conjugate linear pairing of $E$ with itself and gives a conjugate linear isomorphism between $E$ and $E^*$. Thus in the complex case, $E$ need not be isomorphic to $E^*$; this fact can sometimes be detected by the characteristic classes, as we shall see later.

*Examples 4.2*

1. *Tangent and cotangent bundles of* $S^2$: Let $U = S^2 - \{(0, 0, -\frac{1}{2})\}$ and let $U' = S^2 - \{(0, 0, \frac{1}{2})\}$ be spheres of unit diameter minus the south/north poles. We stereographically project these two neighborhoods to the plane to define coordinates $\boldsymbol{x} = (x, y)$ and $\boldsymbol{x}' = (x', y')$. Let $r^2 = x^2 + y^2$. In these coordinates, the standard metric is given by:

$$ds^2 = (1 + r^2)^{-2}(dx^2 + dy^2).$$

The $U'$ coordinates are related to the $U$ coordinates by the inversion

$$\boldsymbol{x}' = r^{-2}\boldsymbol{x},$$

so

$$d\boldsymbol{x}' = r^{-4}(r^2\, d\boldsymbol{x} - 2\boldsymbol{x}\,(\boldsymbol{x} \cdot d\boldsymbol{x})).$$

The transition functions $|\partial x'/\partial x|$ for $T^*(S^2)$ are therefore given by:

$$\Phi_{U'U}(\boldsymbol{x}) = r^{-4}(\delta_{ij}r^2 - 2x_ix_j) \quad \text{on} \quad U' \cap U.$$

We introduce polar coordinates on $\mathbb{R}^2 - (0, 0)$ and restrict to $r = 1$, so that we are effectively working on the equator $S^1$ of the sphere. Then we find

$$\Phi_{U'U}(\cos\theta, \sin\theta) = \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}. \tag{4.13}$$

(The transposed inverse matrix is of course the transition matrix for $T(S^2)$.) $\Phi_{U'U}$ represents a non-trivial map of $S^1 \to \mathrm{GL}(2, \mathbb{R})$. This map is just twice the generator of $\pi_1(\mathrm{GL}(2, \mathbb{R})) = \mathbb{Z}$.

The bundles $T(S^2)$ and $T^*(S^2)$ are non-trivial and isomorphic. Let $I$ denote the trivial bundle over $S^2$. We can identify $I$ with the normal bundle of $S^2$ in $\mathbb{R}^3$ so $T(S^2) \oplus I = T(\mathbb{R}^3) = I^3$ is the trivial bundle of dimension 3 over $S^2$. Similarly $T^*(S^2) \oplus I = T^*(\mathbb{R}^3) = I^3$. If we regard the transition map $\Phi_{U'U} \oplus I$ given above as a map from $S^1$ to $\mathrm{GL}(3, \mathbb{R})$, then it is still twice the generator. Since $\pi_1(\mathrm{GL}(3, \mathbb{R})) = Z_2$, the map is null homotopic and $T^*(S^2) \oplus I$ is trivial.

2. *The natural line bundle over projective space.* We defined $P_n(\mathbb{C})$ to be the set of lines through the origin in $\mathbb{C}^{n+1}$. Let $I^{n+1} = P_n(\mathbb{C}) \times \mathbb{C}^{n+1}$ be the trivial bundle of dimension $n + 1$ over $P_n(\mathbb{C})$. We denote a point of $I^{n+1}$ by the pair $(p, z)$; scalar multiplication and addition are performed on the second factor while leaving the first factor unchanged in this expression. Let $L$ be the sub-bundle of $I^{n+1}$ defined by:

$$L = \{(p, z) \in I^{n+1} = P_n(\mathbb{C}) \times \mathbb{C}^{n+1} \text{ such that } z \in p\}. \tag{4.14}$$

In other words, the fiber of $L$ over a point $p$ of $P_n(\mathbb{C})$ is just the set of points in $\mathbb{C}^{n+1}$ which belong to the line $p$.

In example 2.1.3 we defined coordinates $\zeta_i^{(j)} = z_i/z_j$ on neighborhoods $U_j = \{p: z_j(p) \neq 0\}$. On $U_j$, we define the section $s_j$ to $L$ by:

$$s_j(p) = (\zeta_0^{(j)}(p), \ldots, 1, \ldots, \zeta_n^{(j)}(p)).$$

The transition functions are $1 \times 1$ complex matrices – i.e. scalars:

$$s_k(p) = (\zeta_k^{(j)})^{-1} s_j(p).$$

Since the transition functions are holomorphic, $L$ is a holomorphic line bundle.

The dual bundle $L^*$ has sections $s_j^*$ so that $s_j^*(s_j) = 1$. (*Note*: since we have a line bundle, a frame is given by a single section. The subscripts here refer to different coordinate systems and not to elements of a frame.) The transition functions act as

$$s_k^* = s_j^* \zeta_k^{(j)}.$$

We now interpret the $\{s_j^*\}$ as homogeneous coordinates on $P_n(\mathbb{C})$, since it is clear that

$$s_j^*(p) = z_j.$$

Note that $s_j^* = 0$ whenever $z_j = 0$, i.e. whenever $p$ is *not* in the neighborhood $U_j$. The ratio of these global sections may be used to define the inhomogeneous coordinates $\zeta_i^{(j)}$.

*Note*: $L^*$ has global holomorphic sections $s_j^*$ whose zeroes lie in the complement of $U_j$, which is just a projective space of dimension $(n-1)$. The bundle $L$ does *not* have any global holomorphic sections; since $s_j s_j^* = 1$ and $s_j^* = 0$ on the complement of $U_j$, $s_j$ must blow up like $z_j^{-1}$ on the complement of $U_j$. The $s_j$ are therefore *meromorphic* sections of $L$.

We define the line bundle $L^k$ by:

$$\begin{aligned} &L^* \otimes \cdots \otimes L^* \quad \text{if } k < 0 \\ &L^0 = I \quad \text{(the trivial line bundle)} \\ &L \otimes \cdots \otimes L \quad \text{if } k > 0. \end{aligned} \tag{4.15}$$

Because $L \otimes L^* = I$, $L^j \otimes L^k = L^{j+k}$ for all integers $j$, $k$. Any line bundle over $P_n(\mathbb{C})$ is isomorphic to $L^k$ for some uniquely defined integer $k$. The integer $k$ is related to the first Chern class of $L^k$ as we shall see later.

Let $T_c(P_n(\mathbb{C}))$ and $T_c^*(P_n(\mathbb{C})) = \Lambda^{1,0}(P_n(\mathbb{C}))$ be the complex tangent and cotangent spaces. Then:

$$I \oplus T_c(P_n(\mathbb{C})) = L^* \oplus \cdots \oplus L^* \quad \text{(a total of } n+1 \text{ times)}$$

$$I \oplus T_c^*(P_n(\mathbb{C})) = L \oplus \cdots \oplus L \quad \text{(a total of } n+1 \text{ times).}$$

(This identity does not preserve the holomorphic structures but is an isomorphism between complex vector bundles.)

3. *Relationship between $T(S^2)$ and $L^k$*. Using the relations $S^2 = P_1(\mathbb{C})$ and $T(S^2) = T_c(P_1(\mathbb{C}))$, we may combine the two previous examples for $n = 1$ to show

$$T^*(S^2) = L \otimes L, \qquad T(S^2) = L^* \otimes L^*. \tag{4.16}$$

We prove these relationships by recalling that we may choose complex coordinates on $S^2$ of the form $\zeta_0 = z_1/z_0$ on $U_0$ and $\zeta_1 = z_0/z_1 = \zeta_0^{-1}$ on $U_1$. We choose the basis of $T^*(S^2)$ to be $\mathrm{d}\zeta_0$ on $U_0$ and $-\mathrm{d}\zeta_1$ on $U_1$. The transition functions are given by

$$(-\mathrm{d}\zeta_1) = \zeta_0^{-2}(\mathrm{d}\zeta_0).$$

The local sections

$$s_0 = (1, \zeta_0), \qquad s_1 = (\zeta_0^{-1}, 1)$$

of $L$ give the transition function $s_1 = \zeta_0^{-1} s_0$. The $L \otimes L$ transition functions are thus

$$s_1 \otimes s_1 = \zeta_0^{-2} s_0 \otimes s_0,$$

so $T^*(S^2)$ and $L \otimes L$ are isomorphic bundles. The isomorphism between $T(S^2)$ and $L^* \otimes L^*$ is obtained by dualizing the preceding argument.

### 4.3. *Principal bundles*

A vector bundle is a fiber bundle whose fiber $F$ is a linear vector space and whose transition functions belong to the general linear group of $F$. A *principal bundle* $P$ is a fiber bundle whose fiber is a Lie group $G$ (which is a manifold); the transition functions of $P$ belong to $G$ and act on $G$ by *left* multiplication. We can define a *right* action of $G$ on $P$ because left and right multiplication commute. This action is a map from $P \times G \to P$ which commutes with the projection $\pi$, i.e.

$$\pi(p \cdot g) = \pi(p) \quad \text{for any } g \in G \text{ and } p \in P.$$

We remind the reader that the roles of left and right multiplication may be reversed if desired.

We can construct a principal bundle $P$ known either as the *frame bundle* or as the *associated principal bundle* from a given vector bundle $E$. The fiber $G_x$ of $P$ at $x$ is the set of all frames of the vector space $F_x$ which is the fiber of $E$ over the point $x$. In order to be specific, let us consider the case of the complex vector space of $k$ dimensions, $F = \mathbb{C}^k$. Then the fiber $G$ of the frame bundle $P$ is the collection of the $k \times k$ non-singular matrices which form the group $\mathrm{GL}(k, \mathbb{C})$; i.e., $G$ is the structure group of the vector bundle $E$.

The associated principal bundle $P$ has the same transition functions as the vector bundle $E$. These transition functions are $\mathrm{GL}(k, \mathbb{C})$ group elements and they act on the fiber $G$ by *left* multiplication. On the other hand the *right* action of the group $G = \mathrm{GL}(k, \mathbb{C})$ on the principal $G$ bundle $P$ takes a frame $e = \{e_1, \ldots, e_k\}$ to a new frame in the same fiber

$$e \cdot g = \{e_i g_{i1}, \ldots, e_i g_{ik}\} \qquad \text{(sum over } i \text{ is implied)} \tag{4.17}$$

for $|g_{ij}| \in \mathrm{GL}(k, \mathbb{C})$.

If $P$ is a principal $G$ bundle and if $\rho$ is a representation of $G$ on a finite-dimensional vector space $V$, we can define the *associated vector bundle* $P\times_\rho V$ by introducing the equivalence relation on $P\times V$:

$$(p, \rho(g)\cdot v) \simeq (p\cdot g, v) \quad \text{for all } p\in P, v\in V, g\in G. \tag{4.18}$$

The transition functions on $P\times_\rho V$ are given by the representation $\rho(\Phi)$ applied to the transition functions $\Phi$ of $P$. If $P$ is the frame bundle of $E$ and if $\rho$ is the identity representation of $G$ on the fiber $F$, then $P\times_\rho F = E$. In this way we may pass from a vector bundle $E$ to its associated principal bundle $P$ and back again by changing the space on which the transition functions act from the vector space to the general linear group and back.

*Unitary frame bundles*: If $E$ is a vector bundle with an inner product, we can apply the Gram–Schmidt process to construct unitary frames. The bundle of unitary frames is a $U(k)$ principal bundle if $E$ is complex and an $O(k)$ principal bundle if $E$ is real. If $E$ is an oriented real bundle, we may consider the set of oriented frames to define an $\mathrm{SO}(k)$ principal bundle.

If $E$ is a complex vector bundle with an inner product and if the transition functions are unitary with determinant 1, we can define an $\mathrm{SU}(k)$ principal bundle associated with $E$. However, not every vector bundle admits $\mathrm{SU}(k)$ transition functions; the first Chern class must vanish.

*Local sections*: If $\gamma(x)$ is a local section to $P$ over a neighborhood $U$ in $M$, we can use right multiplication to define a map

$$\Phi: U\times G \to \pi^{-1}(U),$$

where $\Phi(x, g) = \gamma(x)\cdot g$. This gives a local trivialization of $P$. A principal bundle $P$ is trivial if and only if it has a global section; non-trivial principal bundles do not admit global sections. (The identity element of $G$ is *not* invariant so there is no analog of the zero section to a vector bundle.)

*Lie algebras*: The Lie algebra $\mathcal{G}$ of $G$ is the tangent space $T_e(G)$ at the identity element $e$ of $G$. By using *left* translation in the group, we may identify $\mathcal{G}$ with the set of left-invariant vector fields on $G$. Let $\mathcal{G}^*$ be the dual space. We can identify $\mathcal{G}^*$ with the left-invariant 1-forms on $G$. Let $\{L_a\}$ be a basis for $\mathcal{G}$ and let $\{\phi_a\}$ be the dual basis for $\mathcal{G}^*$. The $\{L_a\}$ obey the Lie bracket algebra

$$[L_a, L_b] = f_{abc}L_c, \tag{4.19}$$

where the $f_{abc}$ are the structure constants for $\mathcal{G}$. The Maurer–Cartan equation

$$\mathrm{d}\phi_a = \tfrac{1}{2}f_{abc}\phi_b \wedge \phi_c \tag{4.20}$$

is the corresponding equation for $\mathcal{G}^*$.

*Examples 4.3*

1. *Principal $Z_2$ bundle.* One of the simplest examples of a principal fiber bundle is obtained from the Möbius strip example with $M = S^1$ by replacing the line-interval fiber $F = [-1, 1]$ by its end points $\pm 1$. These end points form a group under multiplication

$$\mathbf{Z}_2 = S^0 = \{+1, -1\},$$

and we have a fiber which is a group manifold. The transition functions $\Phi$ are $\mathbb{Z}_2$ group elements and act on the fiber $F = \mathbb{Z}_2$ by the group multiplication. We let $M = S^1$ be covered by two neighborhoods, so there are two overlapping regions I and II. Then we can construct two different types of bundles in the following way;

$$\text{trivial bundle: } \Phi_{\text{I}} = \Phi_{\text{II}} \qquad E = S^1 \times \mathbb{Z}_2 = \text{two circles;}$$

$$\text{non-trivial bundle: } \Phi_{\text{I}} = -\Phi_{\text{II}}, \qquad E = \text{double covering of a circle.}$$

These bundles correspond to the boundaries of a cylinder and a Möbius strip.

2. *Magnetic monopole bundle.* We shall see later that Dirac's magnetic monopole corresponds to a principal $U(1)$ bundle over $S^2$. We construct this bundle by taking

$$\text{Base } M = S^2; \qquad \text{coordinates } (\theta, \phi),\ 0 \le \theta < \pi,\ 0 \le \phi < 2\pi$$

$$\text{Fiber } F = U(1) = S^1; \qquad U(1) \text{ coordinate } \mathrm{e}^{\mathrm{i}\psi}.$$

We break $S^2$ into two hemispherical neighborhoods $H_\pm$ with $H_+ \cap H_-$ a thin strip parametrized by the equatorial angle $\phi$, as shown in fig. 4.5. Locally, the bundle looks like

$$H_+ \times U(1), \quad \text{coordinates } (\theta, \phi; \mathrm{e}^{\mathrm{i}\psi_+})$$

$$H_- \times U(1), \quad \text{coordinates } (\theta, \phi; \mathrm{e}^{\mathrm{i}\psi_-}).$$

The transition functions must be functions of $\phi$ along $H_+ \cap H_-$ and must be elements of $U(1)$ to give a principal bundle. We therefore choose to relate the $H_+$ and $H_-$ fiber coordinates as follows:

$$\mathrm{e}^{\mathrm{i}\psi_-} = \mathrm{e}^{\mathrm{i}n\phi}\, \mathrm{e}^{\mathrm{i}\psi_+}. \tag{4.21}$$

*n must be an integer* for the resulting structure to be a manifold; the fibers must fit together exactly when we complete a full revolution around the equator in $\phi$. This is in essence a topological version of the Dirac monopole quantization condition.

For $n = 0$, we have a trivial bundle

$$P(n = 0) = S^2 \times S^1.$$

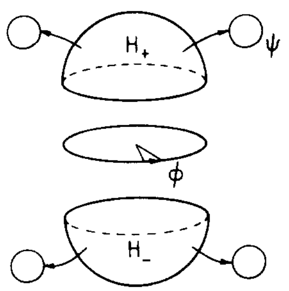


Fig. 4.5. The magnetic monopole bundle, showing the two hemispherical neighborhoods $H_\pm$ covering the base manifold $M = S^2$. A fiber $U(1) = S^1$ parametrized by $\psi$ is attached to each point of $H_\pm$. The intersection of $H_\pm$ at $\theta \sim \pi/2$ is a strip parametrized by $\phi$.

The case $n = 1$ is the famous Hopf fibering (Steenrod [1951]; Trautman [1977]) of the three-sphere

$$P(n=1)=S^3$$

and describes a singly-charged Dirac monopole. For general $n$, we have a more complicated bundle corresponding to a monopole of charge $n$.

*Remark*: $n$ corresponds to the *first Chern class* and characterizes inequivalent monopole bundles.

3. *Instanton bundle*. Another interesting principal bundle corresponds to the Yang–Mills instanton. We take the base space to be compactified Euclidean space-time, namely the four-sphere, and the fiber to be the group SU(2):

$$\text{Base } M = S^4; \qquad \text{coordinates } (\theta, \phi, \psi, r)$$

$$\text{Fiber } F = \mathrm{SU}(2) = S^3; \quad \text{coordinates } (\alpha, \beta, \gamma).$$

We split $S^4$ into two "hemispheres" $H_\pm$ whose boundaries are $S^3$'s. Thus we may parametrize the thin intersection of $H_+$ with $H_-$ along the "equator" of $S^4$ by the Euler angles $(\theta, \phi, \psi)$ of $S^3$. Using the standard construction, we have a representation $h(\theta, \phi, \psi)$ of SU(2),

$$h = \frac{t - \mathrm{i}\boldsymbol{\lambda}\cdot\boldsymbol{x}}{r}, \qquad \begin{cases} x + \mathrm{i}y = r\cos\dfrac{\theta}{2}\exp\dfrac{\mathrm{i}}{2}(\psi+\phi) \\ z + \mathrm{i}t = r\sin\dfrac{\theta}{2}\exp\dfrac{\mathrm{i}}{2}(\psi-\varphi), \end{cases}$$

where the $\boldsymbol{\lambda}$ are the Pauli matrices. The fiber coordinates are similarly given by SU(2) matrices $g(\alpha, \beta, \gamma)$ depending on the group Euler angles $(\alpha, \beta, \gamma)$.

Thus we have the local bundle patches

$$H_+ \times \mathrm{SU}(2), \quad \text{coordinates } (\theta, \phi, \psi, r; \alpha_+, \beta_+, \gamma_+)$$

$$H_- \times \mathrm{SU}(2), \quad \text{coordinates } (\theta, \phi, \psi, r; \alpha_-, \beta_-, \gamma_-).$$

In $H_+ \cap H_-$, we construct the transition from the SU(2) fibers $g(\alpha_+, \beta_+, \gamma_+)$ to $g(\alpha_-, \beta_-, \gamma_-)$ using multiplication by the SU(2) matrix $h(\theta, \phi, \psi)$;

$$g(\alpha_-, \beta_-, \gamma_-) = h^k(\theta, \phi, \psi)\, g(\alpha_+, \beta_+, \gamma). \tag{4.22}$$

The power $k$ of the matrix $h(\theta, \phi, \psi)$ must be an integer to give a well-defined manifold.

For $k = 1$, we get the Hopf fibering of $S^7$ (Steenrod [1951]; Trautman [1977]),

$$P(k=1) = S^7.$$

This is the bundle described by the single-instanton solution (Belavin et al. [1975]). More general instanton solutions describe bundles with other values of $k$.

*Remark*: $k$ corresponds to the *second Chern class* and characterizes the equivalence classes of instanton bundles.

### 4.4. Spin bundles and Clifford bundles

We have concentrated in most of this section on vector bundles and principal bundles whose fibers had structure groups such as $O(k)$ and $U(k)$. Another important type of vector space which may appear as a fiber is a space of *spinors*. The structure group of a spinor space is the spin group, Spin($k$). For example, the spin group corresponding to SO(3) is just its double covering, Spin(3) = SU(2). The principal spin bundles associated with a bundle of spinors have fibers lying in Spin($k$). We note that not all base manifolds admit well-defined spinor structures; spinors arising from the tangent space can only be defined for manifolds where the second Stiefel–Whitney class (described in section 6) vanishes.

Spinors must in general belong to an algebra of anticommuting variables. Such variables are a special case of the more general notion of a *Clifford algebra*, which may also be used to define a type of fiber bundle. For example, if we start with a real vector bundle $E$ of dimension $k$, we can construct the corresponding *Clifford bundle*, Cliff($E$), as follows. The sections of Cliff($E$) are constructed from sections $s(x)$ and $s'(x)$ of $E$ by introducing the Clifford multiplication

$$s \cdot s' + s' \cdot s = 2(s, s'), \tag{4.23}$$

where $(s, s')$ is the vector bundle inner product. Cliff($E$) is then a $2^k$-dimensional bundle containing $E$ as a sub-bundle. The Clifford algebra acts on itself by Clifford multiplication; relative to a matrix basis, this action admits a $2^k \times 2^k$ dimensional representation of the algebra. For $k = 1$, we find a $2 \times 2$ Pauli matrix representation, while for $k = 2$, we have the $4 \times 4$ Dirac matrices.

We note that there is a natural isomorphism between the exterior algebra bundle $\Lambda^*(E)$ and the Clifford bundle, Cliff($E$). For example, the 16 independent Dirac matrix components 1, $\gamma_5$, $\gamma_\mu$, $\gamma_\mu\gamma_5$ and $[\gamma_\mu, \gamma_\nu]$ can be matched with the elements 1, $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$, $dx^\mu$, $\epsilon_{\mu\nu\lambda\sigma}\, dx^\nu \wedge dx^\lambda \wedge dx^\sigma$ and $dx^\mu \wedge dx^\nu$ of $\Lambda^*$.

For further details, see Chevalley [1954] and Atiyah, Bott and Shapiro [1964].

## 5. Connections on fiber bundles

So far, we have only considered fiber bundles as global geometric constructions. The notion of a connection plays an essential role in the local differential geometry of fiber bundles. A connection defines a covariant derivative which contains a gauge field and specifies the way in which a vector in the bundle $E$ is to be parallel-transported along a curve lying in the base $M$. We shall first describe connections on vector bundles and then proceed to treat connections on principal bundles. We shall give several examples, including the Dirac monopole and the Yang–Mills instanton.

### 5.1. Vector bundle connections

*The Levi-Civita connection on a surface in $\mathbb{R}^3$*

The modern concept of a connection arose from the attempt to find an intrinsic definition of differentiation on a curved two dimensional surface embedded in the three dimensional space $\mathbb{R}^3$ of physical experience. We take the unit sphere $S^2$ in $\mathbb{R}^3$ as a specific example. Let the coordinates

$$x(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$$

parametrize the sphere. We observe that $\boldsymbol{x}(\theta, \phi)$ is also the unit normal. The Riemannian metric induced by the chosen embedding is given by:

$$g_{ij} = \begin{pmatrix} \partial_\theta \boldsymbol{x} \cdot \partial_\theta \boldsymbol{x} & \partial_\theta \boldsymbol{x} \cdot \partial_\phi \boldsymbol{x} \\ \partial_\theta \boldsymbol{x} \cdot \partial_\phi \boldsymbol{x} & \partial_\phi \boldsymbol{x} \cdot \partial_\phi \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

so that

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 .$$

The two vector fields

$$\boldsymbol{u}_1 = \partial_\theta \boldsymbol{x} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\boldsymbol{u}_2 = \partial_\phi \boldsymbol{x} = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$$

are tangent to the surface and span the tangent space provided that $0 < \theta < \pi$ (i.e., away from the north and south poles, where this parametrization is singular). Clearly, any derivative can be decomposed as shown in fig. 5.1 into tangential components proportional to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, and a normal component $\hat{n}$ proportional to $\boldsymbol{x}$. We identify $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ with the bases $\partial/\partial\theta$ and $\partial/\partial\phi$ for the tangent space because

$$\partial f(\boldsymbol{x})/\partial\theta = \boldsymbol{u}_1 \cdot \partial f/\partial \boldsymbol{x}, \qquad \partial f(\boldsymbol{x})/\partial\phi = \boldsymbol{u}_2 \cdot \partial f/\partial \boldsymbol{x}$$

where $f(\boldsymbol{x})$ is a function on $\mathbb{R}^3$.

Our goal is now to differentiate tangential vector fields in a way which is intrinsic to the surface and not to the particular *embedding* involved.

First we compute the ordinary partial derivatives

$$\partial_\theta(u_1) = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta) = -\boldsymbol{x}$$

$$\partial_\phi(u_1) = \partial_\theta(u_2) = (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0) = \frac{\cos\theta}{\sin\theta}\boldsymbol{u}_2$$

$$\partial_\phi(u_2) = (-\sin\theta, \cos\phi, -\sin\theta\sin\phi, 0) = -\sin^2\theta\,\boldsymbol{x} - \cos\theta\sin\theta\,\boldsymbol{u}_1 .$$

We define intrinsic covariant differentiation $\nabla_X$ with respect to a given tangent vector $X$ by taking the

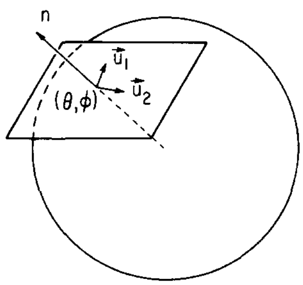


Fig. 5.1. Normal direction $\hat{n}$ and tangential directions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ at a point $(\theta, \phi)$ of $S^2$ embedded in $R^3$.

ordinary derivative and projecting back to the surface. $\nabla_X$ is then the directional derivative obtained by throwing away the normal component of the ordinary partial derivatives:

$$\nabla_{u_1}(u_1) = 0$$
$$\nabla_{u_1}(u_2) = \nabla_{u_2}(u_1) = \cot\theta\, u_2$$
$$\nabla_{u_2}(u_2) = -\cos\theta\sin\theta\, u_1.$$

$\nabla$ is the *Levi–Civita connection* on $S^2$. Using the identification of $(u_1, u_2)$ with $(\partial/\partial\theta, \partial/\partial\phi)$, we write

$$\nabla_{\partial/\partial\theta} \equiv \nabla_{u_1}, \qquad \nabla_{\partial/\partial\phi} \equiv \nabla_{u_2}.$$

Now the Christoffel symbol is defined by

$$\nabla_{u_i}(u_j) = u_k\Gamma^k{}_{ij} \quad \text{or} \quad \nabla_{\partial_i}(\partial_j) = \Gamma^k{}_{ij}\,\partial_k$$

where $\partial_1 = \partial/\partial\theta$, $\partial_2 = \partial/\partial\phi$. Then, in our example, we find

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot\theta, \qquad \Gamma^1{}_{22} = -\cos\theta\sin\theta, \qquad \Gamma^k{}_{ij} = 0 \text{ otherwise.}$$

*Geodesic equation*: Suppose $x(t)$ is a curve lying on $S^2$. This curve is a *geodesic* if there is no shear, i.e., the acceleration $\ddot{x}$ has only components normal to the surface. This condition may be written

$$\nabla_{\dot{x}}(\dot{x}) = 0. \tag{5.1}$$

For example, if we consider a parallel to latitude $x(t) = x(\theta = \theta_0,\ \phi = t)$ then $\dot{x} = u_2$ and $\nabla_{\dot{x}}(\dot{x}) = -\cos\theta_0\sin\theta_0\, u_1$. This curve is a geodesic on the equator, $\theta_0 = \pi/2$. The curves $x(t) = x(\theta = t, \phi = \phi_0)$ always satisfy the geodesic equations because $\dot{x} = u_1$ and $\nabla_{\dot{x}}(\dot{x}) = 0$; these are great circles through the north and south poles.

*Parallel transport*: The Levi–Civita connection provides a rule for the parallel transport of vectors on a surface. Let $x(t)$ be a curve in $S^2$ and let $s(t)$ be a vector field defined along the curve. We say that $s$ is parallel transported along the curve if it satisfies the equation

$$\nabla_{\dot{x}}(s) = 0,$$

i.e., $\dot{s}$ is normal to the surface. Given an initial vector $s(t_0)$ and the connection, $s(t)$ is uniquely determined by the parallel transport equation.

Parallel translation around a closed curve need not be the identity. For example, let $x$ be the geodesic triangle in $S^2$ connecting the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. $x$ consists of 3 great circles:

$$x(t) = \begin{cases} (\cos(t), \sin(t), 0) & t \in [0, \pi/2] \\ (0, \sin(t), -\cos(t)) & t \in [\pi/2, \pi] \\ (-\sin(t), 0, -\cos(t)) & t \in [\pi, 3\pi/2]. \end{cases}$$

Let $s(0)$ be the initial tangent vector

$$s(0) = (0, \alpha, \beta)$$

at $(1, 0, 0)$. When we parallel transport $s(0)$ along $x(t)$ using the Levi–Civita connection we find

$$s(t) = \begin{cases} (-\alpha \sin(t), \alpha \cos(t), \beta) & t \in [0, \pi/2] \\ (-\alpha, \beta \cos(t), \beta \sin(t)) & t \in [\pi/2, \pi] \\ (\alpha \cos(t), -\beta, -\alpha \sin(t)) & t \in [\pi, 3\pi/2]. \end{cases}$$

One may verify that $s(t)$ is continuous at the corners $\pi/2$, $\pi$ and satisfies $\nabla_{\dot{x}}(s) = 0$, since $\partial s/\partial t$ is normal to the surface. Parallel translation around the geodesic triangle changes $s$ from $s(0) = (0, \alpha, \beta)$ to $s(3\pi/2) = (0, -\beta, \alpha)$, which represents a rotation through $\pi/2$ (see fig. 5.2). Note that $\pi/2$ is the area of the spherical triangle.

*Holonomy*: Holonomy is the process of assigning to each closed curve the linear transformation measuring the rotation which results when a vector is parallel transported around the given curve. In our example, the holonomy matrix changing $s(0)$ to $s(3\pi/2)$ is

$$H_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} .$$

The set of holonomy matrices forms a group called the *holonomy group*. The non-triviality of holonomy is related to the existence of curvature on the sphere: parallel transport around a closed curve in a plane gives no rotation.

*General definitions of the connection*

In the general case, there is no natural embedding of a manifold $M$ in Euclidean space. Thus, even for the tangent bundle, it is meaningless to talk about normals to $M$. The problem is even more difficult for a general vector bundle. Therefore, we now proceed to abstract the intrinsic features of the Levi–Civita connection which allowed us to discuss parallel translation.

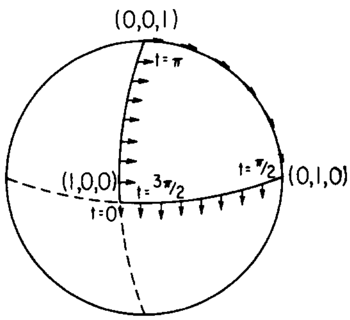


Fig. 5.2. Parallel transport of a vector around a spherical triangle.

*Background*: Let $E$ be a general vector bundle. On each neighborhood $U$ we choose a local frame $\{e_1, e_2, \ldots, e_k\}$ and express vectors in $\pi^{-1}(U)$ in the form

$$Z = \sum_{i=1}^{k} e_i z^i .$$

This gives a local trivialization of $\pi^{-1}(U) \approx U \times F$ and defines local coordinates $(x, z)$. The vectors $e_i$ themselves have the form

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

in each *local* frame. This, however, does not mean that $e_i$ is a constant vector on $M$ since the local frames may be different in each neighborhood. The dependence of $e_i$ on $x$ due to the change of the local frame is dictated by the rule of covariant differentiation described below. A *local section* to the bundle is a smooth map from $U$ to the fiber and can be regarded as a vector-valued function,

$$s(x) = \sum_{i=1}^{k} e_i(x)\, z^i(x).$$

The *tangent space* $T(E)$ and the *cotangent space* $T^*(E)$ of the bundle may be assigned the local bases

$$T(E): \quad (\partial/\partial x^\mu, \partial/\partial z^i)$$

$$T^*(E): \quad (\mathrm{d}x^\mu, \mathrm{d}z^i).$$

We now give a series of equivalent definitions of a connection on a vector bundle.

(1) *Parallel transport approach.* The Levi–Civita connection lets us take the directional derivative of a tangent vector field and get another tangent vector field. We generalize this concept for vector bundles as follows: Let $X$ be a tangent vector and let $s$ be a section to $E$. A connection $\nabla$ is a rule $\nabla_X(s)$ for taking the directional derivative of $s$ in the direction $X$ and getting another section to $E$. The assignment of a connection $\nabla$ in a general vector bundle $E$ provides a rule for the parallel transport of sections.

Let $x(t)$ be a curve in $M$; we say that $s(t)$ is parallel-transported along $x$ if $s$ satisfies the differential equation

$$\nabla_{\dot{x}}(s) = 0. \tag{5.2}$$

There always exists a unique solution to this equation for given initial conditions. The generalized Christoffel symbols $\Gamma^j{}_{\mu i}$ giving the action of a connection $\nabla$ on a frame of the bundle $E$ are defined by

$$\nabla_{\partial/\partial x^\mu}(e_i) = e_j \Gamma^j{}_{\mu i}.$$

We recall that we may associate the operator $\mathrm{d}/\mathrm{d}t$ with $\dot{x}^\mu$ because

$$\mathrm{d}f(x)/\mathrm{d}t = \dot{x}^\mu\, \partial f/\partial x^\mu .$$

In terms of the Christoffel symbols, the parallel transport equation takes the form

$$\begin{aligned}\nabla_{\dot{x}}(s) &= \nabla_{\mathrm{d}/\mathrm{d}t}(e_i z^i) = \nabla_{\mathrm{d}/\mathrm{d}t}(e_i)z^i + e_j \dot{z}^j \\ &= \dot{x}^\mu(\nabla_{\partial/\partial x^\mu}(e_i)z^i + e_j\,\partial_\mu z^j) \\ &= \dot{x}^\mu e_j(\Gamma^j{}_{\mu i}z^i + \partial_\mu z^j) = 0.\end{aligned}$$

*Note*: we have implicitly made use of various properties of $\nabla_X(s)$ which we will formalize later.

(2) *Tangent space approach.* Parallel transport along a curve $x(t)$ lets us compare the fibers of the bundle $E$ at different points of the curve. Thus it becomes natural to think of *lifting* a curve $x(t)$ in $M$ to a curve

$$c(t) = (x^\mu(t), z^i(t))$$

in the bundle. Differentiation along $c(t)$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{z}^i \frac{\partial}{\partial z^i} \quad ,$$

where $\dot{z}^i$ is given by solving the parallel transport equation:

$$\dot{z}^i + \Gamma^i{}_{\mu j}\dot{x}^\mu z^j = 0. \tag{5.3}$$

Thus we may write

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{x}^\mu\left(\frac{\partial}{\partial x^\mu} - \Gamma^i{}_{\mu j}z^j\frac{\partial}{\partial z^i}\right) = \dot{x}^\mu \mathrm{D}_\mu \quad ,$$

where

$$\mathrm{D}_\mu = \frac{\partial}{\partial x^\mu} - \Gamma^i{}_{\mu j}z^j\frac{\partial}{\partial z^i} \tag{5.4}$$

is the operator in $T(E)$ known as the *covariant derivative.*

We are thus led to define a splitting of $T(E)$ at $x \in U$ into *vertical* component $V(E)$ with basis $\{\partial/\partial z^i\}$ lying strictly in the fiber and a *horizontal* component $H(E)$ with basis $\{\mathrm{D}_\mu\}$:

$$T_x(E) = V_x(E) \oplus H_x(E)$$

$$\text{basis} = \left(\frac{\partial}{\partial z^i}, \mathrm{D}_\mu\right).$$

This splitting is illustrated schematically in fig. 5.3.

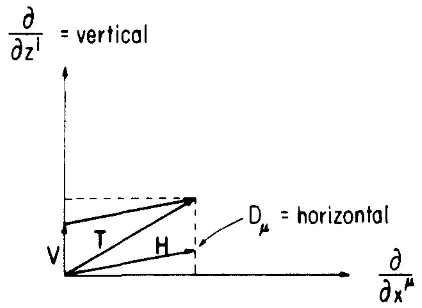


Fig. 5.3. Splitting the tangent space $T(E)$ of the bundle into vertical and horizontal components.

(3) *Cotangent space approach.* In the cotangent space approach, one considers a vector-valued one-form

$$\omega^i = \mathrm{d}z^i + \Gamma^i{}_{\mu j}\,\mathrm{d}x^\mu\, z^j \tag{5.5}$$

in $T^*(E)$ which is essentially the deviation from the parallel transport law given above. We observe that $\omega^i$ is the unique non-trivial solution to the conditions

$$\begin{aligned} &\langle \omega^i, \mathrm{D}_\mu \rangle = 0. \\ &\langle \omega^i, \partial/\partial z^j \rangle = \delta_{ij}. \end{aligned} \tag{5.6}$$

Conversely, these conditions determine $\mathrm{D}_\mu$ if $\omega^i$ is given. The connection one-form $\omega^i$ annihilates the horizontal subspace of $T(E)$, and is, in some sense, dual to it.

We now introduce the matrix-valued connection one-form $\Gamma$, where

$$\Gamma^i{}_j = \Gamma^i{}_{\mu j}\,\mathrm{d}x^\mu.$$

The *total covariant derivative* $\nabla(s)$ is defined by

$$\nabla(s) = e_i \otimes \mathrm{d}z^i(x) + e_i \otimes \Gamma^i{}_j z^j(x) \tag{5.7}$$

which maps $C^\infty(E)$ to $C^\infty(E \otimes T^*(M))$. Note that this is the pullback to $M$ (using the section $z^i(x)$) of a covariant derivative in the *bundle* given by

$$\nabla(Z) = e_i \otimes \omega^i, \qquad (Z = e_i z^i \in \pi^{-1}(U)),$$

where $\omega^i$ belongs to $T^*(E)$ rather than $T^*(M)$. The total covariant derivative contains all the directional derivatives at the same time in the same way that $\mathrm{d}f = (\partial f/\partial x^\mu)\,\mathrm{d}x^\mu$ contains all the partial derivatives of $f$.

(4) *Axiomatic approach.* We began this section by discussing covariant differentiation as a directional derivative. We now formalize the properties of covariant differentiation that we have been using implicitly in the previous approaches. The axiomatic properties of the connection $\nabla_X(s)$ are

1. Linearity in $s$:

$$\nabla_X(s+s')=\nabla_X(s)+\nabla_X(s')$$

2. Linearity in $X$:

$$\nabla_{X+X'}(s)=\nabla_X(s)+\nabla_{X'}(s)$$

3. Behaves like a first-order differential operator:

$$\nabla_X(sf)=s\cdot X(f)+(\nabla_X(s))f$$

4. Tensoriality in $X$:

$$\nabla_{fX}(s)=f\nabla_X(s)$$

where $s(x)$ is a section to $E$, $X$ is vector field on $M$ and $f(x)$ is a scalar function. These are clearly desirable properties which are straightforward generalizations of the features of the Levi–Civita connection.

The axiomatic properties of the *total* covariant derivative $\nabla$ are:

1. Linearity in $s$:

$$\nabla(s+s')=\nabla(s)+\nabla(s')$$

2. Behaves like a first-order differential operator:

$$\nabla(sf)=s\otimes \mathrm{d}f+\nabla(s)f.$$

The relationship between these two differential operators is given by

$$\begin{aligned}&1.\ \nabla(s)=\nabla_{\partial/\partial x^\mu}(s)\otimes \mathrm{d}x^\mu\\ &2.\ \nabla_X(s)=\langle\nabla(s),X\rangle,\end{aligned}\tag{5.8}$$

where $X\in C^\infty(T(M))$ and $\nabla(s)\in C^\infty(E\otimes T^*(M))$.

One can extend total covariant differentiation to $p$-form-valued sections of $E$ by the rule

$$\nabla(s\otimes\theta)=\nabla(s)\wedge\theta+s\otimes \mathrm{d}\theta\tag{5.9}$$

where $s\in C^\infty(E)$ and $\theta\in C^\infty(\Lambda^p(M))$. $\nabla$ thus extends to a differential operator with the following domain and range:

$$\nabla\colon C^\infty(E\otimes\Lambda^p(M))\to C^\infty(E\otimes\Lambda^{p+1}(M)).$$

(5) *Change of frame approach.* Under a change of frame,

$$e'_j = e_i \Phi^{-1}_{ij}(x), \qquad z'^i = \Phi_{ij}(x) z^j,$$

and sections are invariant:

$$s(x) = e_i z^i = e'_i z'^i = s'(x).$$

We see that

$$\nabla(e'_j) = \nabla(e_i) \otimes \Phi^{-1}_{ij} + e_i \otimes \mathrm{d}\Phi^{-1}_{ij} = e'_i \Gamma^i{}_j{}'$$

where

$$\Gamma'^i{}_j = \Phi_{ik} \Gamma^k{}_l \Phi^{-1}_{lj} + \Phi_{ik}\, \mathrm{d}\Phi^{-1}_{kj}, \tag{5.10}$$

so the connection 1-form $\Gamma^i{}_j$ transforms as a gauge field rather than as a tensor. We may in fact *define* a connection as a collection of one-forms $\Gamma^i{}_j$ obeying the transformation law (5.10).

Using eq. (5.10), we can check that $\nabla$ is independent of the choice of frame and is thus well-defined in the overlap region $U \cap U'$. We find

$$\nabla(s) = \nabla(e_i z^i) = e_j \otimes \Gamma^j{}_i z^i + e_j \otimes \mathrm{d}z^j = e'_j \otimes \Gamma'^j{}_i z'^i + e'_j \otimes \mathrm{d}z'^j.$$

### 5.2. *Curvature*

The curvature of a fiber bundle characterizes its geometry. It can be calculated in several different equivalent ways corresponding to the different approaches to the connection.

(1) *Parallel transport.* Curvature measures the extent to which parallel transport is path-dependent. If the curvature is zero and $x(t)$ is a path lying in a coordinate ball of $M$, then the result of parallel transport is always the identity transformation (this need not be true if the path encloses a hole, as we shall see later when we discuss locally flat bundles). For curved manifolds, we get non-trivial results: parallel transport around a geodesic triangle on $S^2$ gives a rotation equal to the area of the spherical triangle.

A quantitative measure of the curvature can be calculated using parallel transport as follows: Let $(x^1, x^2, \ldots)$ be a local coordinate chart and take a square path $x(t)$ with vertices, say, in the 1–2 plane. Let $H_{ij}(\tau)$ be the holonomy matrix obtained by traversing the path with vertices $(0, 0, 0, \ldots)$, $(0, \tau^{1/2}, 0, \ldots)$, $(\tau^{1/2}, \tau^{1/2}, 0, \ldots)$, $(\tau^{1/2}, 0, 0, \ldots)$. Then the curvature matrix in the 1–2 plane is

$$R_{ij}(1, 2) = \frac{\mathrm{d}}{\mathrm{d}\tau} H_{ij}(\tau)\big|_{\tau=0}. \tag{5.11}$$

The correspondence between this curvature and those to be introduced below may be found by expanding the connection in Taylor series.

(2) *Tangent space.* The curvature is defined as the commutator of the components $\mathrm{D}_\mu$ of the basis for

the horizontal subspace of $T(E)$,

$$[D_\mu, D_\nu] = -R^i{}_{j\mu\nu} z^j \, \partial/\partial z^i, \tag{5.12}$$

where $R^i{}_{j\mu\nu}$ can be expressed in terms of Christoffel symbols as

$$R^i{}_{j\mu\nu} = \partial_\mu \Gamma^i{}_{\nu j} - \partial_\nu \Gamma^i{}_{\mu j} + \Gamma^i{}_{\mu k} \Gamma^k{}_{\nu j} - \Gamma^i{}_{\nu k} \Gamma^k{}_{\mu j}.$$

Note that the right-hand side of eq. (5.12) has only *vertical* components. $R^i{}_{j\mu\nu}$ is interpretable as the obstruction to integrability of the horizontal subspace.

(3) *Cotangent space.* In this approach, the curvature appears as a matrix-valued 2-form

$$R^i{}_j = \mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j = \tfrac{1}{2} R^i{}_{j\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{5.13}$$

We observe that $R^i{}_j z^j$ is the covariant differential of the one-form $\omega^i \in T^*(E)$:

$$R^i{}_j z^j = \mathrm{d}\omega^i + \Gamma^i{}_j \wedge \omega^j.$$

Note that although $\omega^i$ has $\mathrm{d}z^k$ components, they cancel out in $R^i{}_j$.

(4) *Axiomatic formulation.* Curvature measures the extent to which covariant differentiation fails to commute. We define the *curvature operator* as

$$R(X, Y)(s) = \nabla_X \nabla_Y (s) - \nabla_Y \nabla_X (s) - \nabla_{[X,Y]}(s), \tag{5.14}$$

where

$$R\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)(e_i) = e_j R^j{}_{i\mu\nu}.$$

The axiomatic properties of the curvature operator are

1. Multilinearity:

$$R(X + X', Y)(s) = R(X, Y)(s) + R(X', Y)(s)$$

2. Anti-symmetry:

$$R(X, Y)(s) = -R(Y, X)(s)$$

3. Tensoriality:

$$\begin{aligned} R(fX, Y)(s) &= R(X, fY)(s) = R(X, Y)(fs) \\ &= fR(X, Y)(s) \end{aligned}$$

where $X$ and $Y$ are vector fields, $s(x)$ is a section and $f(x)$ is a scalar function.

The *total curvature* $R$ is a matrix-valued 2-form given by

$$\begin{aligned} R(s) &= \nabla^2(s) = \nabla(e_j \otimes \Gamma^j{}_i z^i + e_j \otimes \mathrm{d}z^j) \\ &= e_k \otimes \Gamma^k{}_j \wedge \Gamma^j{}_i z^i + e_k \otimes (\mathrm{d}\Gamma^k{}_i z^i - \Gamma^k{}_j \wedge \mathrm{d}z^j) + e_k \otimes \Gamma^k{}_j \wedge \mathrm{d}z^j + 0 \\ &= e_k \otimes R^k{}_i z^i . \end{aligned} \tag{5.15}$$

The matrix $R = \|R^i{}_j\|$ is also given by

$$R = \tfrac{1}{2} R\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

acting on a section $s$. The axiomatic property of $R$ is just the statement that it is a 2-form valued linear map from $E \to E$.

(5) *Change of frame.* By using (5.13), we find that $R^i{}_j$ transforms as

$$R'^i{}_j = \Phi^i{}_k R^k{}_l (\Phi^{-1})^l{}_j$$

under the change of frame (4.6) and (5.10). Hence by (5.15) $R(s)$ is in fact invariant under a change of frame.

The curvature can be regarded as an obstruction to finding locally flat (i.e., covariant constant) frames. Given $e_i$, let us attempt to find a new frame $e'_i = e_j \Phi^{-1}_{ji}$ which is locally flat. If we set $\nabla(e'_i) = 0$, we find the matrix differential equation

$$\Phi \Gamma \Phi^{-1} + \Phi \,\mathrm{d}\Phi^{-1} = 0.$$

This equation is solved if $\Gamma$ is a pure gauge,

$$\Gamma^i{}_j = -(\mathrm{d}\Phi^{-1})^i{}_k \,\Phi^k{}_j = (\Phi^{-1})^i{}_k \,\mathrm{d}\Phi^k{}_j .$$

If $\Gamma$ obeys this equation, the curvature vanishes. Conversely, by the Frobenius theorem, if the curvature vanishes, $\Gamma$ can be written as a pure gauge.

### *5.3. Torsion and connections on the tangent bundle*

One advantage of the cotangent space formulation (5.7) of the vector bundle connection $\nabla$ is that it is independent of the coordinate system $\{x^\mu\}$ on $M$. Furthermore, multiple covariant differentiation of an invariant one-form such as $p_\mu \,\mathrm{d}x^\mu$ is independent of the connection chosen on the cotangent bundle $T^*(M)$. However, if we choose to differentiate the individual tensor components $z^i{}_{;\mu}$ of the covariant derivative of a section $s(x) = e_i z^i(x)$ of a vector bundle, we must specify in addition a connection on $T^*(M)$ to treat the "$\mu$" index. (We will show in the next section that connections on $T(M)$ give natural connections on $T^*(M)$, and vice-versa.) *Torsion* is a property of the connection on the tangent bundle which must be introduced when we examine the double covariant derivative. We have already encountered torsion in section 3 when we studied metric geometry on Riemannian manifolds. Here we extend the notion to general vector bundles.

Let $\{\Gamma^i{}_{\mu j}\}$ be the Christoffel symbols on the vector bundle $E$, and let $\{\gamma^\nu{}_{\mu\lambda}\}$ be the Christoffel symbols on $T(M)$. We define the components of the double covariant derivative of a section $s(x) = e_i z^i(x)$ as

$$z^i{}_{;\mu;\nu} = \partial_\nu(\partial_\mu z^i + \Gamma^i{}_{\mu j} z^j) + \Gamma^i{}_{\nu j}(\partial_\mu z^j + \Gamma^j{}_{\mu k} z^k) - \gamma^\lambda{}_{\mu\nu}(\partial_\lambda z^i + \Gamma^i{}_{\lambda j} z^j).$$

(The sign in front of $\gamma^\lambda{}_{\mu\nu}$ follows from the requirement for lowering indices to get the connection on $T^*(M)$.) The commutator of double covariant differentiation on a section yields the formula

$$z^i{}_{;\mu;\nu} - z^i{}_{;\nu;\mu} = -R^i{}_{j\mu\nu} z^j - T^\lambda{}_{\mu\nu} z^i{}_{;\lambda}, \tag{5.16}$$

where we have introduced a new tensor, the *torsion*,

$$T^\lambda{}_{\mu\nu} = \gamma^\lambda{}_{\mu\nu} - \gamma^\lambda{}_{\nu\mu}.$$

Multiple covariant differentiation can be written schematically in the form

$$C^\infty(E) \xrightarrow{\nabla} C^\infty(E \otimes T^*(M)) \xrightarrow{\nabla} C^\infty(E \otimes T^*(M) \otimes T^*(M)),$$

which again emphasizes the requirement for a connection on $T^*(M)$, or equivalently on $T(M)$.
*Note*: We remark that the multiple covariant derivative treated here is *not* the operator $\nabla^2$ used to define the curvature 2-form, since $\nabla^2$ is independent of the connection on $T^*(M)$ and has values in $C^\infty(E \otimes \Lambda^2(T^*(M)))$.

*Axiomatic approach to torsion*: We define the torsion operator on $T(M)$ by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

This is a vector field with components

$$T\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = (\gamma^\lambda{}_{\mu\nu} - \gamma^\lambda{}_{\nu\mu}) \frac{\partial}{\partial x^\lambda}.$$

*Levi-Civita connection*: Once a metric $(X, Y) = g_{\mu\nu} x^\mu y^\nu$ has been chosen, the Levi-Civita connection on $T(M)$ is uniquely defined by the properties

$$\begin{aligned} &1.\ \text{Torsion-free:} \quad T(X, Y) = 0 \\ &2.\ \text{Covariant constancy of metric:} \quad \mathrm{d}(X, Y) = (\nabla X, Y) + (X, \nabla Y). \end{aligned} \tag{5.17}$$

These conditions were discussed in detail in section 3.

### 5.4. Connections on related bundles

*Dual bundles*: If $E$ and $E^*$ are dual vector bundles with dual frame bases $\{e_i\}$ and $\{e^{*i}\}$, the connection

$\nabla^*$ on $E^*$ is defined by the requirement that the natural inner product between sections $s$ and $s^*$ be differentiated according to the following rule:

$$\mathrm{d}\langle s, s^*\rangle = \langle \nabla(s), s^*\rangle + \langle s, \nabla^*(s^*)\rangle.$$

In other words,

$$\nabla(e_i) = e_j \Gamma^j{}_{\mu i}\, \mathrm{d}x^\mu$$

$$\nabla^*(e^{*i}) = -e^{*j}\Gamma^i{}_{\mu j}\, \mathrm{d}x^\mu.$$

If $E$ has a fiber metric, we may identify $E$ with $E^*$ using a conjugate linear isomorphism. The connection $\nabla$ is said to be *Riemannian* if $\nabla = \nabla^*$, i.e.,

$$\Gamma^i{}_{\mu j} = -\Gamma^j{}_{\mu i} \tag{5.18}$$

relative to an orthonormal frame basis. The curvature of a Riemannian connection relative to an orthonormal frame basis is anti-symmetric:

$$R^i{}_j = -R^j{}_i. \tag{5.19}$$

The Levi–Civita connection on $T(M)$ is the unique torsion-free Riemannian connection.

*Whitney sum bundle*: If $E$ and $F$ are vector bundles with connections $\nabla$ and $\nabla'$, there is a natural connection $\nabla \oplus \nabla'$ defined on $E \oplus F$ by the following rule:

$$(\nabla \oplus \nabla')(s \oplus s') = \nabla(s) \oplus \nabla'(s').$$

In other words,

$$(\nabla \oplus \nabla')(e_i \oplus f_j) = e_k \otimes \Gamma^k{}_{\mu i}\, \mathrm{d}x^\mu \oplus f_l \otimes \Gamma'^l{}_{\mu j}\, \mathrm{d}x^\mu. \tag{5.20}$$

The curvature is given by the direct sum of the curvatures of $E$ and $F$.

*Tensor product bundle*: There is a natural connection $\nabla''$ defined on $E \otimes F$ by the following rule:

$$\nabla''(s \otimes s') = (\nabla \otimes 1 + 1 \otimes \nabla')(s \otimes s') = \nabla(s) \otimes s' + s \otimes \nabla'(s').$$

The curvature of $\nabla''$ is given by

$$R'' = R \otimes 1 + 1 \otimes R'. \tag{5.21}$$

*Pullback bundle*: Let $f$: $M \to M'$ and let $\nabla'$ be a connection on the vector bundle $E'$ over $M'$. There is a natural pullback connection $\nabla = f^*\nabla'$ with Christoffel symbols which are the pullback of the Christoffel symbols of $\nabla'$, that is:

$$\Gamma^i{}_{\mu j} = \Gamma'^i{}_{\alpha j} \frac{\partial x'^\alpha}{\partial x^\mu}.$$

The curvature of $\nabla$ is the pullback of the curvature of $\nabla'$:

$$R^i{}_{j\mu\nu} = \tfrac{1}{2} R'^i{}_{j\alpha\beta} \left( \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} - \frac{\partial x'^\alpha}{\partial x^\nu} \frac{\partial x'^\beta}{\partial x^\mu} \right).$$

*Projected connections*: Let $E$ be a sub-bundle of $F$ and let $\pi$: $F \to E$ be a projection. If $\nabla$ is a connection on $F$, we can define the projected connection $\nabla^\pi$ on $E$ by

$$\nabla^\pi(s) = \pi(\nabla(s))$$

where $s$ is a section of $F$ belonging to the *sub-bundle* $E$. Note that the curvature of $\nabla^\pi$ may be non-trivial even if the curvature of $\nabla$ is zero. (Our introductory example deriving the Levi–Civita connection on $S^2$ embedded in $\mathbb{R}^3$ was in fact of this type.)

If $\pi$ is an orthogonal projection relative to some fiber metric and $\nabla$ is Riemannian, then $\nabla^\pi$ is Riemannian.

*Examples 5.4*

1. *Complex line bundle of* $P_1(\mathbb{C})$. Let $L$ be the line bundle over $P_1(\mathbb{C})$ defined in example 4.2.2. This is a natural sub-bundle of $P_1(\mathbb{C}) \times \mathbb{C}^2$. We denote a point of the bundle $L$ by $(x; z_0, z_1)$, where $(z_0, z_1)$ lie on the line in $\mathbb{C}^2$ corresponding to the point $x$ in $P_1(\mathbb{C})$. The natural fiber metric on $L$ is given by

$$((x; z_0, z_1), (x; w_0, w_1)) = z_0 \bar{w}_0 + z_1 \bar{w}_1.$$

(This is induced by the canonical metric on $\mathbb{C}^2$.)

Now let

$$h(x; z_0, z_1) = |z_0|^2 + |z_1|^2$$

be the length of a point in $L$ and form a connection $\omega$ lying in $T^*(L)$ given by

$$\omega = h^{-1} \, \partial h = \frac{\bar{z}_0 \, \mathrm{d}z_0 + \bar{z}_1 \, \mathrm{d}z_1}{|z_0|^2 + |z_1|^2}.$$

The curvature then is

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega = (\partial + \bar{\partial})(h^{-1} \, \partial h) + 0 = -\partial \bar{\partial} \ln h.$$

In order to carry out practical computations, we choose a gauge (that is a local section of $L$) with coordinates $(x; \zeta_0^{(1)}, 1)$.

Here

$$\zeta_0^{(1)} = z_0/z_1 = u + \mathrm{i}v$$

for $u, v \in \mathbb{R}$. Then we compute

$$h = 1 + u^2 + v^2$$

$$\Omega = -\partial\bar{\partial}\ln(1 + u^2 + v^2) = \frac{2\mathrm{i}\,\mathrm{d}u \wedge \mathrm{d}v}{(1 + u^2 + v^2)^2} .$$

We recognize this from section 3.4 on Kähler manifolds as (2i) times the Kähler form for $S^2 = P_1(\mathbb{C})$. We thus can read off the metric directly from $\Omega$.

*Remark 1*: In some sense $\omega = h^{-1}\,\partial h$ is a "pure gauge" with respect to a curvature involving only $\partial$. We find non-trivial full curvature because $\Omega$ involves $\mathrm{d} = (\partial + \bar{\partial})$.

*Remark 2*: The Fubini–Study metric on $P_n(\mathbb{C})$ can be defined in this same manner by taking

$$h(x; z_0, z_1, \ldots, z_n) = \sum_{i=0}^{n} |z_i|^2.$$

*Remark 3*: The same construction works for an arbitrary holomorphic line bundle over an arbitrary complex manifold once a fiber metric is chosen.

2. *Vector bundles over $S^n$*. If we let $n = 2l$, the trivial bundle $S^n \times \mathbb{C}^{2^l}$ can be split into a sum of non-trivial bundles $E_\pm$ by constructing a projection operator $\Pi_\pm$: $S^n \times \mathbb{C}^{2^l} \to E_\pm$. To accomplish this, we embed $S^n$ in $\mathbb{R}^{n+1}$ using coordinates $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ and consider the set of $2^l \times 2^l$ self-adjoint complex matrices $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ obeying

$$\lambda_i\lambda_j + \lambda_j\lambda_i = 2\delta_{ij}$$

$$\lambda_0\lambda_1 \ldots \lambda_n = \mathrm{i}^l I$$

where $I$ is the identity matrix. The $\{\lambda_i\}$ are Pauli matrices ($\lambda_0 = \tau_3, \lambda_1 = \tau_1, \lambda_2 = \tau_2$) for $l = 1$ and Dirac matrices ($\lambda_0 = \gamma_5, \lambda_1 = \gamma_1, \lambda_2 = \gamma_2, \lambda_3 = \gamma_3, \lambda_4 = \gamma_4$) for $l = 2$. We now define the complex matrix

$$\lambda(x) = \sum_{j=0}^{n} x_j\lambda_j$$

with $\{x_j\}$ lying on $S^n$, so that

$$\lambda^2(x) = I.$$

$\lambda(x)$ is a map from $\mathbb{C}^{2^l}$ to $\mathbb{C}^{2^l}$ which depends on the point $x$ of the base space $S^n$. Since $\lambda^2(x) = I$, we may decompose its action on vectors $z \in \mathbb{C}^{2^l}$ into the two eigenspaces with eigenvalues $\pm 1$,

$$\lambda(x) \cdot z = \pm z.$$

We then choose as our projection the matrix

$$\Pi_{\pm}(x) = \tfrac{1}{2}(1 \pm \lambda(x))$$

which selects the $2^{l-1}$ dimensional vector space in $\mathbb{C}^{2^l}$ with $\lambda \cdot z = \pm z$.

We denote by $E_{\pm}$ the complex vector bundles over $S^n$ whose fibers at each point $x \in S^n$ are defined by the action of $\pi_{\pm}(x)$. If $l = 1$, we obtain complex line bundles over $S^2$. Clearly

$$E_+ \oplus E_- = S^n \times \mathbb{C}^{2^l}.$$

We choose as our connections on $E_{\pm}$ the projection $\nabla_{\pm}$ of the flat connection $\nabla$ acting on a section of $E_{\pm}$. To carry out this procedure, we choose a constant frame $e^0_+$ of $E_+$ at a point $x_0$ and generalize it to arbitrary $x$ using the projection;

$$e_+(x) = \Pi_+(x) e^0_+$$

is a frame of $E_+$ everywhere. Since $\Pi_+(x_0)\, e_+(x_0) = \Pi_+(x_0)\, e^0_+ \equiv e_+(x_0)$, we may take $e_+(x_0) = e^0_+$. The flat connection just acts by exterior differentiation, $\nabla(e) = \mathrm{d}e$; while the projected connection is difficult to calculate in general, it can be evaluated at $x_0$ as the projection of the flat connection since $e_{\pm}(x_0) = e^0_{\pm}$,

$$\nabla_{\pm}(e_{\pm})|_{x_0} = \Pi_{\pm}\, \mathrm{d}e_{\pm}|_{x_0} = (\Pi_{\pm}\, \mathrm{d}\Pi_{\pm})|_{x_0} e^0{}_{\pm}.$$

The curvature is obtained in a similar way;

$$(\nabla_{\pm})^2 (e_{\pm})|_{x_0} = \Pi_{\pm}\, \mathrm{d}(\Pi_{\pm}\, \mathrm{d}\Pi_{\pm}\, \mathrm{e}^0{}_{\pm}) = \Pi_{\pm}\, \mathrm{d}\Pi_{\pm} \wedge \mathrm{d}\Pi_{\pm}\, e^0{}_{\pm}.$$

Hence the curvature 2-form at $x_0$ is

$$\Omega_{\pm}(x_0) = \Pi_{\pm}(x_0)\, \mathrm{d}\Pi_{\pm}(x_0) \wedge \mathrm{d}\Pi_{\pm}(x_0).$$

*Remark 1*: Note that although the connection and curvature matrices used here are double the correct dimension, all traces of products of these matrices involve only the meaningful portion of the matrices. The rank of the matrices equals the fiber dimension.

*Remark 2*: To evaluate an invariant polynomial of $\Omega_{\pm}$, it in fact suffices to perform the calculation at $x_0$ alone. One may thus show that

$$\mathrm{Tr}(\Omega_{\pm}{}^l) = \frac{n!(2\mathrm{i})^l}{2^{n+1}}\, \mathrm{d(vol)},$$

where d(vol) is the $n$-form volume element of $S^n$. This formula will be used later to examine the characteristic classes of this bundle.

*Remark 3*: If $l = 1$, the associated principal bundles to $E_{\pm}$ describe the Dirac magnetic monopole.

### *5.5. Connections on principal bundles*

We recall that a principal bundle $P$ is a fiber bundle whose fiber and transition functions both belong to the same matrix group. The gauge potentials of Maxwell's theory of electromagnetism and Yang–Mills gauge theories are identifiable with connections on principal bundles. Here we give a brief treatment of the special aspects of connections on principal bundles.

*Maurer–Cartan forms and the Lie algebra*: We let $G$ be a matrix group and $\mathcal{G}$ be its Lie algebra. The Maurer–Cartan form $g^{-1}\,\mathrm{d}g$ is a matrix of one-forms belonging to the Lie algebra $\mathcal{G}$. This form is invariant under the left action by a constant group element $g_0$,

$$(g_0 g)^{-1}\,\mathrm{d}(g_0 g) = g^{-1}\,\mathrm{d}g.$$

Let $\{\Phi_a\}$ be a basis for the left-invariant one-forms. We then express the Cartan–Maurer form as

$$g^{-1}\,\mathrm{d}g = \Phi_a \frac{\lambda_a}{2\mathrm{i}}, \tag{5.22}$$

where $\lambda_a/2\mathrm{i}$ is a constant matrix in $\mathcal{G}$. Since $\mathrm{d}(g^{-1}\,\mathrm{d}g) + g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g = 0$, we find that $\Phi_a$ obeys the *Maurer–Cartan equations*

$$\mathrm{d}\Phi_a + \tfrac{1}{2} f_{abc}\Phi_b \wedge \Phi_c = 0, \tag{5.23}$$

where the $f_{abc}$ are the structure constants of $\mathcal{G}$.

The *dual* of $\Phi_a$ is the differential operator

$$L_a = \mathrm{Tr}\left(g\,\frac{\lambda_a}{2\mathrm{i}}\,\frac{\partial}{\partial g^{\mathrm{T}}}\right) = \frac{1}{2\mathrm{i}}\, g_{jk}[\lambda_a]_{kl}\,\frac{\partial}{\partial g_{jl}}$$

obeying

$$\langle \Phi_a, L_b\rangle = \delta_{ab}, \qquad [L_a, L_b] = f_{abc}L_c. \tag{5.24}$$

$\{L_a\}$ is a left-invariant basis for the tangent space of $G$.

The corresponding *right* invariant objects are defined by

$$\mathrm{d}g\, g^{-1} = \frac{\lambda_a}{2\mathrm{i}}\,\bar{\Phi}_a, \qquad \bar{L}_a = \mathrm{Tr}\left(\frac{\lambda_a}{2\mathrm{i}}\, g\,\frac{\partial}{\partial g^{\mathrm{T}}}\right) \tag{5.25}$$

where

$$\mathrm{d}\bar{\Phi}_a - \tfrac{1}{2} f_{abc}\bar{\Phi}_b \wedge \bar{\Phi}_c = 0$$

$$\langle \bar{\Phi}_a, \bar{L}_b\rangle = \delta_{ab}, \qquad [\bar{L}_a, \bar{L}_b] = -f_{abc}\bar{L}_c. \tag{5.26}$$

That is, all structure equations have a reversed sign. Note that $L_a$ and $\bar{L}_b$ *commute*:

$$[L_a, \bar{L}_b] = 0. \tag{5.27}$$

$L_a$ and $\bar{L}_b$ generalize the familiar physical distinction between the space-fixed and body-fixed rotation generators of a quantum-mechanical top.

*Parallel transport*: Let $P$ be a principal bundle. If we choose a local trivialization, then we have coordinates $(x, g)$ for $P$, where $g \in G$. A local section of $P$ is a smooth map from a neighborhood $U$ to $G$. The assignment of a connection on a principal bundle provides a rule for the parallel transport of sections. A connection $A$ of a principal fiber bundle is a Lie-algebra valued matrix of 1-forms in $T^*(M)$,

$$A(x) = A^a{}_\mu(x) \frac{\lambda_a}{2\mathrm{i}} \, \mathrm{d}x^\mu. \tag{5.28}$$

If $x(t)$ is a curve in $M$, the section $g_{ij}(t)$ is defined to be parallel-transported along $x$ if the following differential equation is satisfied:

$$\dot{g}_{ik} + A_{\mu ij}(x) \, \dot{x}^\mu \, g_{jk} = 0, \tag{5.29}$$

where $A_\mu$ is the connection on $P$. We may rewrite this as:

$$g^{-1} \frac{\mathrm{d}g}{\mathrm{d}t} + g^{-1}\left(A^a{}_\mu(x) \frac{\lambda_a}{2\mathrm{i}} \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right) g = 0.$$

*Tangent space approach*: Parallel transport along a curve $x(t)$ lets us compare the fibers of $P$ at different points of the curve. In analogy to the methods used for vector bundle connections, we may lift curves $x(t)$ in $M$ to curves in $P$. We define differentiation along the lifted curve by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{g}_{ij} \frac{\partial}{\partial g_{ij}} = \dot{x}^\mu \left(\frac{\partial}{\partial x^\mu} - A^a{}_\mu(x) \frac{(\lambda^a)_{ik}}{2\mathrm{i}} g_{kj} \frac{\partial}{\partial g_{ij}}\right) = \dot{x}^\mu \left(\frac{\partial}{\partial x^\mu} - A^a_\mu(x) \bar{L}_a\right)$$

where we have used the parallel transport equation for $\dot{g}_{ij}$. Now the covariant derivative is defined as

$$\mathrm{D}_\mu = \frac{\partial}{\partial x^\mu} - A^a_\mu(x) \bar{L}_a. \tag{5.30}$$

We are thus led to define a splitting of $T(P)$ into horizontal component $H(P)$ with basis $\mathrm{D}_\mu$, and a vertical component $V(P)$ lying in $T(G)$:

$$T(P) = H(P) \oplus V(P).$$

This splitting is invariant under right multiplication by the group.

The curvature is defined by

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] = -F^a_{\mu\nu} \bar{L}_a,$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc} A^b_\mu A^c_\nu. \tag{5.31}$$

As expected, the commutator of covariant derivatives has only vertical components.

*Cotangent space approach*: We may regard the connection on $P$ as a $\mathcal{G}$-valued one-form $\omega$ in $T^*(P)$ whose vertical component is the Maurer–Cartan form $g^{-1}\,\mathrm{d}g$. In local coordinates, we may write

$$\omega = g^{-1}Ag + g^{-1}\,\mathrm{d}g,$$

where $A(x) = A_\mu^a(x)\,(\lambda_a/2\mathrm{i})\,\mathrm{d}x^\mu$. We observe that, as in the vector bundle case, $\omega$ annihilates the horizontal basis of $T(P)$ and is constant on the vertical basis:

$$\langle \omega, \mathrm{D}_\mu \rangle = 0, \qquad \langle \omega, L_a \rangle = \lambda_a/2\mathrm{i}. \tag{5.32}$$

Under the right action of the group, $g \to gg_0$, $A$ remains invariant and $\omega$ transforms tensorially,

$$\omega \to g_0^{-1}\omega g_0.$$

The curvature in this approach is a Lie-algebra valued matrix 2-form defined by

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega = g^{-1}Fg \tag{5.33}$$

where

$$F = \mathrm{d}A + A \wedge A = \tfrac{1}{2}F_{\mu\nu}{}^a \frac{\lambda_a}{2\mathrm{i}}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

$\Omega$ obeys the Bianchi identity,

$$\mathrm{d}\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0. \tag{5.34}$$

Note that $\Omega$ has no vertical components. It transforms tensorially under right action,

$$\Omega \to g_0^{-1}\Omega g_0.$$

*Gauge transformation*: The transition functions of a principal bundle act on fibers by left multiplication. Let us consider two overlapping neighborhoods $U$ and $U'$ and a transition function $\Phi_{U'U} = \Phi$. The local fiber coordinates $g$ and $g'$ in $U$ and $U'$ are related by

$$g' = \Phi g.$$

Then, in order for the connection 1-form $\omega$ to be well-defined in the overlapping region $U \cap U'$, $A$ must transform as

$$A' = \Phi A \Phi^{-1} + \Phi\,\mathrm{d}\Phi^{-1}. \tag{5.35}$$

We verify that

$$\omega = g^{-1}Ag + g^{-1}\,\mathrm{d}g = g'^{-1}A'g' + g^{-1}\,\mathrm{d}g',$$

so $\omega$ is indeed well-defined in $T^*(P)$. The transformation (5.35) is the *gauge transformation* of $A$. Using

(5.31), we find the gauge transformation of $F$ to be

$$F' = \Phi F \Phi^{-1}.$$

It is easy to check that the curvature 2-form $\Omega$ is also consistently defined over the manifold,

$$\Omega = g^{-1}Fg = g'^{-1}F'g'.$$

*Pullback to base space*: By choosing a section $g = g(x)$, one can pull back $\omega$ and $\Omega$ to the base space. $A$ and $F$ are equivalent to the pullbacks $g^*\omega$ and $g^*\Omega$, which are sometimes denoted simply as $\omega$ and $\Omega$. Gauge transformations of $A$ and $F$ correspond to changes of the section.

In the theory of gauge fields, the structure group $G$ is called the *gauge group*: the choice $G = U(1)$, for instance, gives the theory of electricity and magnetism and $G = \mathrm{SU}(3)$ gives the color theory of strong interactions. The (pulled-back) connection $A$ of a principal bundle is the gauge potential and the (pulled-back) curvature $F$ gives the strength of the gauge field. When matter fields are present in the gauge theory, they are described by the associated vector bundles.

*Examples 5.5*

1. *Dirac magnetic monopole.* We now put a connection on the $U(1)$ principal fiber bundle over the base space $S^2$ described in example 4.3.2. If we choose a particular connection which satisfies Maxwell's equations, the physical system described corresponds to Dirac's magnetic monopole. As before, we split $S^2$ into hemispheres $H_\pm$ and assign $U(1)$ connection 1-forms to each half of the bundle,

$$\omega = \begin{cases} A_+ + \mathrm{d}\psi_+ & \text{on } H_+ \\ A_- + \mathrm{d}\psi_- & \text{on } H_- \end{cases} .$$

(For $U(1)$, we conventionally factor out the (i) arising from our convention that Lie algebras are represented by antihermitian matrices: $g^{-1}\,\mathrm{d}g = \mathrm{e}^{-\mathrm{i}\psi}\,\mathrm{d}\mathrm{e}^{\mathrm{i}\psi} = \mathrm{i}\,\mathrm{d}\psi \to \mathrm{d}\psi$.) Then the choice of the transition function (4.21)

$$\mathrm{e}^{\mathrm{i}\psi_-} = \mathrm{e}^{\mathrm{i}n\phi}\,\mathrm{e}^{\mathrm{i}\psi_+}$$

implies the gauge transformation,

$$A_+ = A_- + n\phi.$$

Gauge potentials which satisfy Maxwell's equations (in $\mathbb{R}^3 - \{0\}$) and are regular in $H_+$ and $H_-$ are given by (see example 2.4.3),

$$A_\pm = \frac{n}{2}(\pm 1 - \cos\theta)\,\mathrm{d}\phi = \frac{n}{2r}\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{z \pm r}.$$

The curvature is given by

$$F = \mathrm{d}A_\pm = \frac{n}{2}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi = \frac{n}{2r^3}(x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y).$$

It is easy to see that although the $A_\pm$ are regular in $H_\pm$, they have a string singularity in $H_\mp$. We will allow $A_\pm$ to be used only in its regular neighborhood. It is clear that $F$ is *closed* but not exact, since $dA_\pm$ is only defined locally in $H_\pm$.

*Remark 1*: We shall see in the next section that the monopole charge is minus the first Chern number $C_1$ characterizing the bundle:

$$-C_1 = -\int_{S^2} c_1 = +\frac{1}{2\pi}\int_{S^2} F = +\frac{1}{2\pi}\left[\int_{H_+} F_+ + \int_{H_-} F_-\right] = n.$$

*Remark 2*: It is instructive to carry out the above calculations using the $S^2$ metric $(dx^2 + dy^2)/(1 + x^2 + y^2)^2 = (dr^2 + r^2\,d\phi^2)/(1 + r^2)^2$ obtained by projection from the north or south pole onto $\mathbb{R}^2$. In this case the "string singularity" occurs at $r = 0$ or $r = \infty$. This treatment closely resembles the instanton case described below.

2. *BPST Instanton in* SU(2) *Yang–Mills* (Belavin et al. [1975]). The instanton solution of Euclidean SU(2) Yang–Mills theory is a connection on a principal bundle with

$$\text{Base } M = S^4, \qquad \text{Fiber } G = \mathrm{SU}(2) = S^3.$$

We take the $S^4$ metric (see example 3.2.3)

$$ds^2 = \frac{dx_\mu\,dx_\mu}{(1 + r^2/a^2)^2} = \frac{dr^2 + r^2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)}{(1 + r^2/a^2)^2} = \sum_{a=0}^{3}(e^a)^2$$

obtained by projection from the north or south pole onto $\mathbb{R}^4$.

As in example 4.3.3, we split $S^4$ into "hemispheres" $H_\pm$. In the overlap region

$$H_+ \cap H_- \simeq S^3,$$

we relate the SU(2) fibers by the transition functions

$$g_- = [h(x)]^k \cdot g_+,$$

where $k$ is an integer, $h = (t - i\boldsymbol{\lambda}\cdot\boldsymbol{x})/r$ and $\boldsymbol{\lambda}$ are the SU(2) Pauli matrices. We note that

$$h^{-1}\,dh = i\lambda_k\sigma_k = i\lambda_k\eta^k{}_{\mu\nu}x_\mu\,dx_\nu/r^2$$

$$dh\,h^{-1} = -i\lambda_k\bar{\sigma}_k = -i\lambda_k\bar{\eta}^k{}_{\mu\nu}x_\mu\,dx_\nu/r^2$$

where $\eta$, $\bar{\eta}$ are 't Hooft's eta tensors ('t Hooft [1977]; see appendix C).

The connection 1-forms in the two neighborhoods of the bundle then may be written as

$$\omega = \begin{cases} g_+^{-1}Ag_+ + g_+^{-1}\,dg_+ & \text{on } H_+ \\ g_-^{-1}A'g_- + g_-^{-1}\,dg_- & \text{on } H_- \end{cases}$$

where

$$A'(x) = h^k(x)\,A(x)\,h^{-k}(x) + h^k(x)\,\mathrm{d}h^{-k}(x).$$

In the case $k = 1$, we have the single instanton solution,

$$H_+ : A = \frac{r^2}{r^2 + a^2} \cdot h^{-1}\,\mathrm{d}h = \frac{r^2}{r^2 + a^2}\,\mathrm{i}\,\lambda_k \sigma_k$$

which is singular at the "south-pole" at $r = \infty$, and the gauge-transformed solution,

$$H_- : A' = h\left[\frac{r^2}{r^2 + a^2} h^{-1}\,\mathrm{d}h\right] h^{-1} + h\,\mathrm{d}h^{-1} = -\frac{\mathrm{d}h\,h^{-1}}{1 + r^2/a^2} = \frac{\mathrm{i}\,\lambda_k \bar{\sigma}_k}{1 + r^2/a^2}$$

which is singular at the "north-pole" at $r = 0$. (*Note*: $A$ and $A'$ are the Yang–Mills analogs of the two gauge-equivalent Dirac monopole solutions with Dirac strings in the upper and lower hemispheres of $S^2$.)

The field strengths in $H_\pm$ are easily computed to be

$$H_+ : \quad F_+ = \mathrm{d}A + A \wedge A = \mathrm{i}\,\lambda_k \frac{2}{a^2}\left(e^0 \wedge e^k + \frac{1}{2}\epsilon_{kij} e^i \wedge e^j\right)$$

$$H_- : \quad F_- = \mathrm{d}A' + A' \wedge A' = hF_+h^{-1}.$$

Since $F$ is self-dual,

$$*F = F,$$

the Bianchi identities imply that the Yang–Mills equations

$$\mathrm{D}_A * F = \mathrm{d} * F + A \wedge *F - *F \wedge A = 0$$

are satisfied. Replacing $h(x)$ by $h^{-1}(x)$ and interchanging $\sigma_k$ and $\bar{\sigma}_k$ throughout would give us an anti-self-dual solution.

*Remark 1*: In the next section, we will see that the "instanton number" $k$ is minus the second Chern number $C_2$ characterizing the bundle:

$$k = -C_2 = -\int_{S^4} c_2 = -\frac{1}{8\pi^2}\int_{S^4} \mathrm{Tr}\,F \wedge F$$

$$= -\frac{1}{8\pi^2}\left[\int_{H_+} \mathrm{Tr}\,F_+ \wedge F_+ + \int_{H_-} \mathrm{Tr}\,F_- \wedge F_-\right] = -\frac{1}{8\pi^2}\left(-\frac{48}{a^4}\right)\int_{S^4} e^0 \wedge e^1 \wedge e^2 \wedge e^3 = +1.$$

(Recall that the volume of $S^4$ with radius $a/2$ is $\pi^2 a^4/6$. See appendix A.)

*Remark 2*: Note that $A_{(\pm)} = A^a_{(\pm)}(\lambda^a/2i)$ for $k = \pm 1$ are derivable from the self-dual or anti-self-dual combinations of the $O(4)$ connections $\omega_{ab}$ of $S^4$ given in example 3.2.3,

$$A^1_{(+)} = -\omega_{01} - \omega_{23} = -2\sigma_x \frac{(r/a)^2}{1+(r/a)^2}, \qquad \text{cyclic},$$

$$A^1_{(-)} = +\omega_{01} - \omega_{23} = -\frac{2\sigma_x}{1+(r/a)^2}, \qquad \text{cyclic}.$$

Here the diameter $2R$ of $S^4$ is identified with the instanton size $a$. This is related to the fact that the $k=1$ bundle is the Hopf fibration of $S^7$.

*Remark 3*: Under an $O(4)$ transformation, the $k=1$ instanton transforms into itself up to a gauge transformation. Under an $O(5)$ transformation of $S^4$, it also transforms into itself up to a gauge transformation; the BPST instanton solution is unique in possessing the $O(5)$ symmetry (see, e.g., Jackiw and Rebbi [1976a]).

## 6. Characteristic classes

We have now seen explicitly how the construction of nontrivial fiber bundles involves certain integers characterizing the transition functions. Furthermore, we observed in passing that when we put connections on the bundles, these same integers corresponded to integrals involving a bundle's curvature. In this section, we will develop more thoroughly the concept of the *characteristic classes* distinguishing inequivalent fiber bundles. The manipulation of characteristic classes plays an essential role in index theory, which is the subject of the next section.

In the preceding sections we have been careful to distinguish among connection 1-forms and curvature 2-forms used for different purposes: $\omega^a{}_b$ and $R^a{}_b$ were used for Riemannian geometry in an orthonormal frame basis, $\Gamma^i{}_j$ and $R^i{}_j$ were used for vector bundles, and $A$ and $F$ were used for principal bundles. The notation $\omega$ was also used for connections lying in $T^*$ of the bundle rather than in $T^*$ of the base, while $\Omega$ was used for the corresponding curvature. In this section, we loosen these distinctions for notational convenience and employ the symbols $\omega$ and $\Omega$ to denote the values of the connection and curvature forms pulled back using sections of a bundle.

We shall deal with the following four categories of characteristic classes.

1. *Chern classes* $c_1, \ldots, c_k$ are defined for a complex vector bundle of dimension $k$ (or equivalently for $GL(k, \mathbb{C})$ principal bundles). $c_i \in H^{2i}(M)$.

2. *Pontrjagin classes* $p_1, \ldots, p_j$ are defined for a real vector bundle of dimension $k$ (or equivalently for $GL(k, \mathbb{R})$ principal bundles). $p_i \in H^{4i}(M)$. ($j = [k/2]$ is the greatest integer in $k/2$.)

3. *The Euler class* $e$ is defined for an oriented bundle of even dimension $k$ with a fiber metric (or equivalently for $SO(k)$ principal bundles). $e \in H^k(M)$.

4. *Stiefel–Whitney classes* $w_1, \ldots, w_k$ are defined for a real vector bundle of dimension $k$ (or equivalently for $GL(k, \mathbb{R})$ principal bundles). They are $\mathbb{Z}_2$ characteristic classes and are not given by curvature. $w_i \in H^i(M; \mathbb{Z}_2)$.

### 6.1. General properties of Chern classes

We begin our study of characteristic classes by examining the Chern classes associated with bundles having GL($k$, $\mathbb{C}$) transition functions. Many of the methods we discuss will then be applicable to other groups and characteristic classes.

*Invariant polynomials*: Let $\alpha$ be a complex $k \times k$ matrix and $P(\alpha)$ be a polynomial in the components of $\alpha$. $P(\alpha)$ is called an *invariant polynomial* or a *characteristic polynomial* if

$$P(\alpha) = P(g^{-1}\alpha g) \tag{6.1}$$

for all $g \in \mathrm{GL}(k, \mathbb{C})$. If $\alpha$ has eigenvalues $\{\lambda_1, \dots, \lambda_k\}$, $P(\alpha)$ is a symmetric function of the eigenvalues. If $S_j(\lambda)$ is the $j$th symmetric polynomial,

$$S_j(\lambda) = \sum_{i_1 < i_2 < \cdots < i_j} \lambda_{i_1}\lambda_{i_2}, \dots, \lambda_{i_j},$$

then $P(\alpha)$ is a polynomial in the $S_j(\lambda)$:

$$P(\alpha) = a + bS_1(\lambda) + cS_2(\lambda) + d[S_1(\lambda)]^2 + \cdots.$$

Examples of invariant polynomials are

$$\mathrm{Det}(I + \alpha) = 1 + S_1(\lambda) + S_2(\lambda) + \cdots + S_k(\lambda) \tag{6.2}$$

and Tr(exp $\alpha$), which are used to define the Chern class and the Chern character.

If a matrix-valued curvature 2-form $\Omega$ is substituted for the matrix $\alpha$ in an invariant polynomial, we find the following properties:

(1) *$P(\Omega)$ is closed*

(2) *$P(\Omega)$ has topologically invariant integrals.*

We will prove these assertions following Chern [1967]. Suppose $P(\alpha_1, \dots, \alpha_r)$ is a homogeneous invariant polynomial of degree $r$. Using the invariance of the polynomial under an infinitesimal transformation $g = I + g'$, we can deduce

$$\sum_{1 \le i \le r} P(\alpha_1, \dots, g'\alpha_i - \alpha_i g', \dots, \alpha_r) = 0.$$

Then if $\theta$ is a $k \times k$ Lie-algebra valued matrix of 1-forms and the $\{\alpha_i\}$ are $k \times k$ Lie-algebra valued forms of degree $d_i$, we find

$$\sum_{1 \le i \le r} (-1)^{d_1 + \cdots + d_{i-1}} P(\alpha_1, \dots, \theta \wedge \alpha_i, \dots, \alpha_r) - \sum_{1 \le i \le r} (-1)^{d_1 + \cdots + d_i} P(\alpha_1, \dots, \alpha_i \wedge \theta, \dots, \alpha_r) = 0. \tag{6.3}$$

Therefore if we choose $\theta$ to be the connection 1-form $\omega$, we may write

$$\mathrm{d}P(\alpha_1, \dots, \alpha_r) = \sum_{1 \le i \le r} (-1)^{d_1 + \cdots + d_{i-1}} P(\alpha_1, \dots, \mathrm{D}\alpha_i, \dots, \alpha_r)$$

where

$$\mathrm{D}\alpha_i = \mathrm{d}\alpha_i + \omega \wedge \alpha_i - (-1)^{d_i}\alpha_i \wedge \omega$$

is the covariant derivative of the form $\alpha_i$.

If $\alpha_i = \Omega$, the curvature 2-form, we conclude that

$$\mathrm{d}P(\Omega) = 0$$

because of the Bianchi identity (5.34).

Now let $\omega$ and $\omega'$ be two connections on the bundle and $\Omega$ and $\Omega'$ their curvatures. We consider the interpolation between $\omega$ and $\omega'$,

$$\omega_t = \omega + t\eta \qquad 0 \le t \le 1,$$

where $\eta = \omega' - \omega$.

Then

$$\Omega_t = \mathrm{d}\omega_t + \omega_t \wedge \omega_t = \Omega + t\mathrm{D}\eta + t^2\eta \wedge \eta,$$

where $\mathrm{D}\eta = \mathrm{d}\eta + \omega \wedge \eta + \eta \wedge \omega$.

Let $P(\alpha_1, \dots, \alpha_r)$ be a symmetric polynomial and let

$$q(\beta, \alpha) = rP(\beta, \underbrace{\alpha, \dots, \alpha}_{r-1}).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\Omega_t) = q(\mathrm{D}\eta, \Omega_t) + 2tq(\eta \wedge \eta, \Omega_t).$$

On the other hand

$$\begin{aligned} \mathrm{D}\Omega_t &= t\mathrm{D}^2\eta + t^2(\mathrm{D}\eta \wedge \eta - \eta \wedge \mathrm{D}\eta) = t(\Omega \wedge \eta - \eta \wedge \Omega) + t^2(\mathrm{D}\eta \wedge \eta - \eta \wedge \mathrm{D}\eta) \\ &= t(\Omega_t \wedge \eta - \eta \wedge \Omega_t), \end{aligned}$$

so that

$$\begin{aligned} \mathrm{d}q(\eta, \Omega_t) &= q(\mathrm{D}\eta, \Omega_t) - r(r-1)P(\eta, \mathrm{D}\Omega_t, \Omega_t, \dots, \Omega_t) \\ &= q(\mathrm{D}\eta, \Omega_t) - r(r-1)tP(\eta, (\Omega_t \wedge \eta - \eta \wedge \Omega_t), \Omega_t, \dots, \Omega_t). \end{aligned}$$

Eq. (6.3) with $\theta = \alpha_1 = \eta$, $\alpha_2 = \cdots = \alpha_r = \Omega_t$ gives

$$2q(\eta \wedge \eta, \Omega_t) + r(r-1)P(\eta, (\Omega_t \wedge \eta - \eta \wedge \Omega_t), \Omega_t, \dots, \Omega_t) = 0.$$

Combining the last two equations, we get

$$\mathrm{d}q(\eta, \Omega_t) = q(\mathrm{D}\eta, \Omega_t) + 2tq(\eta \wedge \eta, \Omega_t) = \frac{\mathrm{d}}{\mathrm{d}t} P(\Omega_t).$$

Hence

$$P(\Omega') - P(\Omega) = \mathrm{d}\int_0^1 q(\omega' - \omega, \Omega_t)\,\mathrm{d}t \equiv \mathrm{d}Q(\omega', \omega). \tag{6.4}$$

Since $P(\Omega')$ and $P(\Omega)$ differ by an exact form $\mathrm{d}Q$, their integrals over manifolds without boundary give the same results. Thus we have proven both properties (1) and (2).

*Chern form*: The Chern form of a complex vector bundle $E$ over $M$ with $\mathrm{GL}(k, \mathbb{C})$ transition functions and a connection $\omega$ is obtained by substituting the curvature 2-form $\Omega \in \mathrm{gl}(k, \mathbb{C})$ into the invariant polynomial $\mathrm{Det}(1 + \alpha)$. We define the *total Chern form* as

$$c(\Omega) = \mathrm{Det}\left(I + \frac{\mathrm{i}}{2\pi}\Omega\right) = 1 + c_1(\Omega) + c_2(\Omega) + \cdots, \tag{6.5}$$

where the individual Chern forms $c_j(\Omega)$ are polynomials of degree $j$ in $\Omega$:

$$c_0 = 1$$

$$c_1 = \frac{\mathrm{i}}{2\pi}\mathrm{Tr}\,\Omega$$

$$c_2 = \frac{1}{8\pi^2}\{\mathrm{Tr}\,\Omega \wedge \Omega - \mathrm{Tr}\,\Omega \wedge \mathrm{Tr}\,\Omega\}$$

$$c_3 = \frac{\mathrm{i}}{48\pi^2}\{-2\,\mathrm{Tr}\,\Omega \wedge \Omega \wedge \Omega + 3(\mathrm{Tr}\,\Omega \wedge \Omega) \wedge \mathrm{Tr}\,\Omega - \mathrm{Tr}\,\Omega \wedge \mathrm{Tr}\,\Omega \wedge \mathrm{Tr}\,\Omega\}$$

$$\vdots$$

The explicit expressions for $c_j$ are obtained from the eigenvalue expansion of $\alpha = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$:

$$\begin{aligned}\mathrm{Det}\left(I + \frac{\mathrm{i}}{2\pi}\alpha\right) &= \left(1 + \frac{\mathrm{i}}{2\pi}\lambda_1\right)\left(1 + \frac{\mathrm{i}}{2\pi}\lambda_2\right)\cdots\left(1 + \frac{\mathrm{i}}{2\pi}\lambda_k\right) \\ &= 1 + \frac{\mathrm{i}}{2\pi}S_1(\lambda) + \left(\frac{\mathrm{i}}{2\pi}\right)^2 S_2(\lambda) + \cdots\end{aligned}$$

where the $S_j(\lambda)$ are the elementary symmetric functions defined earlier. For example,

$$\left(\frac{\mathrm{i}}{2\pi}\right)^2 \sum_{j<l}^{k} \lambda_j\lambda_l = \left(\frac{\mathrm{i}}{2\pi}\right)^2 \left(\frac{1}{2}\right)((\mathrm{Tr}\,\alpha)^2 - \mathrm{Tr}(\alpha^2))$$

gives $c_2$ if the matrix $\alpha$ is replaced by $\Omega$. Since $c_j(\Omega) \in \Lambda^{2j}(M)$, we see that

$$c_j = 0 \qquad \text{for } 2j > n = \dim M.$$

Thus $c(\Omega)$ is always a finite sum.

Since any invariant polynomial $P(\alpha)$ can be expressed in terms of the elementary symmetric functions, $P(\alpha)$ can be expressed as a polynomial in the Chern forms. Thus the Chern forms generate the characteristic ring.

*Chern classes and cohomology*: Since $P(\Omega)$ is closed, any homogeneous polynomial in the expansion of an invariant polynomial $P(\Omega)$ is closed:

$$\mathrm{d}c_j(\Omega) = 0. \tag{6.6}$$

We may verify this explicitly using the Bianchi identities; for example,

$$\mathrm{d}c_1(\Omega) = \frac{\mathrm{i}}{2\pi}\,\mathrm{Tr}\,\mathrm{d}(\mathrm{d}\omega + \omega \wedge \omega) = \frac{\mathrm{i}}{2\pi}\,\mathrm{Tr}(\Omega \wedge \omega - \omega \wedge \Omega) \equiv 0.$$

We conclude that the Chern forms $c_j(\Omega)$ define $2j$th cohomology classes,

$$c_j(\Omega) \in H^{2j}(M). \tag{6.7}$$

This cohomology class, which we will often denote by $c_j(E)$, is independent of the connection because $P(\Omega) - P(\Omega')$ is *exact* for any characteristic polynomial.

*Chern numbers and topological invariance*: It is a remarkable fact that the cohomology classes to which the Chern forms $c_j(\Omega)$ belong are actually *integer* classes. If we integrate $c_j(\Omega)$ over any $2j$-cycle in $M$ with integer coefficients, we obtain an integer which is independent of the connection. The *Chern numbers* of a bundle are the numbers which result from integrating characteristic polynomials over the entire manifold; for example, if $n = 4$, the only two Chern numbers are

$$C_2(E) = \int_M c_2(\Omega)$$

$$C_1^2(E) = \int_M c_1(\Omega) \wedge c_1(\Omega).$$

*Characteristic classes of unitary bundles*: One can show that the $U(k)$ and $\mathrm{GL}(k, \mathbb{C})$ characteristic polynomials can be identified. Therefore their characteristic classes can be identified. This is *not* true for $\mathrm{GL}(k, \mathbb{R})$ and $O(k)$ or $\mathrm{SO}(k)$. The $\mathrm{SU}(k)$ characteristic classes are generated by $(c_2, \ldots, c_k)$ because $c_1 = 0$. Note that if $c_1 \neq 0$ for a complex vector bundle $E$, there is no associated $\mathrm{SU}(k)$ principal bundle. (Warning: there exist bundles with $c_1 = 0$ which also do not admit an $\mathrm{SU}(k)$ structure.)

*Chern classes of composite bundles*: Let $c(E) = c_0(E) + \cdots + c_k(E)$, with $c_j(E) \in H^{2j}(M)$, denote the total Chern class for a $k$-dimensional complex vector bundle $E$ over $M$. Then we find

(1) Whitney sum: $c(E \oplus F) = c(E) \wedge c(F)$.
(2) $c_1(L \otimes L') = c_1(L) + c_1(L')$ for $L, L'$ = line bundles.
(3) Pullback class: $c(f^*E) = f^*c(E)$, where $f\colon M' \to M$ and $E' = f^*E$ is the pullback of $E$ over $M'$.

These properties plus the requirement that $C_1(L) = -1$ for the line bundle $L$ over $P_1(\mathbb{C})$ are sometimes used as an axiomatic definition of the Chern class.

*The Weil homomorphism*: It is well-known that the "Casimir invariants" or polynomials in the center of a Lie algebra $\mathcal{G}$ with matrix basis $\{X_i\}$ are generated by the determinant

$$\mathrm{Det}(t \cdot I + a_i X_i) = \sum_k t^k P_k(a_i).$$

The Chern classes are thus obtained by substituting the Lie-algebra valued curvature 2-forms into each of the resulting characteristic polynomials.

### 6.2. Classifying spaces

We motivate the concept of classifying spaces for fiber bundles by showing how the standard complex line bundle $L$ over $P_{n-1}(\mathbb{C})$ may be used to classify other line bundles. Let $E$ be a complex line bundle over $M$ and assume that we can find a complementary bundle $\bar{E}$ such that

$$E \oplus \bar{E} = M \times \mathbb{C}^n$$

for some $n > 1$ (this is always possible). The fibers of $E$ are then lines in $\mathbb{C}^n$. We define a map $f(x)$ from the points $x \in M$ to $P_{n-1}(\mathbb{C})$ which associates to each point $x$ the line in $\mathbb{C}^n$ given by the fiber $F_x$. *Then the line bundle E is isomorphic to the pullback of the natural line bundle L over* $P_{n-1}(\mathbb{C})$:

$$E \approx f^* L.$$

We can generalize this construction by considering the *Grassmann manifold* $\mathrm{Gr}(m, k, \mathbb{C})$ of $k$-planes in $\mathbb{C}^m$; just as the points of $\mathrm{Gr}(m, 1, \mathbb{C}) \equiv P_{m-1}(\mathbb{C})$ correspond to lines through the origin in $\mathbb{C}^m$, each point of $\mathrm{Gr}(m, k, \mathbb{C})$ corresponds to a $k$-plane through the origin. The *natural $k$-plane bundle* $L(m, k, \mathbb{C})$ over $\mathrm{Gr}(m, k, \mathbb{C})$ has as its fiber the $k$-plane in $\mathbb{C}^m$ over the corresponding point in $\mathrm{Gr}(m, k, \mathbb{C})$; $L(m, 1, \mathbb{C})$ is just the natural line bundle $L$ over $P_{m-1}(\mathbb{C})$. We now quote without proof a basic theorem (see, e.g., Chern [1972]):

*Theorem*: Let $M$ be a manifold of dimension $n$ and $E$ any $k$-dimensional complex vector bundle over $M$. Then there exists an integer $m_0$ (depending on $n$) such that for $m \geq m_0$,

(a) there exists a map $f$: $M \to \mathrm{Gr}(m, k, \mathbb{C})$ such that $E = f^* L(m, k, \mathbb{C})$;

(b) given any two maps $f$ and $g$ mapping $M \to \mathrm{Gr}(m, k, \mathbb{C})$, then $f^* L(m, k, \mathbb{C}) \approx g^* L(m, k, \mathbb{C})$ if and only if $f$ and $g$ are homotopic.

As a consequence of this theorem, the set of isomorphism classes of $k$-dimensional vector bundles is itself isomorphic to the set of homotopy classes of maps from $M$ to $\mathrm{Gr}(m, k, \mathbb{C})$; in this manner, questions about the classification of vector bundles are reduced to questions about homotopy theory in algebraic topology.

*Classifying spaces of principal bundles*: $P(m, k, \mathbb{C})$, the bundle of frames of $L(m, k, \mathbb{C})$, is a principal $\mathrm{GL}(k, \mathbb{C})$ bundle over $\mathrm{Gr}(m, k, \mathbb{C})$. For $m \geq m_0$, very large, $P(m, k, \mathbb{C})$ and $L(m, k, \mathbb{C})$ are described by the same set of homotopy classes of maps from $M \to \mathrm{Gr}(m, k, \mathbb{C})$. In fact, we can make the identification

$$\begin{aligned} \mathrm{Gr}(m, k, \mathbb{C}) &= \mathrm{GL}(m, \mathbb{C})/\mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(m-k, \mathbb{C}) \\ P(m, k, \mathbb{C}) &= \mathrm{GL}(m, \mathbb{C})/\mathrm{GL}(m-k, \mathbb{C}) \end{aligned} \tag{6.8}$$

where the projection $\pi$: $P(m, k, \mathbb{C}) \to \mathrm{Gr}(m, k, \mathbb{C})$ projects out the fiber $\mathrm{GL}(k, \mathbb{C})$. Clearly similar constructions can be carried out for $\mathrm{GL}(k, \mathbf{R})$ principal bundles, $\mathrm{SO}(k)$ principal bundles, $\mathrm{SU}(k)$ principal bundles, etc.

*Universal classifying spaces*: We define the universal Grassmannian $\mathrm{Gr}(\infty, k, \mathbb{C})$ by taking the union of the natural inclusion maps of $\mathrm{Gr}(m, k, \mathbb{C})$ into $\mathrm{Gr}(m+1, k, \mathbb{C})$. We denote the universal classifying bundles corresponding to $\mathrm{Gr}(\infty, k, \mathbb{C})$ by $L(\infty, k, \mathbb{C})$ and $P(\infty, k, \mathbb{C})$. The cohomology of $\mathrm{Gr}(\infty, k, \mathbb{C})$ is simpler than that of $\mathrm{Gr}(m, k, \mathbb{C})$ and is a polynomial algebra with generators $c_i = c_i(L(\infty, k, \mathbb{C})) = c_i(P(\infty, k, \mathbb{C}))$. Given a $k$-dimensional bundle $E$ and a map

$$\begin{aligned} &f: M \to \mathrm{Gr}(\infty, k, \mathbb{C}) \\ &f^* L(\infty, k, \mathbb{C}) \approx E, \end{aligned}$$

we see that

$$c_i(E) = f^* c_i.$$

$f$ is defined uniquely up to homotopy so the cohomology classes are all well-defined and depend only on the bundle $E$.
*Note*: from this approach, it is obvious that $U(k)$ bundles and $\mathrm{GL}(k, \mathbb{C})$ bundles both have the same classifying space $\mathrm{Gr}(\infty, k, \mathbb{C})$, and thus the same characteristic classes.

## *6.3. The splitting principle*

Algebraic identities involving characteristic classes are a central part of index theory. Such manipulations are made vastly simpler by the use of a tool called the *splitting principle* (see e.g. Hirzebruch [1966]).

We gave above a brief description of the characteristic classes $c_i(E)$ using our knowledge of the cohomology of the classifying spaces $\mathrm{Gr}(m, k, \mathbb{C})$, the Grassmann manifolds. This is an approach based on algebraic topology; from this viewpoint the splitting principle is the idea that even though a given bundle is not, in general, a direct sum of one-dimensional line bundles, characteristic class manipulations can be performed as though this were the case. We also discussed the characteristic classes using invariant polynomials and curvature. From this differential geometric point of view, the splitting principle is simply the assertion that the diagonalizable matrices are dense.

We illustrate the concepts of the splitting principle with the familiar identity

$$\mathrm{Det}[\alpha] = \exp(\mathrm{Tr} \ln[\alpha]).$$