If α is a diagonalizable matrix with eigenvalues $\{\lambda_i\}$, then it is clear that

$$\mathrm{Det}[\alpha] = \prod_i \lambda_i = \exp\Big(\sum_i \ln \lambda_i\Big) = \exp(\mathrm{Tr}\ln[\alpha]).$$

Since both sides of this equation are continuous and since we can approximate any matrix by a diagonalizable matrix, this identity holds true for any matrix. Thus to prove an invariant identity of this sort, we may in fact assume that the matrix α is diagonal.

Now let Ω be an $n \times n$ matrix of curvature 2-forms. If we imagine that Ω is diagonalizable into n 2-forms Ω_j, then the Chern class becomes

$$c(E) = \mathrm{Det}\Big(1 + \frac{\mathrm{i}}{2\pi}\Omega\Big) = \mathrm{Det}\begin{pmatrix} 1+\frac{\mathrm{i}}{2\pi}\Omega_1 & & 0 \\ & \ddots & \\ 0 & & 1+\frac{\mathrm{i}}{2\pi}\Omega_n \end{pmatrix}$$
$$= \prod_{j=1}^{k}\Big(1 + \frac{\mathrm{i}}{2\pi}\Omega_j\Big) = \prod_{j=1}^{k}(1+x_j) \tag{6.9}$$

where we will henceforth use the formal notation

$$x_j = \frac{\mathrm{i}}{2\pi}\Omega_j.$$

Each of the terms $(1+x_j)$ can be interpreted as the Chern class of a one-dimensional line bundle L_j,

$$c(L_j) = 1 + c_1(L_j) = 1 + \frac{\mathrm{i}}{2\pi}\Omega_j.$$

If we imagine that a k-dimensional vector bundle E has a decomposition

$$E = L_1 \oplus \cdots \oplus L_k,$$

then

$$c(E) = \prod_j c(L_j) = \prod_j (1+x_j).$$

Thus $c_l(E)$ can be thought of as the lth elementary symmetric function of the variables $\{x_j\}$:

$$c_1 = \sum_j x_j, \qquad c_2 = \sum_{i<j} x_i x_j, \qquad \dots, \qquad c_k = x_1 x_2 \dots x_k. \tag{6.10}$$

Sums of bundles: If A and B are matrices, then

$$\mathrm{Det}(A \oplus B) = \mathrm{Det}\,A \cdot \mathrm{Det}\,B.$$

Consequently, if E and F are vector bundles, then

$$c(E \oplus F) = c(E) \wedge c(F),$$

since this is true on the form level when we use the Whitney sum connection. From the point of view of algebraic topology, this identity is first proved for bundles E and F which actually split into a sum of line bundles. The splitting principle is then invoked to deduce the identity for the general case.

Chern character: Many essential manipulations in index theory involve not only Whitney sums of bundles but also tensor products of bundles. The total Chern class behaves well for Whitney sums, but not for product bundles. We are thus motivated to put aside $c(E) = \Pi_j(1 + x_j)$ and to find some other polynomial in the $\{x_j\}$ which has simple properties for product bundles as well as Whitney sums. One such polynomial is the Chern character ch(E). In terms of matrices, the Chern character is defined by the following invariant polynomial:

$$\mathrm{ch}(\alpha) = \mathrm{Tr}\exp\left(\frac{\mathrm{i}}{2\pi}\alpha\right) = \sum_j \frac{1}{j!}\mathrm{Tr}\left(\frac{\mathrm{i}}{2\pi}\alpha\right)^j. \tag{6.11}$$

Since

$$\mathrm{ch}(\alpha \oplus \beta) = \mathrm{ch}(\alpha) + \mathrm{ch}(\beta)$$

$$\mathrm{ch}(\alpha \otimes \beta) = \mathrm{ch}(\alpha)\,\mathrm{ch}(\beta),$$

these identities still hold when we substitute the curvature 2-form Ω to define ch(E). *Note*: since $\mathrm{Tr}(\Omega)^j = 0$ for $j > n/2$, we in fact have a finite sum.

The Chern character of E has the splitting principle expansion

$$\mathrm{ch}(E) = \sum_{j=1}^{k} \mathrm{e}^{x_j} = k + c_1(E) + \tfrac{1}{2}(c_1{}^2 - 2c_2)(E) + \cdots. \tag{6.12}$$

Other characteristic polynomials: Using the splitting principle, we may define characteristic classes by their generating functions. For example, the total Chern class has the generating function $\Pi(1 + x_j)$, while the Chern character has the generating function $\Sigma\, \mathrm{e}^{x_j}$. Another class which appears in the index theorem is the *Todd class* which has the generating function

$$\mathrm{td}(E) = \prod_{j=1}^{k} \frac{x_j}{1 - \mathrm{e}^{-x_j}} = 1 + \tfrac{1}{2}c_1(E) + \tfrac{1}{12}(c_1^2 + c_2)(E) + \cdots. \tag{6.13}$$

The Todd class is multiplicative for Whitney sums,

$$\mathrm{td}(E \oplus F) = \mathrm{td}(E)\,\mathrm{td}(F).$$

We can define other multiplicative characteristic classes by using other generating functions. Two other such functions are the *Hirzebruch L-polynomial*

$$L(E) = \prod_j \frac{x_j}{\tanh x_j} \tag{6.14}$$

which appears in the signature index formula, and the $\hat{A}$ *polynomial*

$$\hat{A}(E) = \prod_j \frac{x_j/2}{\sinh(x_j/2)} \tag{6.15}$$

which appears in the spin index formula.

Examples 6.3

1. *Chern class of $P_1(\mathbb{C})$ line bundle.* Let L be the natural line bundle over base manifold $M = S^2 = P_1(\mathbb{C})$ (see example 5.4.1) with the natural curvature

$$\Omega = -\partial\bar{\partial} \ln(1 + |z|^2) = \frac{-\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + z\bar{z})^2}.$$

Then

$$c_1(L) = \frac{\mathrm{i}}{2\pi}\Omega = -\frac{1}{\pi}\frac{\mathrm{d}x \wedge \mathrm{d}y}{(1 + x^2 + y^2)^2} = -\frac{1}{\pi}\frac{r\,\mathrm{d}r \wedge \mathrm{d}\theta}{(1 + r^2)^2},$$

so the Chern number characterizing the bundle is

$$C_1(L) = \int_{S^2} c_1(L) = -\frac{1}{\pi}\int_0^\infty \frac{r\,\mathrm{d}r}{(1 + r^2)^2}\int_0^{2\pi} \mathrm{d}\theta = -1.$$

Dual line bundle: The natural curvature on the dual line bundle L^* is the complex conjugate of that for L,

$$\Omega(L^*) = \bar{\Omega}(L) = -\Omega(L),$$

so that the Chern class reverses in sign:

$$c_1(L^*) = -c_1(L)$$

$$C_1(L^*) = \int c_1(L^*) = +1.$$

Alternatively, we may derive this result from the fact that the tensor product bundle

$$L^* \otimes L = I \text{ [trivial complex line bundle over } P_1(\mathbb{C})]$$

is *trivial* because the transition functions $(z_i/z_j)(z_j/z_i) = 1$ are trivial. Thus we know that the total Chern class is

$$c(L^* \otimes L) = c\ (\text{trivial bundle}) = 1 + 0.$$

But then the Chern character formula for product bundles gives us our result:

$$0 = c_1(L^* \otimes L) = c_1(L^*) + c_1(L).$$

Tangent and cotangent bundles of $S^2 = P_1(\mathbb{C})$: We showed in example 4.2.3 that

$$T(S^2) = T_c(P_1(\mathbb{C})) \approx L^* \otimes L^*$$
$$T^*(S^2) = T_c^*(P_1(\mathbb{C})) \approx L \otimes L.$$

From the product bundle formula we immediately find

$$C_1(T(S^2)) = C_1(L^*) + C_1(L^*) = +2$$

$$C_1(T^*(S^2)) = C_1(L) + C_1(L) = -2.$$

2. *Chern classes of* $P_n(\mathbb{C})$. We next consider the natural line bundle L over $P_n(\mathbb{C})$ and its dual L^*. Choosing the Fubini–Study metric (example 3.4.3) on $P_n(\mathbb{C})$, we compute x from the Kähler form,

$$x = c_1(L^*) = \frac{\mathrm{i}}{2\pi} \Omega(L^*) = \frac{1}{\pi} K\ (\text{Fubini–Study}).$$

The factor $\mathrm{i}/2\pi$ is chosen so that the integral of x over $P_1(\mathbb{C})$ is equal to 1. It can be shown that the integral of x^n over $P_n(\mathbb{C})$ for any n is also 1. The 2-form x generates the cohomology ring of $P_n(\mathbb{C})$ with integer coefficients. The Betti numbers are

$$b_0 = b_2 = b_4 = \cdots = b_{2n} = 1$$

$$b_1 = b_3 = \cdots = b_{2n-1} = 0$$

and the Euler characteristic is

$$\chi(P_n(\mathbb{C})) = \sum (-1)^k b_k = n + 1.$$

To compute the Chern classes of $P_n(\mathbb{C})$, we first consider the bundle E_{n+1} consisting of the Whitney sum of $(n+1)$ copies of L^*. The total Chern class is then

$$c(E_{n+1}) = c(L^* \oplus L^* \oplus \cdots \oplus L^*) = (1 + x)^{n+1}.$$

There is a natural embedding of $T_c(P_n(\mathbb{C}))$ in E_{n+1}. The quotient or complementary bundle is trivial.

Thus

$$E_{n+1} = L^* \oplus \cdots \oplus L^* = T_c(P_n(\mathbb{C})) \oplus I.$$

Therefore we find that

$$c(E_{n+1}) = (1+x)^{n+1} = c(T_c(P_n(\mathbb{C}))) \cdot c(I) = c(T_c(P_n(\mathbb{C}))).$$

It is customary to define the Chern class of a complex manifold to be the Chern class of its complex tangent space,

$$c(M) = c(T_c(M)).$$

Thus in particular,

$$c(P_n(\mathbb{C})) = (1+x)^{n+1}.$$

We note that $c_n(P_n(\mathbb{C})) = (n+1)x^n$, so

$$\int_{P_n(\mathbb{C})} c_n(P_n(\mathbb{C})) = n+1.$$

It is no accident that this is the Euler characteristic of $P_n(\mathbb{C})$. The expression

$$\int c_n(M) = \chi(M)$$

is, in fact, the Gauss–Bonnet theorem for a complex manifold of complex dimension n.

3. *Vector bundles over* S^n. Let $n = 2l$ and let $E_\pm$ be the complex vector bundles over S^n introduced in example 5.4.2. We recall that $E_\pm$ was defined using the projection operator $\Pi_\pm$, that $E_+ \oplus E_- = S^n \times \mathbb{C}^{2^l}$ and that the fiber of $E_\pm$ has dimension 2^{l-1}. Choosing the curvature $\Omega_\pm(x_0) = \Pi_\pm(x_0)\,\mathrm{d}\Pi_\pm(x_0) \wedge \mathrm{d}\Pi_\pm(x_0)$, we recall that

$$\mathrm{Tr}(\Omega_\pm)^l = \pm \frac{n!(2\mathrm{i})^l}{2^{n+1}}\,\mathrm{d(vol)}.$$

Consequently

$$\int_{S^n} \mathrm{ch}(\Omega_\pm) = \pm \mathrm{i}^n,$$

where we take only the lth component (the n-form portion) of the ch polynomial. This shows that the bundles $E_\pm$ are non-trivial.

If $n = 2$ $(l = 1)$, the fiber dimension is one and we have complex line bundles with (the 2-form part of) ch equal to $c_1(E_\pm)$, so

$$\int_{S^2} c_1(E_\pm) = \mp 1.$$

The associated principal bundles for these line bundles are the magnetic monopole bundles discussed earlier with charge ± 1.

Remark: While the matrix $\lambda(x)$ defined in example 5.4.2 maps $S^n \to \mathrm{GL}(2^l, \mathbb{C})$, the projection $\Pi_\pm(x)$ acts as

$$\Pi_\pm(x)\colon S^n \to \mathrm{Gr}(2^l, 2^{l-1}, \mathbb{C}).$$

The vector bundles $E_\pm$ are simply the pullbacks under $\Pi_\pm$ of the classifying bundles, $\Pi_\pm{}^* L(2^l, 2^{l-1}, \mathbb{C})$. This example illustrates the relationship between the homology of the embedding of S^n in $\mathrm{Gr}(2^l, 2^{l-1}, \mathbb{C})$ and the cohomology of the bundles characterized by the Chern classes of the classifying space.

4. *Chern class of* $U(1)$ *bundle*. We now turn from vector bundles to the Chern classes of principal bundles. We recall that for a $U(1)$ principal bundle P the curvature is purely imaginary. Thus we may write

$$\Omega = \mathrm{i}F$$

and so find the total Chern class

$$c(P) = \mathrm{Det}\left(1 + \frac{\mathrm{i}}{2\pi}\Omega\right) = 1 + \frac{\mathrm{i}}{2\pi}\mathrm{i}F = 1 - \frac{F}{2\pi}.$$

Hence

$$c_1(P) = -F/2\pi.$$

We noted in example 5.5.1 that the integral of c_1 for the Dirac monopole $U(1)$ bundle over S^2 was the integer giving the monopole charge,

$$C_1 = \int_{S^2} c_1 = -n.$$

Proof of topological invariance: The first Chern class of the monopole bundle ($M = S^2, F = S^1 = U(1)$) depends only on the bundle transition functions and is independent of whether the connection $A(x)$ satisfies Maxwell's equations.

As before, the gauge transformation on the equator is given by

$$A_+(x) = A_-(x) + n\,\mathrm{d}\phi.$$

Applying Stokes' theorem, we obtain,

$$-\int_{S^2} c_1 = \frac{1}{2\pi}\Big[\int_{H_+} dA_+ + \int_{H_-} dA_-\Big] = \frac{1}{2\pi}\int_{S^1}(A_+ - A_-),$$

where the sign change occurs because $\partial H_- = S^1$ has the opposite orientation from $\partial H_+ = S^1$. Using the relation between A_+ and A_-, we find

$$-C_1 = \frac{1}{2\pi}\int_{S^1} n\, d\phi = n.$$

Only the *gauge transformation* enters into the computation.

5. *Chern class of G-bundle.* Let $\lambda_a/2\mathrm{i}$ be a matrix basis for the adjoint representation of the Lie algebra $\mathscr{G}$ of the group G with $\mathrm{Tr}\,\lambda_a\lambda_b = 2\delta_{ab}$. Then the curvature is written as

$$\Omega = g^{-1}F^a(x)\frac{\lambda_a}{2\mathrm{i}}\,g.$$

Since the factors of g^{-1} and g annihilate one another in the determinant, the Chern class of a principal G-bundle P over M is

$$c(P) = \mathrm{Det}\Big(1 + \frac{1}{4\pi}\lambda_a F^a\Big).$$

For $G = \mathrm{SU}(2)$, we can take the λ_a to be Pauli matrices. We find

$$c(P) = 1 - \frac{1}{(4\pi)^2}F^a \wedge F^a = 1 + \frac{1}{8\pi^2}\mathrm{Tr}(F \wedge F)$$

so that

$$c_1(P) = 0$$

$$c_2(P) = \frac{1}{8\pi^2}\mathrm{Tr}(F \wedge F).$$

We noted in example 5.5.2 that the integral of $c_2(P)$ for the self-dual Yang–Mills instanton connection on an SU(2) bundle over S^4 was

$$-C_2 = -\int_{S^4} c_2 = +k,$$

where k is the "instanton number".

Proof of topological invariance: Let us demonstrate the topological invariance of c_2 for the instanton G-bundle. We take $M = S^4$ to be covered by $H_\pm$, with $H_+ \cap H_- \simeq S^3$, and consider the gauge transformation

$$A_- = \Phi A_+ \Phi^{-1} + \Phi\, \mathrm{d}\Phi^{-1}$$
$$F_- = \Phi F_+ \Phi^{-1}.$$

Using the Bianchi identities and $\mathrm{Tr}(A \wedge A \wedge A \wedge A) = 0$, one can show

$$\mathrm{Tr}(F \wedge F) = \mathrm{d}\,\mathrm{Tr}(F \wedge A - \tfrac{1}{3} A \wedge A \wedge A).$$

Then, by using Stokes' theorem, we see that

$$C_2 = \int_{S^4} c_2 = \frac{1}{8\pi^2}\Big[\int_{H_+} \mathrm{Tr}(F_+ \wedge F_+) + \int_{H_-} \mathrm{Tr}(F_- \wedge F_-)\Big]$$
$$= \frac{1}{8\pi^2} \int_{S^3} \Big[\mathrm{Tr}\Big(F_+ \wedge A_+ - \frac{1}{3}(A_+)^3\Big) - \mathrm{Tr}\Big(F \wedge A_- - \frac{1}{3}(A_-)^3\Big)\Big].$$

When we substitute the expressions for A_- and F_- using the gauge transformation, we find

$$C_2 = \int_{S^4} c_2 = \frac{1}{8\pi^2} \int_{S^3} \mathrm{Tr}\Big[\frac{1}{3} \Phi\, \mathrm{d}\,\Phi^{-1} \wedge \Phi\, \mathrm{d}\Phi^{-1} \wedge \Phi\, \mathrm{d}\Phi^{-1} - \mathrm{d}(A_+ \wedge \mathrm{d}\Phi^{-1}\, \Phi)\Big]$$
$$= \frac{1}{24\pi^2} \int_{S^3} \mathrm{Tr}(\Phi\, \mathrm{d}\Phi^{-1})^3.$$

The entire value of C_2 is given by the winding number of the *gauge transformation* $\Phi\, \mathrm{d}\Phi^{-1}$ at the equator $H_+ \cap H_- \simeq S^3$.

Remark: Clearly the transition functions $\Phi(x)$ of the topological bundle fall into equivalence classes characterized by the value of the integer C_2. If C_2 is unchanged by taking

$$\Phi(x) \to h(x)\, \Phi(x),$$

$h(x)$ is referred to in the physics literature as a *small gauge transformation*; such functions are homotopic to the identity map. If C_2 is altered, $h(x)$ is called a *large gauge transformation*; choosing such a transition function modifies the topology of the bundle. A typical large gauge transformation in an SU(2) bundle is

$$h(x) = \frac{t - \mathrm{i}\boldsymbol{\lambda} \cdot \boldsymbol{x}}{r}, \qquad \{\boldsymbol{\lambda}\} = \text{Pauli matrices}.$$

If $\Phi = h^k$, we find that the bundle has $C_2 = -k$.

6.4. Other characteristic classes

6.4.1. Pontrjagin classes

We now discuss the characteristic classes of real vector bundles and their associated principal bundles. The bundle transition functions and the fibers of the principal bundles then belong to $\mathrm{GL}(k, \mathbb{R})$. If one puts a fiber metric on a real vector bundle, the bundle transition functions can be reduced to $O(k)$. The associated bundle of orthonormal frames is an $O(k)$ principal bundle. There are some subtleties present in the real case which are absent in the complex case. While the characteristic forms of real vector bundles whose structure groups are $O(k)$ and $\mathrm{GL}(k, \mathbb{R})$ are different, their characteristic classes are in fact the same.

Since we can always reduce the structure group to $O(k)$ and choose a Riemannian connection on the bundle, we first consider this case.

The *total Pontrjagin class* of a real $O(k)$ bundle E with curvature Ω lying in the Lie algebra of $O(k)$ is defined by the invariant polynomial

$$p(E) = \mathrm{Det}\left(I - \frac{1}{2\pi}\Omega\right) = 1 + p_1 + p_2 + \cdots . \tag{6.16}$$

Since $\Omega = -\Omega^t$, the only non-zero polynomials are of even degree in Ω. Thus $p_j(\Omega) \in \Lambda^{4j}(M)$ and the series expansion of $p(E)$ terminates either when $4j > n = \dim M$ or when $2j > k = \dim E$. $p_j(\Omega)$ is always closed and the cohomology class it represents is independent of the metric and the connection chosen; we let $p_j(E)$ denote this cohomology class. It is clear that the total Pontrjagin class obeys the Whitney sum formula

$$p(E \oplus F) = p(E)\, p(F).$$

Any invariant polynomial for a real bundle can be expanded in the Pontrjagin forms p_j, in the following sense: if $Q(\Omega)$ is a $\mathrm{GL}(k, \mathbb{R})$-invariant polynomial and Ω is a $\mathrm{gl}(k, \mathbb{R})$-valued curvature 2-form, then

$$Q = R(p_1, p_2, \ldots, p_{\max}) + S(\Omega)$$

where R is a polynomial and $S = 0$ when Ω lies in $O(k)$. Furthermore, the cohomology class represented by $S(\Omega)$ (for example: $S = \mathrm{Tr}\, \Omega$) will always be zero, even though $S(\Omega) \neq 0$ on $\mathrm{gl}(k, \mathbb{R})$. Thus the $\mathrm{GL}(k, \mathbb{R})$ and $O(k)$ characteristic classes are the *same*, while their characteristic forms may differ.

Pontrjagin classes in terms of Chern classes: In many applications, it turns out to be convenient to express the Pontrjagin classes of a real bundle in terms of the Chern classes of a complex bundle. If E is a real bundle, we can define $E_c = E \oplus \mathbb{C}$ as the complexification of E. (This is defined by the natural inclusion of $\mathrm{GL}(k, \mathbb{R})$ into $\mathrm{GL}(k, \mathbb{C})$.) If A is a skew-adjoint real matrix, we have the identity:

$$\det\left(I + \frac{\mathrm{i}}{2\pi} A\right) = 1 - p_1(A) + p_2(A) \ldots$$

where the factors of -1 arise from the i^2 terms. This yields the identity:

$$p_k(E) = (-1)^k c_{2k}(E_c). \tag{6.17}$$

Conversely, given a complex bundle E of dimension k, we can form the corresponding real bundle E_r of dimension $2k$ by forgetting the complex structure on E. (This is called the "forgetful functor".) If we then form $(E_r)_c$, this is a complex vector bundle of complex dimension $2k$. Let $\bar{E}$ denote the complex conjugate bundle, which is, in fact, isomorphic to the dual bundle E^*. Then

$$(E_r)_c = E \oplus \bar{E} = E \oplus E^*.$$

Since

$$c(\bar{E}) = 1 - c_1(E) + c_2(E) - c_3(E) \ldots,$$

we find

$$c((E_r)_c) = 1 - p_1(E_r) + p_2(E_r) - \cdots = c(E)\, c(\bar{E})$$
$$= [1 + c_1(E) + c_2(E) + \cdots][1 - c_1(E) + c_2(E) \ldots].$$

Half the terms cancel out. Identifying the remaining terms yields:

$$p_1(E_r) = (c_1^2 - 2c_2)(E)$$

$$p_2(E_r) = (c_2^2 - 2c_1c_3 + 2c_4)(E), \text{ etc.}$$

Using the splitting principle, we find the equivalent polynomial expressions:

$$p_1(E_r) = \sum_i x_i^2$$
$$p_2(E_r) = \sum_{i<j} x_i^2 x_j^2 \tag{6.18}$$

and so forth. The form of these polynomials is related to the fact that the eigenvalues of a skew-symmetric matrix occur in complex conjugate pairs with purely imaginary eigenvalues.
Example: $P_n(\mathbb{C})$. The total Pontrjagin class of a complex manifold such as $P_n(\mathbb{C})$ is computed by using the forgetful functor to obtain the real tangent space $T(P_n(\mathbb{C}))$ from the complex tangent space $T_c(P_n(\mathbb{C}))$ and computing the Pontrjagin class of $T(P_n(\mathbb{C}))$. From example 6.3.2, we know that

$$c(T_c(P_n(\mathbb{C}))) = (1+x)^{n+1}$$
$$c(\bar{T}_c(P_n(\mathbb{C}))) = (1-x)^{n+1}$$

where x is the generator of the integral cohomology of $P_n(\mathbb{C})$. Then we find

$$c(T(P_n(\mathbb{C}))\otimes\mathbb{C}) = c(T_c(P_n(\mathbb{C})))c(\bar{T}_c(P_n(\mathbb{C}))) = (1-x^2)^{n+1}$$

$$= 1 - p_1(T(P_n(\mathbb{C}))) + p_2 \dots$$

so that the total Pontrjagin class is

$$p(T(P_n(\mathbb{C}))) = 1 + p_1 + p_2 + \cdots = (1+x^2)^{n+1}.$$

6.4.2. The Euler class

The transition functions of an oriented real k-dimensional vector bundle E can always be reduced to $SO(k)$ transition functions. If $k = 2r$ is even, we can define an additional $SO(k)$-invariant polynomial $e(\alpha)$ called the *Pfaffian*. This polynomial is not invariant under the orientation-preserving group $GL_+(k,\mathbb{R})$. Thus the corresponding characteristic class can only be computed using a Riemannian connection, not a general linear connection. There exist bundles E with $e(E)\neq 0$ which nevertheless admit *flat* non-Riemannian connections. We recall that, in contrast, the Pontrjagin forms could be computed using a general linear connection.

Let $|\alpha_{ij}|$ be a real anti-symmetric $k\times k$ matrix in the Lie algebra $SO(k)$. Taking $\{z^i\}$ to be local fiber coordinates in E. We construct the 2-form

$$\alpha = \tfrac{1}{2}\alpha_{ij}\, dz^i \wedge dz^j.$$

$e(\alpha)$ is then defined by the r-fold wedge product

$$\frac{1}{r!}\left(\frac{\alpha}{2\pi}\right)^r = e(\alpha)\, dz^1 \wedge \cdots \wedge dz^k. \tag{6.19}$$

The Pfaffian $e(\alpha)$ is $SO(k)$-invariant. The Euler form of the bundle E is found by substituting the bundle's $SO(k)$-valued curvature 2-form Ω for α:

$$\text{Euler form} = e(\Omega).$$

The Euler form is always closed and the characteristic class $e(E)$ is independent of the particular Riemannian metric and connection chosen.

Properties of the Euler class: While a real anti-symmetric matrix like $|\alpha_{ij}|$ cannot be diagonalized, it can be put in the form

$$\begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x_r \\ & & & -x_r & 0 \end{pmatrix}$$

The splitting-principle formula for $e(E)$ is thus

$$e(E) = x_1 x_2 \dots x_r.$$

Since

$$p_r(E) = x_1^2 x_2^2 \dots x_r^2 = e^2(E),$$

we conclude that $e(E)$ is a *square root* of the highest Pontrjagin class. If we change the orientation of E, we replace $e(E)$ by $-e(E)$, and change the sign of the square root.

It is clear that e is multiplicative for Whitney sums:

$$e(E \oplus F) = e(E)\, e(F),$$

where we define $e(E) = 0$ for odd-dimensional bundles.

Complex bundles: If E is a complex vector bundle of dimension r, then its real $2r$-dimensional counterpart E_r inherits a natural orientation. Then we know that

$$e(E_r)^2 = p_r(E_r) = c_r(E)^2.$$

In fact, the signs work out so that $e(E_r)$ is just the *top Chern class* of E,

$$e(E_r) = [p_r(E_r)]^{1/2} = c_r(E).$$

Gauss–Bonnet theorem: The Gauss–Bonnet theorem for an even-dimensional manifold M relates the Euler characteristic to the Euler class by

$$\chi(M) = \int_M e(T(M)). \tag{6.20}$$

(If M is odd-dimensional, both $e(T(M)) = 0$ and $\chi(M) = 0$.) The example of $P_n(\mathbb{C})$ was worked out in 6.3.2.

Stable and unstable characteristic classes: In some circumstances, the Euler class may be non-zero even for bundles with *vanishing* Pontrjagin classes. For example, consider the tangent bundle of the sphere $T(S^m)$ for even m. Since $\chi = 2$, the Gauss–Bonnet theorem gives

$$e(T(S^m)) = 2 \cdot V(S^m),$$

where $V(S^m) \in H^m(S^m)$ is the normalized S^m volume element. Since

$$T(S^m) \oplus I = I^{m+1}$$

is a trivial bundle over S^m, we find

$$p(T(S^m)\oplus I) = p(T(S^m))\cdot p(I) = p(T(S^m)) = p(I^{m+1}) = 1.$$

Thus the Pontrjagin classes of $T(S^m)$ are trivial. Pontrjagin classes are stable characteristic classes, while the Euler class is an *unstable* characteristic class; *stabilization* is the process of adding a trivial bundle to eliminate low fiber-dimensional pathologies of which the Euler characteristic is an example (see the discussion of K-theory given below).

Examples: The Euler classes for two or four-dimensional Riemannian manifolds M are given by

$$n=2:\quad e(T(M)) = \frac{1}{2\pi}R_{12} = \frac{1}{4\pi}\epsilon_{ab}R^{ab} = \frac{1}{2\pi}R_{1212}e^1\wedge e^2$$

$$n=4:\quad e(T(M)) = \frac{1}{32\pi^2}\epsilon_{abcd}R^{ab}\wedge R^{cd},$$

where R^{ab} is the curvature 2-form in the orthonormal cotangent space basis. Since R^{ab} as a matrix belongs to so(n), we can see from the Weil homorphism construction how $e(T(M))$ emerges as a "square root" of a Pontrjagin class which would itself be zero when curvatures were substituted. For $n=2$, we have

$$\mathrm{Det}\left[I - \frac{1}{2\pi}\begin{bmatrix}0 & \lambda\\ -\lambda & 0\end{bmatrix}\right] = 1 + \frac{\lambda^2}{(2\pi)^2} = 1 + p_1,$$

so we take $\lambda = R_{12}$ to find

$$e = (p_1)^{1/2} = \frac{R_{12}}{2\pi}.$$

For $n=4$, with $R_{i4} = E_i$, $R_{jk} = \frac{1}{2}\epsilon_{ijk}B_i$, we have

$$\mathrm{Det}\left[I - \frac{1}{2\pi}\begin{pmatrix}0 & B_3 & -B_2 & E_1\\ -B_3 & 0 & B_1 & E_2\\ B_2 & -B_1 & 0 & E_3\\ -E_1 & -E_2 & -E_3 & 0\end{pmatrix}\right] = 1 + p_1 + p_2$$

$$= 1 + \frac{1}{(2\pi)^2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) + \frac{1}{(2\pi)^4}(\boldsymbol{E}\cdot\boldsymbol{B})^2$$

$$= 1 - \frac{1}{8\pi^2}R_{ab}R_{ba} + \frac{1}{(2\pi)^4}\frac{1}{64}(\epsilon_{abcd}R^{ab}R^{cd})^2.$$

Hence we find the first Pontrjagin class

$$p_1 = -\frac{1}{8\pi^2}\mathrm{Tr}\, R\wedge R$$

and the Euler class

$$e(T(M)) = (p_2)^{1/2} = \frac{1}{32\pi^2}\epsilon_{abcd}R^{ab} \wedge R^{cd}.$$

Similar formulas hold for all even dimensional cases.
Remark: Clearly the existence of the Euler class as a "square root" follows from the fact that the determinant of an anti-symmetric even-dimensional matrix is a *perfect square*. For odd dimensions, this determinant vanishes, and, in fact, the Euler class for n odd is always zero.

6.4.3. Stiefel–Whitney classes

The Stiefel–Whitney classes of a real bundle E over M with k dimensional fiber are the $\mathbb{Z}_2$ cohomology classes. In contrast to the other characteristic classes we have given earlier, they are not integral cohomology classes and are not given in terms of curvature. We identify the Stiefel–Whitney classes as

$$w_i \in H^i(M;\mathbb{Z}_2) \qquad i = 1, \ldots, n-1.$$

For $i = n$ (n even), w_n has values in $\mathbb{Z}$ rather than $\mathbb{Z}_2$ and is identifiable with the Euler class discussed above. The total Stiefel–Whitney class is, as usual, defined by

$$w(E) = 1 + w_1 + w_2 + \cdots + w_n.$$

The *first Stiefel–Whitney class* $w_1(T(M))$ is zero if and only if M is orientable.

The *second Stiefel–Whitney class* $w_2(T(M))$ is of great importance in physics because it determines whether or not parallel transport of Dirac spinors can be globally defined on $E = T(M)$. If

$$w_1(T(M)) = w_2(T(M)) = 0,$$

then spinors are well-defined and M is a *spin-manifold*. If

$$w_2(T(M)) \neq 0,$$

then there is a sign ambiguity when spinors are parallel-transported around some path in M: *such manifolds do not admit a spin structure.*
Example 1. Stiefel–Whitney classes of $P_n(\mathbb{C})$: The Stiefel–Whitney classes can be computed in closed form from the expression for the cohomology of $T(P_n(\mathbb{C}))$. The total class is just (Milnor and Stasheff [1974])

$$w(T(P_n(\mathbb{C}))) = (1+x)^{n+1} = 1 + w_2 + w_4 + \cdots + w_{2n},$$

where x is the 2-form c_1 of the natural line bundle and all coefficients of x^k are taken *mod 2* except for w_{2n}. Hence we find for $P_n(\mathbb{C})$

$$w_2 = (n+1)|_{\text{mod } 2} \cdot x = \begin{cases} 0 & n \text{ odd} \\ 1 \cdot x & n \text{ even.} \end{cases}$$

In particular, $P_2(\mathbb{C})$, $P_4(\mathbb{C}), \ldots$ do *not* admit a spin structure, while $P_1(\mathbb{C})$, $P_3(\mathbb{C}), \ldots,$ do. Since $w_{2n} = (n+1) \cdot x$, we recover our previous result that the Euler characteristic is $(n+1)$. In addition, all the manifolds $P_n(\mathbb{C})$ are orientable since $w_1 = 0$.
Example 2: The total Stiefel–Whitney class of S^n is

$$w(S^n) = 1 + (1 + (-1)^n)\, V(S^n)$$

where $V(S^n)$ is the normalized n-form volume element. Hence $w_2 = 0$ and all n-spheres are spin manifolds.
Remark: For $S^2 = P_1(\mathbb{C})$, $w_2 = 2x$ plays a double role: the Euler characteristic $= 2$, and $2 \pmod 2) = 0$ implies that a spin structure exists.

6.5. *K-theory*

K-theory is concerned with the study of formal differences of vector bundles and plays an essential role in index theory. From the standpoint of algebraic topology, K-theory is an exotic cohomology theory, although we shall not adopt this viewpoint here (see, Atiyah [1967]).

Problems with formal differences of vector bundles: In the preceding sections we have studied the properties and characteristic classes of Whitney sum bundles such as $E \oplus F$. If $E \oplus F \approx E' \oplus F$, then it is tempting to introduce a formal difference operation which would allow us to cancel the vector bundle F from both sides of this equation and to conclude that $E \approx E'$. Unfortunately this cancellation does not work in general, as we may see from the following example:

Consider the manifold $M = S^2$ to be embedded in $\mathbb{R}^3$, and let $T(S^2)$ and $N(S^2)$ be the tangent and normal bundles, respectively. Letting I^k denote the trivial real vector bundle of dimension k, we note that $N(S^2) \simeq I$, the trivial line bundle. Then we find that

$$T(S^2) \oplus N(S^2) = T(\mathbb{R}^3) \simeq I^3$$

$$I^2 \oplus N(S^2) \simeq I^2 \oplus I = I^3 .$$

If we could perform the formal cancellation of $N(S^2)$, then we would conclude that $T(S^2) \simeq I^2$, which is false. There are similar examples also for complex bundles.

Stable equivalence of vector bundles: The problems with formal differences of vector bundles can be resolved by replacing the notion of vector bundle isomorphism by the broader relationship of stable equivalence. If E and E' are two vector bundles, not necessarily of the same dimension, we say that E and E' are *stably equivalent* and write $E \underset{s}{\approx} E'$ provided that

$$E \oplus I^l \simeq E' \oplus I^j$$

for some integers l and j.

Taking the Whitney sum with trivial bundles serves to eliminate pathologies arising from low fiber dimension; this process is called *stabilization*. Two vector bundles of the same fiber dimension $k > \dim(M)$ are stably equivalent if and only if they are isomorphic; these two notions correspond if the fiber dimension is large enough. Since $E \underset{s}{\simeq} E'$ and $E' \underset{s}{\simeq} E''$ implies $E \underset{s}{\simeq} E''$, then $\underset{s}{\simeq}$ is an *equivalence relation*.

Definition of $K_0(M)$: If $E \oplus F \simeq E' \oplus F$, then E need not be isomorphic to E', but it is stably equivalent to E',

$$E \underset{s}{\simeq} E'.$$

If we define $K_0(M)$ to be the set of stable equivalence classes, then formal differences are well-defined on $K_0(M)$. Thus, for example, $T(S^2) \underset{s}{\simeq} I^2$ and $T(S^2)$ is stably trivial. Let $\mathrm{Vect}_k(M)$ be the set of isomorphism classes of vector bundles of fiber dimension k. We say that k is in the *stable range* provided that:

$k > \dim(M)$ (if we are working with real vector bundles)

$k > \frac{1}{2}\dim(M)$ (if we are working with complex vector bundles),

where $\dim(M)$ denotes the real dimension of M. We can identify $\mathrm{Vect}_k(M)$ with $K_0(M)$ in the stable range. In other words, once k is large enough, given any bundle E there is a bundle E' with fiber dimension k such that $E \underset{s}{\simeq} E'$. Furthermore, if $E \underset{s}{\simeq} E''$ is another such bundle, then E' and E'' are actually isomorphic.

If E is a vector bundle, we can always find a complementary bundle F such that $E \oplus F \simeq I^l$ is trivial for some integer l. The isomorphism class of F is not uniquely defined, but the stable equivalence class of F is unique and defines an element of $K_0(M)$. Since I^l represents the trivial or "zero" element of $K_0(M)$, F is the formal inverse of E. We thus have a group structure on $K_0(M)$. *Formal subtraction* of the bundle E is defined by taking the Whitney sum with the complementary bundle $E^{-1} = F$. Since $K_0(M) = \mathrm{Vect}_k(M)$ for k in the stable range, this also defines a group structure on $\mathrm{Vect}_k(M)$.

Unreduced K-theory: $K_0(M)$ does not distinguish between trivial bundles of different dimension since $I^l \underset{s}{\simeq} I^k$ for any k and l. We define a new group $K(M)$ using the following construction of Grothendieck (see Atiyah [1967]). If E and F are vector bundles, we define the *virtual bundle* $E \ominus F$ representing their formal difference. $K(M)$ is the Abelian group whose elements are virtual bundles. Thus $T(S^2)$ and I^2 represent the same element of $K(S^2)$.

The *virtual dimension* of $E \ominus F$ is $\dim(E) - \dim(F)$. $K_0(M)$ can be identified as the subgroup of $K(M)$ with vanishing virtual dimension.

Note that the tensor product is distributive with respect to the Whitney sum and thus defines a multiplication or ring structure on both $K(M)$ and $K_0(M)$.

Rational K-theory. We define $K(M)$ by allowing objects of the form jE where j could be 0, positive or negative. If j is positive, this is just $E \oplus \cdots \oplus E$, while if j is negative, this is a formal object involving formal differences. It is convenient to consider other coefficient groups in this context just as we did for homology and for cohomology. $K(M; \mathbb{Q})$ and $K_0(M; \mathbb{Q})$ are the groups which arise when we consider objects of the form qE where q is rational:

$$K(M; \mathbb{Q}) = K(M) \times \mathbb{Q} \qquad K_0(M; \mathbb{Q}) = K_0(M) \times \mathbb{Q}.$$

So far, we have not really distinguished between the complex and real case except to note that the stable range is greater in the complex case. We shall reserve the notation $K(M)$ and $K_0(M)$ for the group of complex bundles and shall use the notation $K^r(M)$ and $K_0^r(M)$ for the group of real vector bundles.

The Chern isomorphism: The Chern character provides the bridge between rational K-theory and rational cohomology. We recall that the Chern character satisfies the identities

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F), \qquad \mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F).$$

We can, in fact, extend the Chern character to K theory so that

$$\mathrm{ch}(E \ominus F) = \mathrm{ch}(E) - \mathrm{ch}(F).$$

This relationship is one of the important consequences of the Grothendieck construction.

The Chern character is a ring isomorphism from $K(M; \mathbb{Q})$ to the even-dimensional cohomology of M; it is a map

$$\mathrm{ch}\colon\quad K(M; \mathbb{Q}) \to \bigoplus_j H^{2j}(M; \mathbb{Q}).$$

If we restrict the Chern character to the subgroup $K_0(M; \mathbb{Q})$, then ch provides an isomorphism

$$\mathrm{ch}\colon\quad K_0(M; \mathbb{Q}) \simeq \bigoplus_{j>0} H^{2j}(M; \mathbb{Q}).$$

In other words, if M has non-trivial even cohomology, then M will have non-trivial vector bundles. In the real case, $c_j(E) = 0$ if j is odd so

$$\mathrm{ch}\colon\quad K^r(M; \mathbb{Q}) \simeq \bigoplus_j H^{4j}(M; \mathbb{Q}).$$

Thus, for example, any *real* vector bundle over S^2 is stably trivial since there is no real cohomology in dimensions divisible by 4 above H^0. On S^4, by contrast, there are many non-trivial bundles which are parametrized by the first Pontrjagin class p_1 because $H^4(S^4; \mathrm{Q}) = \mathrm{Q}$.

Torsion in K-theory: Suppose $k > \frac{1}{2}\dim(M)$ is in the stable range and consider the set of all cohomology classes of the form $\mathrm{ch}(E)$ as E ranges over all possible bundles with fiber dimension k. This set spans the even rational cohomology of M. Furthermore, if $\mathrm{ch}(E) = \dim(E)$ (i.e., $c_l(E) = 0$ for $l > 0$), then some multiple of E is stably trivial: there exists an integer j such that

$$E \oplus \cdots \oplus E \simeq I^{j\cdot\dim(E)}.$$

In other words, $jE = 0$ in K-theory so E is a *torsion element* of $K(M)$. The Chern character permits us to compute $K(M)$ *modulo torsion.*

The existence of torsion elements in K-theory can be illustrated by the following example: consider $P_2(\mathbb{R})$, which is S^2 modulo the identification of antipodal points, $x \sim -x$. We define L as the bundle over

$P_2(\mathbb{R})$ obtained by identifying $(x, z) \sim (-x, -z)$ in $S^2 \times \mathbb{C}$ (this is a generalization of the Möbius bundle). A section s of L over $P_2(\mathbb{R})$ is simply a function s on S^2 satisfying the identity $-s(x) = s(-x)$. Since any such function must have a zero, L is non-trivial (and in fact is not stably trivial so L represents a non-zero element of $K(P_2(\mathbb{R}))$). A frame for $L \oplus L$ is just a map g: $S^2 \to \mathrm{GL}(2, \mathbb{C})$ such that $g(x) = -g(-x)$. If we define:

$$g(x) = \begin{pmatrix} x_1 & x_2 + \mathrm{i}x_3 \\ x_2 - \mathrm{i}x_3 & -x_1 \end{pmatrix}$$

then $g(x)^2 = I$ for $x \in S^2$. Thus $L \oplus L \simeq I^2$ on $P_2(\mathbb{R})$ and L represents a torsion element of $K(P_2(\mathbb{R}))$.

If M has only even dimensional free cohomology, then there are *no* torsion elements in $K(M)$ so we can identify $K(M)$ with $\oplus_j H^{2j}(M; \mathbb{Z})$ additively (the ring structures are different). Since both S^n and $P_n(\mathbb{C})$ satisfy these hypotheses, we conclude that:

$$K(S^n) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \text{ if } n \text{ is even} \\ \mathbb{Z} \text{ if } n \text{ is odd} \end{cases} \qquad K(P_n(\mathbb{C})) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (n+1 \text{ times})$$

$$K_0(S^n) = \begin{cases} \mathbb{Z} \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases} \qquad K_0(P_n(\mathbb{C})) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (n \text{ times}).$$

For example: if n is odd and if $\dim(E) > \frac{1}{2}n$, then E is trivial on S^n since $K_0(S^n) = 0$. If n is even and if $k > \frac{1}{2}n$, we may identify

$$\mathrm{Vect}_k(S^n) = K_0(S^n) = \mathbb{Z}.$$

In other words, the stable equivalence class of any bundle E over S^n can be determined from the integer

$$\int_{S^n} c_l(E) \qquad \text{for } l = n/2.$$

The bundles constructed in example 5.4.2 give the generators for $K_0(S^n)$ if n is even.

Bott periodicity is the statement that the stable homotopy groups of $U(k)$ are periodic. This means that:

$$\pi_j(U(k)) = \begin{cases} \mathbb{Z} \text{ for } j \text{ odd and } j < 2k \\ 0 \text{ for } j \text{ even and } j < 2k. \end{cases}$$

This is related to the calculation of $K_0(S^{n+1}) = \mathrm{Vect}_k(S^{n+1})$ as follows: let E be a k-dimensional bundle over S^{n+1} and let $D_\pm$ be the upper and lower hemispheres of S^{n+1}. These are contractible so E is trivial over these sets. Let $e_\pm$ be unitary frames for E over $D_\pm$ and let $e_- = g(x)\, e_+$ on $S^n = D_+ \cap D_-$. $g(x)$ is the transition function defining E and gives a map g: $S^n \to U(k)$ which represents an element of $U(k)$. This map is in fact an isomorphism in the stable range. Therefore:

$$\pi_n(U(k)) = K_0(S^{n+1}) = \begin{cases} \mathbb{Z} & \text{if } n+1 \text{ is even} \quad (\text{i.e. } n \text{ is odd}) \\ 0 & \text{if } n+1 \text{ is odd} \quad (\text{i.e. } n \text{ is even}). \end{cases}$$

For example, we find that $\pi_1(U(k)) = \mathbb{Z}$, so $\mathrm{Vect}_k(S^2) = \mathbb{Z}$ for all k. Since $\pi_2(U(k)) = 0$, we conclude that $\mathrm{Vect}_k(S^3) \simeq 0$ for all k.

Another way of stating Bott periodicity is to take $k = \infty$ and write

$$\pi_n(U(\infty)) = \pi_{n+2}(U(\infty)).$$

A similar but somewhat more involved argument for the real groups $O(k)$ leads to the formula

$$\pi_n(O(\infty)) = \pi_{n+8}(O(\infty)).$$

Remark: Difference bundles of the type treated by K-theory play an essential role in the mathematical definition of high-spin fields, such as the spin $\frac{3}{2}$ Rarita–Schwinger field. K-theory is implicitly used in the applications of index theorems to high-spin fields described in section 10.

7. Index theorems: Manifolds without boundary

The index theorem states the existence of a relationship between the analytic properties of differential operators on fiber bundles and the topological properties of the fiber bundles themselves. The simplest example is the Gauss–Bonnet theorem, which relates the number of harmonic forms on the manifold (Betti numbers) to the topological Euler characteristic given by integrating the Euler form over the manifold. In this case, the relevant differential operator is the exterior derivative mapping $C^\infty(\Lambda^p) \to C^\infty(\Lambda^{p+1})$, and the analytic property in question is the number of zero-frequency solutions to Laplace's equation. In general, the index theorem gives analogs of the Gauss–Bonnet theorem for other differential operators. The index of an operator, determined by the number of zero-frequency solutions to a generalized Laplace's equation, is expressed in terms of the characteristic classes of the fiber bundles involved. Thus the index theorem gives us useful information concerning various types of differential equations provided we understand the topology of the fiber bundles upon which the differential operators are defined.

We will first discuss the general formulation of the index theorem and then apply it to the classical elliptic complexes. We work out the index theorem explicitly in dimensions two and four for the de Rham, signature, Dolbeault and spin complexes. The index theorems for these complexes correspond to the Gauss–Bonnet theorem, the Hirzebruch signature theorem, the Riemann–Roch theorem, and the index theorem for the $\hat{A}$-genus. We conclude with a discussion of the Lefschetz fixed point theorem and the G-index theorem.

7.1. The index theorem

We begin for the sake of completeness with a fairly abstract description of the index theorem of Atiyah and Singer [1968a, b; 1971a, b]. The reader who is interested in specific applications may proceed directly to the appropriate subsequent sections. For an alternative treatment using heat equation methods, see, for example, Gilkey [1974], and references quoted therein.

Let M be a compact smooth manifold without boundary of dimension n. We will consider the case of manifolds with boundary in section 8. Let E and F be vector bundles over M and let D: $C^\infty(E) \to C^\infty(F)$ be a first-order differential operator. We choose local bundle coordinates for E and for F, with

$\{x_j\}$ being local coordinates on M. Then we can decompose D in the form

$$D = a_j(x)\, \partial/\partial x_j + b,$$

where the a_j and b are matrix-valued.

Symbol of an operator: The symbol of an operator is its *Fourier transform*. Let (x, k) be local coordinates on $T^*(M)$; we regard k as the Fourier-transform variable. Let $\tilde{f}(k)$ be the Fourier transform of $f(x)$ and recall that

$$Df(x) = a_j(x) \frac{\partial f(x)}{\partial x_j} + bf$$

$$= \int [\mathrm{i}a_j(x)\, k_j + b]\, \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}x \cdot k}\, \mathrm{d}k.$$

The *leading symbol* $\tilde{D}$ of D is the highest-order part of its Fourier transform,

$$\tilde{D}(x, k) = \sigma_{\mathrm{L}}(D)\,(x, k) = \mathrm{i}a_j(x) k_j.$$

This is a linear map from E to F.

Elliptic complexes: If $E = F$ and if $\tilde{D}(x, k)$ is invertible for $k \neq 0$, then D is said to be an *elliptic operator*. A similar definition holds for higher order operators.

Let $\{E_p\}$ be a finite sequence of vector bundles over M and let D_p: $C^\infty(E_p) \to C^\infty(E_{p+1})$ be a sequence of differential operators. We assume that this sequence is a *complex*, i.e., $D_{p+1} D_p = 0$. Figure 7.1 gives the standard graphical depiction of such a complex. Now let D_p^*: $C^\infty(E_{p+1}) \to C^\infty(E_p)$ be the dual map and let

$$\Delta_p = D_p^* D_p + D_{p-1} D_{p-1}^*$$

be the associated Laplacian. The complex is *elliptic* if Δ_p is an elliptic operator on $C^\infty(E_p)$. Equivalently, the complex is elliptic (or exact on the symbol level) if

$$\operatorname{Ker} \tilde{D}_p(x, k) = \operatorname{image} \tilde{D}_{p-1}(x, k), \qquad k \neq 0.$$

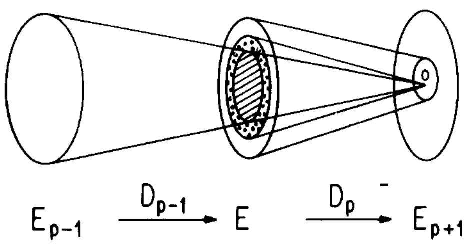

Fig. 7.1. A piece of a complex with $D_p D_{p-1} = 0$. The hatched area is Im D_{p-1}. The dotted area is Ker D_p/Im D_{p-1}.

Here the exactness on the symbol level plays a role analogous to that played by the Poincaré lemma in de Rham cohomology. These properties define an *elliptic complex*, denoted by $(E, D) = (\{E_p\}, \{D_p\})$.

Cohomology of elliptic complexes: There is a generalization of the Hodge decomposition theorem for an elliptic complex (E, D). If $f_p \in C^\infty(E_p)$, then f_p can be uniquely decomposed as a sum

$$f_p = D_{p-1} f_{p-1} + D_p^* f_{p+1} + h_p$$

where h_p is harmonic in the sense that $\Delta_p h_p = 0$.

We observe that

$$\operatorname{Ker} D_p \supset \operatorname{Image} D_{p-1}$$

because $D_p D_{p-1} = 0$. We may thus define cohomology groups for the elliptic complex (E, D) by (see fig. 7.1)

$$H^p(E, D) = \operatorname{Ker} D_p / \operatorname{Image} D_{p-1}. \tag{7.1}$$

As in de Rham cohomology, each cohomology class contains a unique harmonic representative, so we have an isomorphism

$$H^p(E, D) \approx \operatorname{Ker} \Delta_p. \tag{7.2}$$

These cohomology groups are finite-dimensional.

The index of an elliptic complex (E, D) is

$$\begin{aligned} \operatorname{index}(E, D) &= \sum_p (-1)^p \dim H^p(E, D) \\ &= \sum_p (-1)^p \dim \operatorname{Ker} \Delta_p. \end{aligned} \tag{7.3}$$

Example: Let $E_p = \Lambda^p(M)$ and let $D_p = \mathrm{d}$ be exterior differentiation on p-forms. Then

$$H^p(E, D) = H^p_{\mathrm{DR}}(M) = H^p(M; \mathbf{R})$$

by the de Rham theorem. The index of this complex is therefore the Euler characteristic,

$$\operatorname{index}(\Lambda^*, \mathrm{d}) = \sum_p (-1)^p \dim H^p(M; \mathbf{R}) = \sum_p (-1)^p b_p = \chi(M). \tag{7.4}$$

Note that the leading symbol of the Laplacian is $\tilde{D}(x, k) = +|k|^2$, so the complex is indeed elliptic.

Rolling up the complex: It is possible to construct a convenient two-term elliptic complex with the same index as a given complex (E, D). Let

$$F_0 = \bigoplus_p E_{2p} \qquad F_1 = \bigoplus_p E_{2p+1}$$

be the *even* and *odd* bundles, respectively. Then we consider the operators

$$A = \bigoplus_p (D_{2p} + D^*_{2p-1})$$

$$A^* = \bigoplus_p (D^*_{2p} + D_{2p-1})$$

where $A\colon C^\infty(F_0) \to C^\infty(F_1)$ and $A^*\colon C^\infty(F_1) \to C^\infty(F_0)$. The associated Laplacians are

$$\Box_0 = A^*A = \bigoplus_p \Delta_{2p}$$

$$\Box_1 = AA^* = \bigoplus_p \Delta_{2p+1}.$$

Therefore

$$\begin{aligned} \text{index}\,(F, A) &= \dim \operatorname{Ker} \Box_0 - \dim \operatorname{Ker} \Box_1 \\ &= \sum_p (-1)^p \dim \operatorname{Ker} \Delta_p = \text{index}\,(E, D). \end{aligned} \tag{7.5}$$

We note that if $k \neq 0$, the leading symbol $\tilde{A}(x, k)$ of A is an invertible matrix mapping F_0 to F_1. In particular, these two bundles have the same dimension.

Example: Let $(E, D) = (\Lambda^*, \mathrm{d})$ be the de Rham complex. Then F_0 is the bundle of even forms, F_1 is the bundle of odd forms, and $A = d + \delta$. The Euler characteristic is the sum of the *even* Betti numbers minus the sum of the *odd* Betti numbers.

The index theorem: The general index theorem may be described as follows: Let (x, k) be local coordinates for $T^*(M)$ and choose the "symplectic orientation" $\mathrm{d}x_1 \wedge \mathrm{d}k_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}k_n$. Let $D(M)$ be the unit disk bundle in $T^*(M)$ defined

$$D(M) = \{(x, k)\colon |k|^2 \leq 1\}$$

and let the unit sphere bundle $S(M)$,

$$S(M) = \{(x, k)\colon |k|^2 = 1\}$$

be its boundary. Now take *two* copies $D_\pm(M)$ of the unit disk bundles and glue them together along their common boundary $S(M)$ to define a new fiber bundle $\Psi(M)$ over M with *fiber* S^n. $\Psi(M)$ is the *compactified tangent bundle* of M. The orientation on $\Psi(M)$ is chosen to be that of $D_+(M)$. Finally, let ρ be the projection,

$$\rho\colon \Psi(M) \to M \tag{7.6}$$

and let $\rho_\pm$ be the restrictions of ρ to the "hemisphere bundles" $D_\pm(M)$,

$$\rho_\pm\colon D_\pm(M) \to M. \tag{7.7}$$

Given this structure, we wish to compute the index of an elliptic complex (E, D), which we roll up to form a two-term elliptic complex (F, A). Let $\tilde{A}(x, k)$ be the leading symbol of the operator A. Now consider the pullback bundles

$$F_+ = \rho_+^*(F_0) \quad \text{over } D_+(M)$$
$$F_- = \rho_-^*(F_1) \quad \text{over } D_-(M). \tag{7.8}$$

Intuitively, we are placing the two bundles of the complex over the two hemispheres of $\Psi(M)$. We would now like to glue these bundles together to form a smooth bundle over $\Psi(M)$.

We can regard $\tilde{A}(x, k) = \sigma_L(A)(x, k)$ as a map from F_+ to F_- over $S(M) = D_+(M) \cap D_-(M)$. Because the complex is elliptic, $\tilde{A}(x, k)$ is an isomorphic map from F_+ to F_- over $S(M)$. We use this isomorphism to define the vector bundle $\Sigma(A)$ obtained by gluing F_+ to F_- using the transition function $\tilde{A}(x, k)$ over $S(M)$. $\Sigma(A)$ is sometimes called the *symbol bundle.*

Let $\mathrm{td}(M)$ be the Todd class of $T(M)$ and $\mathrm{ch}(\Sigma(A))$ be the Chern character of the symbol bundle. Then the *Atiyah–Singer index theorem* states that

$$\mathrm{index}(E, D) = \mathrm{index}(F, A) = \int_{\Psi(M)} \mathrm{ch}(\Sigma(A)) \wedge \rho^* \,\mathrm{td}(M). \tag{7.9}$$

We include in the integrand only those terms of dimension $2n = \dim \Psi(M)$. For the four classical elliptic complexes, this formula reduces to the form

$$\mathrm{index}(E, D) = (-1)^{n(n+1)/2} \int_M \mathrm{ch}\Big(\bigoplus_p (-1)^p E_p\Big) \frac{\mathrm{td}(M)}{e(M)}, \tag{7.10}$$

where $e(M)$ is the Euler form and the division is heuristic.

Note: The index of any elliptic complex over an odd-dimensional manifold is zero; this would not be true if we considered *pseudo*-differential operators. For example, let

$$M = S^1$$
$$F_0 = F_1 = S^1 \times \mathbb{C}$$
$$A = \mathrm{e}^{-\mathrm{i}\theta}(-\mathrm{i}\,\partial_\theta + (-\partial_\theta^2)^{1/2}) - (\mathrm{i}\,\partial_\theta + (-\partial_\theta^2)^{1/2})$$
$$\tilde{A}(\theta, k) = \mathrm{e}^{-\mathrm{i}\theta}(k + |k|) + (k - |k|).$$

This is a pseudo-differential elliptic complex with index = 1.

7.2. The de Rham complex

The exterior algebra $\Lambda^*(M)$ can be split into two distinct elliptic complexes. In this subsection we discuss the first, the de Rham complex, which is related to the Euler characteristic. We will discuss the second, the signature complex, in the following subsection.

The de Rham complex arises from the decomposition of the exterior algebra into even and odd forms:

$$\Lambda^{\text{even}} = \Lambda^0 \oplus \Lambda^2 \oplus \cdots$$
$$\Lambda^{\text{odd}} = \Lambda^1 \oplus \Lambda^3 \oplus \cdots$$

The operator for this elliptic complex is $\mathrm{d}+\delta$ where

$$(\mathrm{d}+\delta)\colon C^\infty(\Lambda^{\text{even}}) \to C^\infty(\Lambda^{\text{odd}}).$$

The index of the de Rham complex is the Euler characteristic $\chi(M)$,

$$\operatorname{index}(\Lambda^{\text{even,odd}}, \mathrm{d}+\delta) = \chi(M). \tag{7.11}$$

When we apply the index theorem to the de Rham complex, we recover the Gauss–Bonnet theorem,

$$\chi(M) = \int_M e(M), \tag{7.12}$$

where $e(M)$ is the Euler form. Using the results of the previous section, we may express $e(M)$ explicitly to show

$n = 2$:

$$\chi(M) = \frac{1}{4\pi} \int_M R_{ijij}\, \mathrm{d\,vol}$$

$$= \frac{1}{4\pi} \int_M \epsilon_{ab} R_{ab} = \frac{1}{2\pi} \int_M R_{12},$$

$n = 4$:

$$\chi(M) = \frac{1}{16\pi^2} \int_M \left(\tfrac{1}{2} R_{ijij} R_{klkl} - 2 R_{ijik} R_{ljlk} + \tfrac{1}{2} R_{ijkl} R_{ijkl}\right) \mathrm{d\,vol}$$

$$= \frac{1}{32\pi^2} \int_M \epsilon_{abcd} R_{ab} \wedge R_{cd},$$

where R_{ab} is the curvature 2-form of M.

It is worth noting that we can use these integrals to evaluate $\chi(M)$ even if M is *not* orientable by regarding (d vol) as a measure rather than as an n-form. The remaining index theorems will only apply to oriented manifolds.

Examples: (1) If $M = S^n$, then $\chi = 0$ for n = odd, $\chi = 2$ for n = even. (2) If $M = P_n(\mathbb{C})$, then $\chi = n + 1$.

7.3. The signature complex

The second splitting of the exterior algebra leads to the signature complex. We restrict ourselves henceforth to oriented manifolds of even dimension, $n = 2l$. We recall that the Euler characteristic $\chi(M)$ can be regarded either as a topological invariant or as the index of the de Rham complex. Similarly, the signature can be regarded either topologically, or as the index of an elliptic complex.

Topological signature. Let θ and ϕ belong to the middle cohomology group $H^l(M;\mathbb{R})$ and define the inner product

$$\sigma(\theta,\phi) = \int_M \theta \wedge \phi.$$

This inner product is *symmetric* if l = even (so n is divisible by 4) and *anti-symmetric* if l = odd. By Poincaré duality, this inner product is non-degenerate: for any $\theta \neq 0$, there is a ϕ such that $\sigma(\theta,\phi) \neq 0$. The *topological signature* $\tau(M)$ is defined as the signature of this quadratic form, i.e., the number of positive eigenvalues minus the number of negative eigenvalues. Note that if l = odd (i.e., n was not divisible by 4), then $\tau(M)$ vanishes automatically.

If $n = 4k$, we may relate the signature to the space of harmonic forms $H^{2k}(M;\mathbb{R})$. Since $*^2 = 1$ on $H^{2k}(M;\mathbb{R})$, we may decompose the harmonic forms into subspaces $H^{2k}_{\pm}(M;\mathbb{R})$ with eigenvalues ± 1 under the action of Hodge $*$. Since $\sigma(\theta,\phi)$ is related to the standard inner product by

$$\sigma(\theta,\phi) = (\theta, *\phi) = \int_M \theta \wedge \phi,$$

the decomposition of H^{2k} into $H^{2k}_{\pm}$ diagonalizes the quadratic form. Therefore, we may express the signature of M as

$$\begin{aligned}\tau(M) &= \dim H^{2k}_{+}(M;\mathbb{R}) - \dim H^{2k}_{-}(M;\mathbb{R})\\ &= b^{+}_{2k} - b^{-}_{2k},\end{aligned} \tag{7.13}$$

where we have split the middle dimension Betti number into $b_{2k} = b^{+}_{2k} + b^{-}_{2k}$.

Examples: (1) If $M = S^{2l}$, then $n = 2l$ and $b_l = 0$, so $\tau = 0$. (2) If $M = P_{2l}(\mathbb{C})$, then $n = 4l$ and $b_{2l} = b^{+}_{2l} = 1$, so $\tau = 1$.

Signature complex: We may use the above relationship to compute $\tau(M)$ as the index of an elliptic complex. We define an operator ω acting on p-forms by

$$\omega = \mathrm{i}^{p(p-1)+n/2}\,*,$$

where $\omega = *$ on Λ^{2k} if $n = 4k$. It is easy to show that

$$\omega(\mathrm{d}+\delta) = -(\mathrm{d}+\delta)\omega$$

$$\omega^2 = +1.$$

(Note that $(-\mathrm{i})^{n/2}\omega$ is just Clifford multiplication by the volume form.) Now let $\Lambda^{\pm}$ be the ± 1 eigenspaces of ω. Since ω anticommutes with $\mathrm{D} = \mathrm{d} + \delta$, we may define the elliptic complex

$$(\mathrm{d}+\delta):\quad C^{\infty}(\Lambda^{+}) \to C^{\infty}(\Lambda^{-}).$$

This is the *signature complex*. The contributions of the harmonic forms with eigenvalues ± 1 under ω cancel except in the middle dimension. *The index of the signature complex is the signature* $\tau(M)$,

$$\operatorname{index}(\Lambda^{\pm}, \mathrm{d}+\delta) = \dim H^{2k}_{+}(M; \mathbb{R}) - \dim H^{2k}_{-}(M; \mathbb{R}) = \tau(M). \tag{7.14}$$

When we apply the index theorem to the signature complex, we recover the Hirzebruch signature theorem,

$$\tau(M) = \int_M L(M), \tag{7.15}$$

where $L(M)$ is the Hirzebruch L-polynomial

$$L(M) = \prod_j \frac{x_j}{\tanh x_j} = 1 + \frac{1}{3} p_1 + \frac{1}{45}(7p_2 - p_1^2) + \cdots .$$

We only evaluate the integral for the part of $L(M)$ which is an n-form, and so $\tau(M) = 0$ if n is not a multiple of 4. Since the formula depends on the orientation of M, $\tau(M)$ changes sign when we reverse the orientation. Using the results of the previous section, we may express $L(M)$ explicitly to show

$n = 2$:

$$\tau(M) = 0$$

$n = 4$

$$\tau(M) = \frac{1}{3}\int_M p_1(T(M)) = -\frac{1}{24\pi^2}\int_M \operatorname{Tr}(R \wedge R).$$

Twisted signature complex (Atiyah, Bott and Patodi [1973, 1975]). Although $\tau(M) = 0$ for $n = 2, 6, 10, \ldots$, we can obtain a non-trivial index problem by taking coefficients in another vector bundle V. We can extend $(\mathrm{d}+\delta)$ to an operator $(\mathrm{d}+\delta)_V$, where

$$(\mathrm{d}+\delta)_V:\quad C^{\infty}(\Lambda^{+} \otimes V) \to C^{\infty}(\Lambda^{-} \otimes V).$$

The index theorem then becomes

$$\operatorname{index}(\Lambda^{\pm} \otimes V, (\mathrm{d}+\delta)_V) = \int_M L(M) \wedge \widetilde{\mathrm{ch}}(V), \tag{7.16}$$

where $\widetilde{\mathrm{ch}}$ is the Chern character with Ω replaced by 2Ω, i.e.,

$$\widetilde{\mathrm{ch}}(V) = \sum_k \left(\frac{\mathrm{i}}{2\pi}\right)^k \frac{2^k}{k!} \mathrm{Tr}(\Omega^k). \tag{7.17}$$

Thus, in particular, we find

$n = 2$:

$$\text{index} = \int_M 2c_1(V) = \frac{\mathrm{i}}{\pi} \int_M \mathrm{Tr}\,\Omega$$

$n = 4$:

$$\text{index} = \dim(V)\frac{1}{3}\int_M p_1 + \int_M (2c_1^2(V) - 4c_2(V))$$

$$= -\frac{\dim(V)}{24\pi^2}\int_M \mathrm{Tr}\,R \wedge R - \frac{1}{2\pi^2}\int_M \mathrm{Tr}\,\Omega \wedge \Omega$$

where Ω is the curvature of the bundle V. (Recall that if F is a 2-form corresponding to physical gauge field strengths, then $\Omega = \mathrm{i}F$ for $U(1)$ bundles, $\Omega = (\lambda^a/2\mathrm{i})F_a$ for $\mathrm{SU}(n)$ bundles, etc.)

If we perform the corresponding construction for the de Rham complex to define $(\mathrm{d}+\delta)_V$: $C^\infty(\Lambda^{\text{even}} \otimes V) \to C^\infty(\Lambda^{\text{odd}} \otimes V)$, then the index of this elliptic complex is just $\dim(V)\chi(M)$; the twisting is not detected by the de Rham complex. However, the signature complex is quite sensitive to the twisting, which can be used to produce an elliptic complex with non-zero index even in dimensions not divisible by 4.

7.4. *The Dolbeault complex*

If M is a complex manifold of real dimension n (complex dimension $n/2$), we may split the exterior algebra in yet another way. In section 3.4, we examined complex manifolds and defined the operator

$$\bar{\partial}: C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1}).$$

The Dolbeault complex is obtained by taking $p = 0$. We write the index of this complex as

$$\text{index}(\bar{\partial}) = \sum_{q=0}^{n/2} (-1)^q \dim H^{0,q}(M),$$

where $H^{p,q}$ is the cohomology group of $\bar{\partial}$ on $C^\infty(\Lambda^{p,q})$. *The index of the Dolbeault complex is the arithmetic genus* of the manifold and is the complex analog of the Euler characteristic. If the metric is Kähler, there is a natural identification

$$H^k(M;\mathbb{R}) = \bigoplus_{p+q=k} H^{p,q}(M),$$

so that the $H^{p,q}$ can be regarded as a refinement of de Rham cohomology.

When we apply the index theorem to the Dolbeault complex, we recover the Riemann–Roch theorem:

$$\text{index}(\bar\partial) = \int_M \text{td}(T_c(M)), \tag{7.18}$$

where $T_c(M)$ is the complex tangent space introduced in section 3.4 and td is the Todd class:

$$\text{td}(T_c(M)) = \prod_j \frac{x_j}{1-e^{-x_j}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_2 + c_1^2) + \cdots .$$

In the special cases $n = 2$ and $n = 4$, we can relate the arithmetic genus to the signature and the Euler characteristic as follows:

$n = 2$:

$$\text{index}(\bar\partial) = \tfrac{1}{2}\chi(M)$$

$n = 4$:

$$\text{index}(\bar\partial) = \tfrac{1}{4}(\chi(M) + \tau(M)).$$

Examples: (1) If $M = P_n(\mathbb{C})$, index $(\bar\partial) = 1$. (2) If $M = S^1 \times S^1$, index $(\bar\partial) = 0$.
Remark: We can use these formulas to show that certain manifolds do not admit complex structures. For example, if $M = S^4$, then $\chi = 2$, $\tau = 0$ and index $(\bar\partial) = \frac{1}{2}$, which shows S^4 is not complex. $P_2(\mathbb{C})$ with the proper orientation has index $(\bar\partial) = \frac{1}{4}(3+1) = 1$ and is complex; $P_2(\mathbb{C})$ with the opposite orientation is *not* complex since index $(\bar\partial) = \frac{1}{4}(3-1) = \frac{1}{2}$.

Twisted Dolbeault complex: Just as in the case of the signature complex, we can consider the tensor product bundle $\Lambda^{0,q} \otimes V$ and obtain a corresponding elliptic complex. The index theorem then becomes

$$\text{index}(\bar\partial_V) = \int_M \text{td}(T_c(M)) \wedge \text{ch}(V) \tag{7.19}$$

where ch(V) is the *ordinary* Chern character of V without any additional powers of 2. Thus, in particular, we find

$n = 2$:

$$\begin{aligned}\text{index}(\bar\partial_V) &= \frac{1}{2}\dim(V) \int_M c_1(T_c(M)) + \int_M c_1(V) \\ &= \frac{1}{2}\dim(V)\,\chi(M) + \frac{i}{2\pi}\int_M \text{Tr}\,\Omega\end{aligned}$$

$n = 4$:

$$\operatorname{index}(\bar\partial_V) = \frac{1}{12}\dim(V)\int_M [c_2(T_c(M)) + c_1^2(T_c(M))] + \frac{1}{2}\int_M [c_1(T_c(M)) \wedge c_1(V) + c_1^2(V) - 2c_2(V)].$$

In particular, if we take $V = \Lambda^{p,0}$, then we can compute $\Sigma_q (-1)^q \dim H^{p,q}(M)$ for any value of p, not just for $p = 0$.

7.5. The spin complex

The spin complex is perhaps the most subtle and interesting of the classical elliptic complexes. The deepest insight into its mathematical structure can be achieved using Clifford algebra bundles (Atiyah, Bott and Shapiro [1964]). Clifford algebras also provide a unified context for treating *all four* of the classical elliptic complexes. In fact, one may use the Clifford algebra approach to show that the spin complex is interpretable as the *square-root of plus or minus the de Rham complex*. Here we shall give a more mundane treatment of the spin complex.

We begin by restricting ourselves to a four-dimensional Euclidean-signature Riemannian spin manifold M. We choose Dirac matrices obeying

$$\{\gamma^a, \gamma^b\} \equiv \gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta_{ab}$$

and take the representation

$$\gamma^a = \begin{pmatrix} 0 & \mathrm{i}\alpha_a \\ -\mathrm{i}\bar\alpha_a & 0 \end{pmatrix}; \qquad \alpha_a = (I, -\mathrm{i}\boldsymbol{\lambda}), \quad \bar\alpha_a = (I, \mathrm{i}\boldsymbol{\lambda})$$

where $\{\boldsymbol{\lambda}\}$ are the 2×2 Pauli matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the chiral operator γ_5 is diagonal,

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and we may split the space of Dirac spinors $\{\psi_\alpha\}$ into two eigenspaces of chirality ± 1:

$$\gamma_5\psi_\pm = \pm\psi_\pm.$$

The Dirac operator D is defined using the covariant derivative with respect to the basis of orthonormal frames of $T_x^*(M)$. Thus we take

$$\begin{aligned} D &= \gamma^a E_a{}^\mu(x)\, D_\mu(x) \\ &= \gamma^a E_a{}^\mu(x)\left(\frac{\partial}{\partial x^\mu} + \tfrac{1}{4}[\gamma_b, \gamma_c]\,\omega_\mu{}^{bc}(x)\right), \end{aligned}$$

where $E_a{}^\mu$ is an inverse vierbein of the metric on M and $\omega_\mu{}^{bc}\,dx^\mu$ is the *spin connection* introduced in section 3. We observe that

$$D^\dagger D = DD^\dagger = -g^{\mu\nu}D_\mu D_\nu + \tfrac{1}{4}[\gamma_a, \gamma_b]\tfrac{1}{4}[\gamma_c, \gamma_d]R^{abcd}$$
$$= -g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + \cdots,$$

so the leading part of the operator is *elliptic* for metrics with Euclidean signature.

Clearly the spinors $\psi_+(x)$ upon which D acts are the analogs of C^∞ sections of the *fibers* of the bundles we treated in previous examples. We therefore must introduce a pair of corresponding *spin bundles* $\Delta_\pm$ over M with local coordinates

$$\Delta_\pm: (x^\mu, \psi_\pm).$$

Thus we finally arrive at the following definition of the *spin complex*

$$D: \quad C^\infty(\Delta_+)\to C^\infty(\Delta_-)$$
$$D^\dagger: \quad C^\infty(\Delta_-)\to C^\infty(\Delta_+).$$

The index of the spin complex is

$$\text{index}(\Delta_\pm, D) = \dim \operatorname{Ker} D - \dim \operatorname{Ker} D^\dagger$$
$$= n_+ - n_- \tag{7.20}$$

where

$$n_\pm = (\text{number of chirality} = \pm 1 \text{ normalizable zero-frequency Dirac spinors}).$$

When we apply the index theorem to the spin complex, we find

$$n_+ - n_- = \int_M \hat{A}(M) \tag{7.21}$$

where the *A-roof genus* is given by

$$\hat{A}(M) = \prod_{i=1}^{n/2} \frac{x_i/2}{\sinh(x_i/2)} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1{}^2 - 4p_2) + \cdots$$

when $n = \dim M$ is a multiple of 4. For $n = 4$, we find

$$n_+ - n_- = -\frac{1}{24}P_1 \equiv -\frac{1}{24}\int_M p_1(T(M)) = +\frac{1}{24\cdot 8\pi^2}\int_M \operatorname{Tr}(R\wedge R).$$

Hence P_1 is a multiple of 24 for any compact 4-dimensional spin manifold without boundary.

Twisted spin complex: As for the other complexes, we can take the tensor product of the spin complex with a vector bundle V to produce a twisted spin complex,

$$\Delta_{\pm} \otimes V.$$

The Dirac spinors then have two sets of indices, one set of spinor indices for $\Delta_{\pm}$ and one set of "isospin" indices for V. In a typical physical application, the connection on V would be taken as

$$A^a_\mu(x) \frac{t^a}{2\mathrm{i}}$$

where A_μ is the Yang–Mills connection on the associated principal bundle and $\{t^a\}$ are $\dim(V) \times \dim(V)$ matrices giving a representation of the corresponding Lie algebra. When the Dirac operator D is extended to the operator D_V including the connection on V, the index theorem becomes

$$\mathrm{index}(\Delta_{\pm} \otimes V, D_V) = \int_M \hat{A}(M) \wedge \mathrm{ch}(V). \tag{7.22}$$

The index itself is the difference between the number of positive and negative chirality spinors in the kernel of the combined Dirac–Yang–Mills operator D_V,

$$\mathrm{index}(\Delta_{\pm} \otimes V) \equiv \nu_+ - \nu_-.$$

For $n = 2$, the index theorem for the twisted spin complex reduces to

$$\nu_+ - \nu_- = \int_M c_1(V) = \frac{\mathrm{i}}{2\pi} \int_M \mathrm{Tr}\, \Omega.$$

For $n = 4$, we find

$$\nu_+ - \nu_- = -\frac{\dim V}{24} \int_M p_1(T(M)) + \tfrac{1}{2} \int_M (c_1(V)^2 - 2c_2(V))$$

$$= +\frac{\dim V}{24 \cdot 8\pi^2} \int_M \mathrm{Tr}(R \wedge R) - \frac{1}{8\pi^2} \int_M \mathrm{Tr}(\Omega \wedge \Omega).$$

Examples: 1. *$U(1)$ principal bundle in 2 dimensions*. Since $\Omega = \mathrm{i}F$ where $F = \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is the Maxwell-field 2-form, we have

$$\nu_+ - \nu_- = -\frac{1}{2\pi} \int_M F.$$

2. *SU(2) principal bundle over S^4*. We choose spinors transforming according to the spin $\frac{1}{2}$ representation of SU(2), so dim $V = 2$. Since $\mathrm{Tr}\, R \wedge R = 0$ for S^4 and $c_1 = 0$, we find for the index

$$\nu_+ - \nu_- = -\int_M c_2(V) = -\frac{1}{8\pi^2}\int_M \mathrm{Tr}(\Omega \wedge \Omega) \equiv k$$

where $\Omega = \frac{1}{2}(\lambda_a/2\mathrm{i})F^a_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Note that the "instanton number" k defined by $k = \nu_+ - \nu_-$ is minus the 2nd Chern number; k is positive if Ω is self-dual and negative for anti-self-dual Ω. In the actual instanton solutions, $\nu_\pm = 0$ for $k < 0$, $k > 0$, respectively.

For spinors ψ belonging to a $(2t+1)$-dimensional representation of SU(2) labeled by $t = 0, 1/2, 1, 3/2, \ldots$, the curvature Ω_t must be expressed as a matrix in the representation of ψ. If we define

$$k = -\frac{1}{8\pi^2}\int_M \mathrm{Tr}(\Omega \wedge \Omega)$$

where $\Omega = \Omega_{1/2}$ is a matrix in the spin 1/2 representation, then the index theorem for $(2t+1)$-dimensional SU(2) spinors can be shown to be

$$\nu_+(t) - \nu_-(t) = -\frac{1}{8\pi^2}\int_M \mathrm{Tr}(\Omega_t \wedge \Omega_t) = \tfrac{2}{3}t(t+1)(2t+1)k.$$

See Grossman [1977] for solutions of the Dirac equation with arbitrary k and an explicit verification of the index theorem for the twisted spin complex.

7.6. G-index theorems

The G-index theorem is a generalization of the ordinary index theorem. It is applicable when one is given in addition to the elliptic complex a suitable map f which takes the base manifold into itself, f: $M \to M$, and which therefore acts on the cohomology of the complex. For the de Rham complex, f may be any smooth map; for the signature complex, f must be an orientation-preserving isometry; f must be holomorphic for the Dolbeault complex, and, for the spin complex, f must be an orientation-preserving isometry which also preserves the spin structure.

The ordinary index theorem computes the alternating sum of dimensions of the cohomology groups of the elliptic complex in terms of characteristic classes; the G-index theorem computes the alternating sum of the trace of the action of f on the cohomology groups (the Lefschetz number) in terms of generalized characteristic classes.

We first examine the Lefschetz fixed-point theorem, which is a special case of the G-index theorem for the de Rham complex. Then we briefly outline the application of the G-index theorem to each of the classical elliptic complexes and present a number of examples.

7.6.1. Lefschetz fixed point theorem

Lefschetz numbers: Let M be a compact real manifold of dimension n without boundary and let $H^p(M;\mathbb{R})$ be the pth cohomology class of M. Let f: $M \to M$ be a smooth map and let f_p^* be the pull-back map on $H^p(M;\mathbb{R})$. Then if we choose a suitable basis, f_p^*: $H^p(M;\mathbb{R})\to H^p(M;\mathbb{R})$ can be represented as a matrix with integer entries. The *Lefschetz number* $L(f)$ is the integer

$$L(f)=\sum_{p=0}^{n}(-1)^p\,\mathrm{Tr}(f_p^*).$$

$L(f)$ is a homotopy invariant of f. If $f(x)=x$ is the identity map, then $f_p^*=I_{\dim(H^p)}$ is the identity map on $H^p(M,\mathbb{R})$, so

$$L(\text{identity})=\sum_{p=0}^{n}(-1)^p\dim(H^p)=\chi(M)$$

is the index of the de Rham complex. Thus the Lefschetz number can be thought of as a generalization of the Euler characteristic.

Lefschetz fixed-point theorem: We consider first the special case of an isometry f: $M\to M$. Then the fixed point set of f consists of totally geodesic submanifolds μ_j of M. Lefschetz proved that

$$L(f)=\sum_{\{\mu_j\}}\chi(\mu_j). \tag{7.23}$$

(If f is not an isometry, there are additional conditions which f must satisfy; in this situation, the terms in the sum are signed according to the direction of the normal derivative of f.) When f is homotopic to the identity and has only isolated fixed points, then the Euler characteristic of M equals the number of fixed points of f,

$$\chi(M)=(\text{number of fixed points of } f).$$

Vector fields: Let $V=v^\mu(x)\,\partial/\partial x^\mu$ be a vector field with isolated non-degenerate zeroes on a manifold M and let the map $f(t,x)$ be the infinitesimal *flow* of V:

$$f^\mu(0,x)=x^\mu$$
$$\frac{\partial f^\mu}{\partial t}(t,x)=v^\mu(f(t,x)).$$

$f(t,x_0)$ is the trajectory of the flow of V beginning at x_0. Since the flow is homotopic to the identity map, the Lefschetz number of the flow is the Euler characteristic of the manifold M. Furthermore, the *fixed points* of the flow correspond to the *zeroes* of the vector field. We conclude that the Euler characteristic of M is equal to the number of zeroes of V:

$$\chi(M)=(\text{number of zeroes of vector field}). \tag{7.24}$$

We note that if the flow is not an isometry (i.e., V is not a Killing vector field), then the zeroes of V have associated plus or minus signs; the Euler characteristic is then the *signed* sum of the zeroes of V. *Example*: $S^2=P_1(\mathbb{C})$. We know that $\chi(S^2)=2$.

Case (1) The map $z \to e^{i\alpha} z$

is an isometry which is the flow of a vector field $r\alpha\, \partial/\partial\theta$ where $z = r\,e^{i\theta}$. It has two fixed isolated non-degenerate fixed points at $z = 0$ and $z = \infty$, each of which appears with a positive sign.

Case (2) The map $z \to z + 1$

is the flow of the vector field $\partial/\partial x$, where $z = x + iy$, and has a degenerate *double* fixed point at ∞.

7.6.2. G-index theorem

For the remainder of this section, we will only consider maps with non-degenerate isolated fixed points, although there are corresponding formulas for maps with higher dimensional invariant sets. With this restriction, we treat the G-index theorem for the four standard elliptic complexes.

We begin by choosing local coordinates $x^\mu \in U$ on M such that the map f can be written in the form

$$f^\mu(x) = f^\mu(x_0) + (x^\nu - x_0^\nu)\, \partial f^\mu(x_0)/\partial x^\nu + \cdots$$

where x_0 is a fixed point of the map. We denote the *Jacobian matrix* f' of the map by

$$f'(x_0) = |\partial f^\mu(x_0)/\partial x^\nu|.$$

We assume that f is non-degenerate, i.e., there are no tangent vectors left infinitesimally fixed by f' at x_0. This is equivalent to requiring that f' does not have the eigenvalue 1:

$$\mathrm{Det}(I - f') \neq 0.$$

Let (E_+, E_-) denote the rolled-up elliptic complex under consideration, and let f^* denote the pull-back operation mapping $E_\pm \to E_\pm$. Let $H^\pm$ denote the cohomology of the elliptic complex and let f^* act on the cohomology by the pullback. The Lefschetz number of the elliptic complex is then defined to be

$$L_E(f) \equiv \mathrm{Tr}(f^* H^+) - \mathrm{Tr}(f^* H^-).$$

The G-index theorem expresses the global invariant $L_E(f)$ in terms of local geometric information:

$$L_E(f) = \sum_{\{\text{fixed points } x_0\}} \frac{\mathrm{Tr}\, f^*(x_0) E_+ - \mathrm{Tr}\, f^*(x_0) E_-}{|\mathrm{Det}(I - f'(x_0))|}.$$

We next apply this formula to the four classical elliptic complexes; for more details, see Atiyah and Singer [1968b].

de Rham complex. Let $E_+ = \Lambda^{\text{even}}(M, \mathbb{R})$, $E_- = \Lambda^{\text{odd}}(M, \mathbb{R})$. Then the G-index theorem becomes

$$L_{\text{de Rham}}(f) = \sum_{\{\text{fixed points}\}} \frac{\mathrm{Tr}(f^* \Lambda^{\text{even}}) - \mathrm{Tr}(f^* \Lambda^{\text{odd}})}{|\mathrm{Det}(I - f')|}.$$

After some algebra, an application of the splitting principle shows that

$$L_{\text{de Rham}}(f) = \sum_{\substack{\text{fixed}\\\text{points}}} \frac{\operatorname{Det}(I-f')}{|\operatorname{Det}(I-f')|} = \sum_{\substack{\text{fixed}\\\text{points}}} \operatorname{sign}\operatorname{Det}(I-f').$$

When f is an isometry, $\operatorname{Det}(I-f') = 1$, so $L_{\text{de Rham}}(f)$ is just the number of fixed points of f.

Example: Analysis of local behavior of an isometry near a fixed point. Let $n = 2$ and let f be an isometry which has the local form

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(*Note*: We need not specify M globally, because any orientation-preserving isometry has this local form.) f' is a rotation about the fixed point at the origin:

$$f' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

As bases for $\Lambda^{\text{even,odd}}$, we choose

$$\Lambda^{\text{even}} = \begin{pmatrix} 1 \\ dx \wedge dy \end{pmatrix}, \quad \Lambda^{\text{odd}} = \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Then

$$f^*\Lambda^{\text{even}} = \begin{pmatrix} 1 \\ df_1 \wedge df_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ dx \wedge dy \end{pmatrix}$$

$$f^*\Lambda^{\text{odd}} = \begin{pmatrix} df_1 \\ df_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

so $\operatorname{Tr}(f^*\Lambda^{\text{even}}) - \operatorname{Tr}(f^*\Lambda^{\text{odd}}) = 2 - 2\cos\theta$.

We verify that this agrees with $\operatorname{Det}(I - f') = 2 - 2\cos\theta$. There is one local fixed point at $x = y = 0$, so the contribution to the Lefschetz formula is

$$\frac{2 - 2\cos\theta}{|2 - 2\cos\theta|} = 1.$$

Signature complex: Let M be an oriented manifold of even-dimension $n = 2l$ and let f be an orientation-preserving isometry. Let $E_\pm = \Lambda^\pm(T^*(M))$ be the signature complex and let $H^\pm$ be the corresponding cohomology groups. We define

$$L_{\text{sign}}(f) = \operatorname{Tr} f^*H^+ - \operatorname{Tr} f^*H^- = \operatorname{Tr} f^*H^{l,+} - \operatorname{Tr} f^*H^{l,-},$$

since all terms cancel except those in the middle dimensional cohomology class. The G-index theorem is then

$$L_{\text{sign}}(f) = \sum_{\substack{\text{fixed} \\ \text{points}}} \frac{\operatorname{Tr} f^* \Lambda^{l,+} - \operatorname{Tr} f^* \Lambda^{l,-}}{\operatorname{Det}(I - f')},$$

where the determinant is positive because f is an isometry.

Example: Let $n = 2$ and let f be the same local map used in the de Rham complex example. As bases for $\Lambda^{\pm}$ we choose

$$\Lambda^{\pm} = \{\mathrm{d}x \pm \mathrm{i}\,\mathrm{d}y).$$

We verify that under the action of the signature operator $\omega = \mathrm{i}\,*$, the bases behave as they should:

$$\omega(\mathrm{d}x \pm \mathrm{i}\,\mathrm{d}y) = \pm(\mathrm{d}x \pm \mathrm{i}\,\mathrm{d}y).$$

Applying the pullback map, we find

$$f^*\Lambda^+ = \mathrm{d}f_1 + \mathrm{i}\,\mathrm{d}f_2 = \mathrm{e}^{+\mathrm{i}\theta}(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y)$$

$$f^*\Lambda^- = \mathrm{d}f_1 - \mathrm{i}\,\mathrm{d}f_2 = \mathrm{e}^{-\mathrm{i}\theta}(\mathrm{d}x - \mathrm{i}\,\mathrm{d}y).$$

Again, there is one fixed point at the origin, so the contribution to the G-signature theorem reads

$$\frac{\mathrm{e}^{+\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2(1 - \cos\theta)} = +\mathrm{i}\frac{\sin\theta}{1 - \cos\theta} = \mathrm{i}\cot(\theta/2).$$

We may extend this result to higher even dimensions $n = 2l$ as follows: Let f' be an orthogonal matrix which we may think of as a rotation about a fixed point at the origin. We decompose this rotation into a product of commuting 2×2 rotations through angles θ_j, $j = 1, \ldots, l$. Then we may show that the local contribution to the fixed point formula at the fixed point is

$$\prod_{j=1}^{l} \frac{\mathrm{i}\sin\theta_j}{1 - \cos\theta_j} = \prod_{j=1}^{l} \mathrm{i}\cot(\theta_j/2).$$

Dolbeault complex. Let M be a holomorphic manifold and let f be a holomorphic map. Let $E_+ = \Lambda^{0,\text{even}}$ and $E_- = \Lambda^{0,\text{odd}}$ be the bundles of the Dolbeault complex. Then

$$L_{\text{Dol}}(f) = \operatorname{Tr} f^* H^{0,\text{even}} - \operatorname{Tr} f^* H^{0,\text{odd}}$$

The G-index theorem is

$$L_{\text{Dol}}(f) = \sum_{\substack{\text{fixed} \\ \text{points}}} \frac{\operatorname{Tr} f^* \Lambda^{0,\text{even}} - \operatorname{Tr} f^* \Lambda^{0,\text{odd}}}{|\operatorname{Det}(I - f')|}.$$

Example: Let $n = 2$, take f to be the local rotation about the origin used above, and choose the bases

$$\Lambda^{0,\mathrm{even}} = \{1\}, \qquad \Lambda^{0,\mathrm{odd}} = \{\mathrm{d}\bar{z} = \mathrm{d}x - \mathrm{i}\,\mathrm{d}y\}.$$

Then the pullback acts as

$$f^*\Lambda^{0,\mathrm{even}} = 1, \qquad f^*\Lambda^{0,\mathrm{odd}} = \mathrm{d}f_1 - \mathrm{i}\,\mathrm{d}f_2 = \mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}\bar{z}.$$

The contribution to the G-index theorem is therefore

$$\frac{1-\mathrm{e}^{-\mathrm{i}\theta}}{2-2\cos\theta}.$$

In higher dimensions the contribution is given by the product of such terms.

Spin complex: Let M be a spin manifold and let f be an orientation-preserving spin isometry. Let $E_\pm = \Delta_\pm$ be the bundles of the spin complex and let $H^{\mathrm{spin},\pm}$ be the corresponding cohomology groups (or the harmonic spaces) of the Dirac operator. Then

$$L_{\mathrm{spin}}(f) = \mathrm{Tr}\, f^* H^{\mathrm{spin},+} - \mathrm{Tr}\, f^* H^{\mathrm{spin},-},$$

and the G-index theorem becomes

$$L_{\mathrm{spin}}(f) = \sum_{\substack{\text{fixed}\\ \text{points}}} \frac{\mathrm{Tr}\, f^*\Delta_+ - \mathrm{Tr}\, f^*\Delta_-}{|\mathrm{Det}(I-f')|}.$$

Example: As before, let $n = 2$ and take f' to be the local rotation around the origin. The spinor bases for $\Delta_\pm$,

$$\Delta_+ = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \Delta_- = \begin{pmatrix}0\\1\end{pmatrix},$$

transform under the rotation f' as

$$\begin{pmatrix}1\\0\end{pmatrix} \to \mathrm{e}^{+\mathrm{i}\theta/2}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\1\end{pmatrix} \to \mathrm{e}^{-\mathrm{i}\theta/2}\begin{pmatrix}0\\1\end{pmatrix}.$$

Thus the contribution to the G-spin theorem becomes

$$\frac{\mathrm{e}^{+\mathrm{i}\theta/2}-\mathrm{e}^{-\mathrm{i}\theta/2}}{2-2\cos\theta} = +\frac{\mathrm{i}}{2\sin(\theta/2)}.$$

The contribution to the G-spin index for higher dimensions is a product of such terms.

Examples 7.6

1. Let G be a compact Lie group of dimension $n>0$. Let $g(t)$ for $t\in[0,1]$ be a curve in G with $g(0)=I$ and $g(t)\neq I$ for $t>0$. Let $f_t(X)=g(t)\cdot X$. Then $f_0(X)=X$ so f_0 is the identity map and $L(f_0)=\chi(G)$. For $t>0$, $f_t(X)=g(t)\cdot X\neq X$ since $g(t)\neq I$. Thus f has no fixed points, so $L(f_t)=0$. Since $L(f_0)=L(f_t)$, $\chi(G)=0$. This shows that the following Euler characteristics vanish: $\chi(U(k))=\chi(\mathrm{SU}(k))=\chi(O(k))=\chi(\mathrm{SO}(k))=0$ for $k>1$. If $k=1$, then we cannot use this argument; for example, $\chi(O(1))=2$ since $O(1)$ consists of two points ± 1.

2. Let $M=P_n(\mathbb{C})$ for n even (so the dimension of M is divisible by 4). Let $x\in H^2(P_n(\mathbb{C});\mathbb{R})$ be the generator discussed in 6.3.2; $x^k\in H^{2k}(P_n(\mathbb{C});\mathbb{R})$ is a generator for $k=1,\ldots,n$. Let $f\colon M\to M$ and $f^*x=\lambda x$. Since f^* preserves the ring structure, $f^*(x^k)=\lambda^k x^k$. Therefore

$$L(f)=1+\lambda+\cdots+\lambda^n.$$

If n is even, this has no real roots so $L(f)\neq 0$. Therefore f must have a fixed point.

3. Let $M=S^1\times S^1$ and let $f(\theta_1,\theta_2)=(\theta_2,\theta_1)$ be the interchange. Let $\{1,\mathrm{d}\theta_1,\mathrm{d}\theta_2,\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\}$ be the basis for $H^*(M;\mathbb{R})$ discussed earlier. Then

$$f^*(1)=1 \qquad f^*(\mathrm{d}\theta_1)=\mathrm{d}\theta_2 \qquad f^*(\mathrm{d}\theta_2)=\mathrm{d}\theta_1 \qquad f^*(\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2)=-\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2$$

$$\mathrm{Tr}\,f_0^*=1 \qquad \mathrm{Tr}\,f_1^*=0 \qquad \mathrm{Tr}\,f_2^*=-1$$

so $L(f)=1-0+(-1)=0$. The fixed point set of f is the diagonal S^1 so $L(f)=\chi(S^1)=0$. If $g(\theta_1,\theta_2)=(-\theta_2,\theta_1)$ then

$$g^*(1)=1 \qquad g^*(\mathrm{d}\theta_1)=-\mathrm{d}\theta_2 \qquad g^*(\mathrm{d}\theta_2)=\mathrm{d}\theta_1 \qquad g^*(\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2)=\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2$$

so $L(g)=1-0+1=2$. g has two isolated fixed points $(0,0)$ and (π,π).

Let $M=S^2\times S^2$. The cohomology ring of M has generators $1\in H^0(M;\mathbb{R})\simeq\mathbb{R}$, ω_1, $\omega_2\in H^2(M;\mathbb{R})\simeq\mathbb{R}\oplus\mathbb{R}$, $\omega_1\wedge\omega_2\in H^4(M;\mathbb{R})\simeq\mathbb{R}$ where the $\omega_i\in H^2(S^2;\mathbb{R})$ for each factor. If $f(x,y)=(y,x)$ then

$$f^*(1)=1 \qquad f^*(\omega_1)=\omega_2 \qquad f^*(\omega_2)=\omega_1 \qquad f^*(\omega_1\wedge\omega_2)=\omega_2\wedge\omega_1=\omega_1\wedge\omega_2$$

so $L(f)=1-0+1=2$. The fixed point set of f is the diagonal S^2 so $L(f)=\chi(S^2)=2$. If $g(x,y)=(-y,x)$, then

$$g^*(1)=1 \qquad g^*(\omega_1)=-\omega_2 \qquad g^*(\omega_2)=\omega_1 \qquad g^*(\omega_1\wedge\omega_2)=-\omega_1\wedge\omega_2$$

so $L(g)=1-0+(-1)=0$. In this case g has no fixed points.

4. Let $M=S^1\times S^1$ be the 2-torus with generators $\mathrm{d}\theta_1$ and $\mathrm{d}\theta_2$. Then (with $\omega=\mathrm{i}*$)

$$\omega\cdot\mathrm{d}\theta_1=\mathrm{i}\,\mathrm{d}\theta_2, \qquad \omega\cdot\mathrm{d}\theta_2=-\mathrm{i}\,\mathrm{d}\theta_1,$$

so $(\mathrm{d}\theta_1\pm\mathrm{i}\,\mathrm{d}\theta_2)$ spans $H^1_\pm(M;\mathbb{R})$. If $f(\theta_1,\theta_2)=(\theta_1,\theta_2)$ is the identity map, then $\mathrm{Tr}\,f_+^*-\mathrm{Tr}\,f_-^*=1-1=\tau(M)=0$. Suppose that $g(\theta_1,\theta_2)=(-\theta_2,\theta_1)$, then

$$g^* \, d\theta_1 = -d\theta_2 \qquad g^* \, d\theta_2 = d\theta_1 \qquad g^*(d\theta_1 \wedge d\theta_2) = d\theta_1 \wedge d\theta_2$$

$$g^*(d\theta_1 + i\, d\theta_2) = i(d\theta_1 + i\, d\theta_2)$$

$$g^*(d\theta_1 - i\, d\theta_2) = -i(d\theta_1 - i\, d\theta_2)$$

$$L_{\text{sign}}(g) = \ i - (-i) = 2i.$$

Since $L_{\text{sign}}(f)$ is a homotopy invariant, we use an argument similar to that given for the ordinary Lefschetz number to show $\tau(M) = 0$ either if M is a compact Lie group or if M admits a Killing vector field without zeroes.

8. Index theorems: Manifolds with boundary

The applications of the index theorem described in the previous section hold only for bundles with base manifolds M which are closed and compact without boundary. Many interesting physical situations deal with base manifolds M which have nonempty boundaries or which, for M noncompact, can be treated as limiting cases of manifolds with boundary. This section is devoted to the extension of the index theorem to manifolds with boundary carried out by Atiyah, Patodi and Singer [1973, 1975a, 1975b, 1976].

Euler characteristic boundary corrections: In order to understand more clearly the necessity for boundary corrections to a topological index, let us review the familiar case of the Euler characteristic of a two-dimensional disc. The general formula can be written

$$\chi[M, \partial M] = \frac{1}{2\pi} \int_M R + \frac{1}{2\pi} \int_{\partial M} \frac{ds}{\rho} + \frac{1}{2\pi} \sum_i (\pi - \theta_i).$$

Here R is the curvature 2-form (essentially the Gaussian curvature), $1/\rho$ is the geodesic curvature on the boundary and θ_i is the interior angle of each vertex, as shown in fig. 8.1. We illustrate the application of the formula to the three special cases depicted in fig. 8.2:

(a) *Flat, n-sided polygon*: We simply recover the fact that

$$\sum_{i=1}^{n} \theta_i = (n-2)\pi$$

implies

$$\chi = 0 + 0 + 1.$$

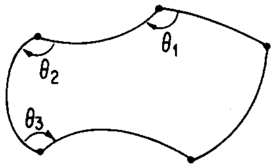

Fig. 8.1. An arbitrary two-dimensional surface with the topology of a disc.

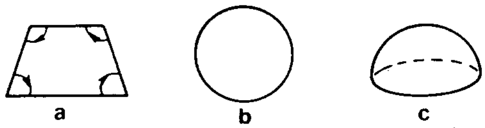

Fig. 8.2. Special cases: (a) polygon, (b) circle, (c) hemisphere.

(b) *Flat circle* of radius r; with $\mathrm{d}s = r\,\mathrm{d}\phi$ and $\rho = r$, only the geodesic term contributes:

$$\chi = 0 + 1 + 0.$$

(c) *Hemisphere*: the geodesics normal to the equator are parallel at the equator, so $\rho = \infty$, $R = (1/r^2)\,r^2\,\mathrm{d}\phi\,\mathrm{d}\cos\theta$ and only the Gaussian curvature term contributes,

$$\chi = 1 + 0 + 0.$$

We conclude that although the Euler characteristic of a disc is always $\chi = 1$, the Gaussian curvature and the boundary terms interact in complicated ways to maintain the topological invariance of the formula.
Remark: The area of a spherical polygon can be computed from the formula above using $\chi = 1$. Taking the sphere to have unit radius, we find

$$\text{Spherical area} = \int_{\text{polygon}} R = \sum \theta_i - (n-2)\pi.$$

For flat polygons (sphere of infinite radius), the "area" vanishes and we recover $\Sigma\,\theta_i = (n-2)\pi$. On a hyperboloid, the curvature is negative and the effective area is the *angular defect*,

$$\text{Hyperboloidal area} = (n-2)\pi - \sum \theta_i.$$

8.1. Index theorem with boundary

When we consider manifolds with boundary, we must first study the boundary conditions which determine the spectra of the operators. Ideally, one would like to find an index theorem using conventional local boundary conditions such as those appearing in ordinary physical problems. However, Atiyah and Bott [1964] have shown that in general there exist topological obstructions to finding good local boundary conditions. The spin, signature, and Dolbeault complexes in particular do not admit local boundary conditions, although the de Rham complex does. Therefore if one wants a general index theorem for a manifold with boundary, one must consider non-local boundary conditions. Atiyah, Patodi and Singer discovered that appropriate non-local boundary conditions could indeed be used to formulate an index theorem for elliptic complexes over manifolds with boundary.

We now outline the general nature of the Atiyah–Patodi–Singer index theorem. We begin by considering a classical elliptic complex (E, D) over a manifold M with nonempty boundary ∂M. For simplicity, we assume that $\{E\}$ is rolled up to a 2-term complex, D: $E_0 \to E_1$. In order to formulate the index theorem, we require analytic information on the boundary in addition to the purely topological information which sufficed in the case without boundary.

Boundary condition: We assume for the time being that M admits a product metric

$$\mathrm{d}s^2 = f(\tau_0)\,\mathrm{d}\tau^2 + g_{ij}(\tau_0, \theta_k)\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j$$

on the boundary, where $\tau = \tau_0$ defines the boundary manifold ∂M. (We will deal later with the case when M does not admit a product metric.) Then we construct from D a Hermitian operator whose eigenfunctions ϕ are subject to the boundary condition

$$\phi \sim e^{-k\tau} \qquad k > 0 \tag{8.1}$$

near the boundary.

The index: We now define cohomology classes $H^p(E, D, \partial M)$ whose representatives obey the required boundary conditions. The corresponding index is then taken to be

$$\text{index}(E, D, \partial M) = \sum_p (-1)^p H^p(E, D, \partial M).$$

Form of the index theorem: The extended index theorem of Atiyah–Patodi–Singer for manifolds with boundary takes the form

$$\text{index}(E, D, \partial M) = V[M] + S[\partial M] + \xi[\partial M]. \tag{8.2}$$

Here

$V[M]$ = the integral over M of the same characteristic classes as in the $\partial M = \emptyset$ case. V is computable from the curvature alone.
$S[\partial M]$ = the integral over ∂M of the Chern–Simons form, described below. S is computable from the connection, the curvature, and the second fundamental form determined by a choice of the normal to the boundary.
$\xi[\partial M] = c\eta[M]$ = a constant c times the Atiyah–Patodi–Singer η-invariant of the boundary, described below. The η-invariant is determined by the eigenvalues of the tangential part of D restricted to the boundary ∂M. For several important cases, η can be computed algebraically.

The surface correction $S[\partial M]$ is present only if one uses a metric on M which does not become a product metric at the boundary. The $\xi[\partial M]$ correction is absent for the de Rham complex, but plays a crucial role in the Dolbeault, signature and spin complex index theorems.

General nature of the boundary corrections: One can develop an intuitive feeling for the nature of the boundary corrections to the index theorem by examining a pair of manifolds M and M' with the same boundary

$$L = \partial M = \partial M'.$$

We give M and M' each a metric and a connection and assume that they admit the same product metric near their boundaries. Thus we may sew M and M' together smoothly along their common boundary to form a new manifold $M \cup M'$ *without* boundary.

Now assume M and M' are 4-dimensional and consider, for example, the signature τ of $M \cup M'$. By the no-boundary index theorem,

$$\tau(M \cup M') = -\frac{1}{24\pi^2} \int_{M \cup M'} \mathrm{Tr}(\Omega \wedge \Omega)$$

where Ω is the curvature of the assumed metrics on M and M'.

Now we break the integral into two parts, one involving M, the other M' with the *opposite* orientation to its orientation in $M \cup M'$ (this gives M and M' the same relative orientation). If we call Ω' the curvature in M' with the new orientation, we find

$$\int_{M \cup M'} \mathrm{Tr}(\Omega \wedge \Omega) = \int_M \mathrm{Tr}(\Omega \wedge \Omega) - \int_{M'} \mathrm{Tr}(\Omega' \wedge \Omega').$$

Since with our chosen orientation the Novikov formula gives (see, e.g. Atiyah and Singer [1968b])

$$\tau(M \cup M') = \tau(M) - \tau(M'),$$

we find

$$\tau(M) + \frac{1}{24\pi^2} \int_M \mathrm{Tr}(\Omega \wedge \Omega) = \tau(M') + \frac{1}{24\pi^2} \int_{M'} \mathrm{Tr}(\Omega' \wedge \Omega').$$

Hence the quantity

$$-\eta_S[L] = \tau(M) + \frac{1}{24\pi^2} \int_M \mathrm{Tr}(\Omega \wedge \Omega)$$

depends only on the metric on $L = \partial M$. The index theorem gives an alternative expression for η_S in terms of the eigenvalues of the signature operator restricted to ∂M.

Next, suppose that we have a metric $\tilde{g}$ on M which is *not* a product metric on the boundary. Let $\tilde{\omega}$ be the connection obtained from $\tilde{g}$ and let $\tilde{\Omega}$ be its curvature. Then, as shown in section 6, the difference between $\mathrm{Tr}\, \tilde{\Omega} \wedge \tilde{\Omega}$ and $\mathrm{Tr}\, \Omega \wedge \Omega$ is a total derivative,

$$\mathrm{d}Q\,(\tilde{\omega}, \omega) = (\mathrm{Tr}\, \tilde{\Omega} \wedge \tilde{\Omega} - \mathrm{Tr}\, \Omega \wedge \Omega),$$

where Ω is the curvature of the metric g which is a product metric on ∂M. This expression gives an additional analytic correction to the index,

$$S[\partial M] = -\frac{1}{24\pi^2} \int_{\partial M} Q.$$

We now turn to a precise definition of the η-invariant.

8.2. The η-invariant

We consider our 2-term elliptic complex (E, D) with D: $E_0 \to E_1$ a linear operator obeying the boundary conditions (8.1). We choose $\partial/\partial\tau$ to represent the outward normal derivative on ∂M. We write D as

$$D = A \cdot \partial + B\,\partial/\partial\tau = B(B^{-1}A \cdot \partial + \partial/\partial\tau)$$

where A and B are matrices and $A \cdot \partial$ represents the tangential part of D. Whereas D itself might not have a true eigenvalue spectrum because $E_0 \neq E_1$ in general, the operator

$$\hat{D} = B^{-1}A \cdot \partial|_{\partial M}$$

maps $E_0 \to E_0$ on ∂M and does have a well-defined spectrum. We let $\{\lambda_i\}$ denote the eigenvalues of the tangential operator $\hat{D}$ acting on ∂M.

The η-invariant of Atiyah–Patodi–Singer is then defined by examining a natural generalization of the spectral Riemann zeta function for non-positive eigenvalues:

$$\eta_D[s, \partial M] = \sum_{\substack{\{\lambda_i\} \\ \lambda_i \neq 0}} \operatorname{sign}(\lambda_i)|\lambda_i|^{-s}, \qquad s > n/2 \quad (n = \dim M).$$

It has been shown that, despite the apparent singularities at $s = 0$, this expression possesses a regular analytic extension to $s = 0$; this continuation defines the η-invariant:

$$\eta_D[\partial M] \equiv \eta_D[s = 0, \partial M]. \tag{8.3}$$

Harmonic correction: If the elliptic operator D in question admits zero eigenvalues (as does the Dirac operator), then one must be careful to account for the missing zero eigenvalues in the definition of η_D. The correct prescription is to add $h_D(\partial M)$, which is the dimension of the space of functions harmonic under $\hat{D}$

$$\eta_D \to \eta_D + h_D.$$

Intuitively, it is clear that η_D counts the asymmetry between the number of positive and negative eigenvalues on the boundary. Furthermore, η_D is independent of the scale of the metric, and hence is independent of the numerical values of the $\{\lambda_i\}$. If the spectrum $\{\lambda_i\}$ varies with some parameter, typically a parameter specifying the location of the boundary surface, the smallest positive eigenvalue (say λ_k), may change sign at some point: one sees immediately that then there is one less positive eigenvalue and one more negative one, so η_D jumps by *two*:

$$\eta_D \to \eta_D - 2.$$

(Clearly many jumps with either sign can occur.) However, we note that *exactly* at the point where $\lambda_k = 0$, we must omit λ_k from the sum and add one, the dimension h_D of the new harmonic space; thus there is no change in η_D until $\lambda_k < 0$.

Computation of η_D: There are variety of special circumstances in which η_D can be calculated directly, e.g., when D = the signature or Dirac operator. The simplest situation is that in which the metric on ∂M possesses an orientation-reversing isometry; in this case

$$\eta_D[\partial M] = 0.$$

(If D is the Dirac operator, one must also assume that M is simply connected.)

Another case which has been calculated directly is that where the metric on ∂M is that of a distorted S^3,

$$\mathrm{d}s^2 = \sigma_x{}^2 + \sigma_y{}^2 + \lambda^2\sigma_z{}^2.$$

Hitchin [1974] has shown by solving for the eigenvalues of the Dirac operator that

$$\eta_{\mathrm{Dirac}} = \tfrac{1}{6}(1-\lambda^2)^2.$$

When $\lambda^2 = 1$, the S^3 metric has an orientation-reversing isometry and η_{Dirac} vanishes as it must.

If one takes the symmetric ($\lambda = 1$) S^3 metric and identifies opposite points to get a metric on $P_3(\mathbb{R})$, η_{S} remains zero but η_{Dirac} may be non-zero because $P_3(\mathbb{R})$ is not simply connected and possesses two inequivalent spin-structures. In fact, the η-invariants for the standard operators can be calculated fairly straightforwardiy using G-index theory when the metric on ∂M is that of S^3 modulo a discrete group. We define the *Lens spaces* of S^3 by taking $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ and identifying the first $\mathbb{R}^2$ with itself when rotated by $e^{i\theta_1}$, then doing the same thing for the second $\mathbb{R}^2$ rotated by $e^{i\theta_2}$, where θ_1 and θ_2 have rational periods. The simplest case, $P_3(\mathbb{R})$, is obtained by setting $\theta_1 = \theta_2 = \pi$.

Let $m\theta_1 = m\theta_2 = 2\pi$. Then the general formulas for the η-invariant corrections to the indices for Lens space boundaries are (Atiyah, Patodi and Singer [1975b]; Atiyah [1978]; Hanson and Römer [1978]):

Signature: $$\xi_{\mathrm{S}} = \frac{1}{m}\sum_{k=1}^{m-1}{}' \cot\tfrac{1}{2}k\theta_1 \cot\tfrac{1}{2}k\theta_2$$

$$(=0 \qquad \text{for } P_3(\mathbb{R}))$$

Dirac: $$\xi_{\mathrm{Dirac}} = -\frac{1}{4m}\sum_{k=1}^{m-1}{}' \frac{1}{\sin\frac{1}{2}k\theta_1 \sin\frac{1}{2}k\theta_2}$$

$$(=-\tfrac{1}{8} \qquad \text{for } P_3(\mathbb{R}))$$

Rarita–Schwinger: $$\xi_{\mathrm{RS}} = -\frac{1}{4m}\sum_{k=1}^{m-1}{}' \frac{2\cos k\theta_1 + 2\cos k\theta_2 - 1}{\sin\frac{1}{2}k\theta_1 \sin\frac{1}{2}k\theta_2}$$

$$(=+\tfrac{5}{8} \qquad \text{for } P_3(\mathbb{R})).$$

(See section 10 for additional cases with physical applications.)

8.3. *Chern–Simons invariants and secondary characteristic classes*

In our treatment of characteristic classes in section 6, we introduced the expression

$$Q(\omega', \omega) = r \int_0^1 P(\omega' - \omega, \Omega_t, \dots, \Omega_t)\,dt$$

derived from an invariant polynomial $P(\Omega)$ of degree r with

$$\Omega_t = d\omega_t + \omega_t \wedge \omega_t.$$
$$\omega_t = t\omega' + (1-t)\omega.$$

The exterior derivative of Q was just the difference of the two invariant polynomials,

$$dQ = P(\Omega') - P(\Omega).$$

If M has no boundary, the integral of dQ vanishes. However, if $\partial M \neq \emptyset$, then by Stokes' theorem,

$$\int_M dQ = \int_{\partial M} Q$$

is not necessarily zero. In this case the forms $Q(\omega', \omega)$ are characteristic classes in their own right and are of independent interest (Chern [1972]; Chern and Simons [1974]).

Yang–Mills surface terms: The Chern–Simons formulas are equally valid for Riemannian connections and for Yang–Mills connections on a principal bundle. In the Yang–Mills case, if we set

$$P(F) = \mathrm{Tr}(F \wedge F)$$
$$F = dA + A \wedge A$$
$$A' = 0,$$

we find

$$Q(A, 0) = \mathrm{Tr}(A \wedge dA + \tfrac{2}{3}A \wedge A \wedge A).$$

Thus the familiar physicists' formula

$$\mathrm{Tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu J_\mu$$

where

$$J_\mu = 2\epsilon_{\mu\alpha\beta\gamma}\,\mathrm{Tr}(A_\alpha \partial_\beta A_\gamma + \tfrac{2}{3}A_\alpha A_\beta A_\gamma)$$

is simply a special case of the Chern–Simons formula.

Other cases of the formula appear in discussions of Yang–Mills "surface terms" (see, e.g., Gervais, Sakita and Wadia [1975]). Choosing $A' \neq 0$ in the Chern–Simons formula for $\mathrm{Tr}(F \wedge F)$ and setting

$\alpha = A - A'$, we find

$$Q(A, A') = \mathrm{Tr}(2\alpha \wedge F - \alpha \wedge \mathrm{d}\alpha - 2\alpha \wedge A \wedge \alpha + \tfrac{2}{3}\alpha \wedge \alpha \wedge \alpha).$$

Second fundamental form: Now let us consider the Levi–Civita connection one-form ω on M following from a metric which is not a product metric on ∂M. Then we choose a product metric on M which agrees with the original metric on ∂M; the connection one-form ω_0 of this metric will have only tangential components on ∂M. The ***second fundamental form***

$$\theta = \omega - \omega_0$$

is a matrix of one-forms which is covariant under changes of frame and has only *normal* components on ∂M. As usual, we take

$$\omega_t = t\omega + (1-t)\omega_0, \qquad R_t = \mathrm{d}\omega_t + \omega_t \wedge \omega_t,$$

and observe that

$$\theta = \mathrm{d}\omega_t/\mathrm{d}t.$$

In four dimensions with $P = \mathrm{Tr}(R \wedge R)$, we find

$$\mathrm{Tr}(R \wedge R) = \mathrm{d}Q(\omega, \omega_0),$$

where

$$Q(\omega, \omega_0) = 2 \int_0^1 \mathrm{Tr}(\theta \wedge R_t)\, \mathrm{d}t$$

$$= \mathrm{Tr}(2\theta \wedge R + \tfrac{2}{3}\theta \wedge \theta \wedge \theta - 2\theta \wedge \omega \wedge \theta - \theta \wedge \mathrm{d}\theta),$$

and we note that $\mathrm{Tr}(R_0 \wedge R_0) = 0$ for a product metric. The formula for Q simplifies considerably at the boundary, where the non-zero components of the matrix θ_{ab} are the normal components of the connection ω_{ab},

$$\theta_{0i} = \omega_{0i}, \qquad \theta_{23} = \theta_{31} = \theta_{12} = 0.$$

Using $R = \mathrm{d}\omega + \omega \wedge \omega$, we find after some algebra that

$$Q(\omega, \omega_0)|_{\text{boundary}} = 2\omega_{0i} \wedge R_{i0} = \mathrm{Tr}(\theta \wedge R).$$

Surface corrections to the index theorem: We now use the Chern–Simons formula to correct the Atiyah–Patodi–Singer index theorem for the case where the metric is not a product metric on ∂M (for a treatment of the signature complex, see Gilkey [1975]). Suppose the standard index theorem integral over curvature can be written in terms of an invariant polynomial $P(R)$ as

$$V[M] = c \int_M P(R)$$

for some constant c. Then the surface correction is

$$S[\partial M] = -c \int_{\partial M} Q(\omega, \omega_0).$$

The correction may be understood intuitively by noting that

$$V[M] + S[\partial M] = c \int_M (P(R) - \mathrm{d}Q(\omega, \omega_0)) \tag{8.4}$$

is *effectively* the integral over $cP(R_0)$. But since M may not admit a product metric with curvature R_0 away from ∂M, $P(R_0)$ cannot always be integrated over M. The surface correction circumvents this difficulty.

Locally flat bundles: The Chern–Simons invariants appear in place of ordinary characteristic classes in a variety of problems involving odd-dimensional manifolds. One interesting case is the study of the holonomy of locally flat bundles; this problem is closely related to the Bohm–Aharonov effect in a region free of electromagnetic fields.

As a simple example, let us take a connection

$$\omega = -\mathrm{i}q\,\mathrm{d}\theta$$

on a bundle $E = S^1 \times \mathbb{C}$, where $0 \le \theta < 2\pi$ are coordinates on the base space S^1. Then we choose sections

$$s(\theta) = \mathrm{e}^{\mathrm{i}q\theta}$$

such that $s(\theta)$ is parallel-transported, $\nabla s = 0$. As θ ranges from 0 to 2π, we find a holonomy or phase shift $\mathrm{e}^{2\pi\mathrm{i}q}$ resulting from the traversal of a circuit around the base space S^1. The secondary characteristic class corresponding to the first Chern class $c_1 = (\mathrm{i}/2\pi)\,\mathrm{Tr}\,\Omega$ is

$$Q(\omega, 0) = \frac{\mathrm{i}}{2\pi} \int_0^1 \omega\,\mathrm{d}t = \frac{q}{2\pi}\,\mathrm{d}\theta.$$

The Chern–Simons invariant is interpretable as a charge:

$$\int_{S^1} Q(\omega, 0) = q.$$

Another example is provided by taking the flat connection on the line bundle $E = S^2 \times \mathbb{C}$ and using the induced connection on the $P_3(\mathbb{R})$ line bundle $\tilde{E}$ obtained by identifying the points (x, z) with $(-x, -z)$ in E. If γ is a path traversing half a great circle in S^3, it is a closed loop in $P_3(\mathbb{R})$ which represents the non-zero element of $\pi_1(P_3(\mathbb{R})) = \mathbb{Z}_2$. A phase factor of -1 is obtained by integrating the secondary characteristic class over γ.

8.4. *Index theorems for the classical elliptic complexes*

Here we briefly summarize the results of the Atiyah–Patodi–Singer index theorem for the classical elliptic complexes in four dimensions.

de Rham complex. Let $R^a{}_b$ be the curvature 2-form and $\theta^a{}_b = \omega^a{}_b - (\omega_0)^a{}_b$ the second fundamental form. Then the index theorem for the de Rham complex is (see Chern [1945]),

$$\chi(M) = \frac{1}{32\pi^2} \int_M \epsilon_{abcd} R^a{}_b \wedge R^c{}_d - \frac{1}{32\pi^2} \int_{\partial M} \epsilon_{abcd} (2\theta^a{}_b \wedge R^c{}_d - \tfrac{4}{3}\theta^a{}_b \wedge \theta^c{}_e \wedge \theta^e{}_d). \tag{8.5}$$

Signature complex. For the Hirzebruch signature complex, we find the index theorem

$$\tau(M) = -\frac{1}{24\pi^2} \int_M \mathrm{Tr}(R \wedge R) + \frac{1}{24\pi^2} \int_{\partial M} \mathrm{Tr}(\theta \wedge R) - \eta_S(\partial M). \tag{8.6}$$

Dolbeault complex. The index theorem for the Dolbeault complex with boundary involves additional subtleties which we will not discuss here. See Donnelly [1977] for further details.

Spin complex. The index theorem for the spin complex takes the form

$$\mathrm{index}(\Delta_\pm, D) = \frac{1}{24 \cdot 8\pi^2} \int_M \mathrm{Tr}(R \wedge R) - \frac{1}{24 \cdot 8\pi^2} \int_{\partial M} \mathrm{Tr}(\theta \wedge R) - \tfrac{1}{2}[\eta_{\mathrm{Dirac}}(\partial M) + h(\partial M)]. \tag{8.7}$$

Explicit examples are worked out at the end of this subsection.

Twisted spin complex. The treatment of twisted complexes over manifolds with boundary is straightforward in principle. We work out the index formulas for the twisted spin complex as an illustration. One first chooses a connection and a combined Dirac–Yang–Mills operator D_V on the twisted complex $\Delta_\pm \otimes V$. The index is the difference in the number of positive and negative chirality spinors in the kernel of D_V *obeying the Atiyah–Patodi–Singer boundary conditions*. (Recall that these are *nonlocal* boundary conditions and thus may not correspond to those which one might be tempted to use from physical considerations.) We write

$$\mathrm{index}(\Delta_\pm \otimes V, \partial M) = \nu_+(\partial M) - \nu_-(\partial M).$$

The twisted η-invariant $\eta(\Delta_\pm \otimes V, \partial M)$ must be computed from the appropriate spectrum $\{\lambda_i\}$ of D_V restricted to ∂M; computing η could in general be quite difficult. If the given metric is not a product metric on the boundary, we choose the desired second fundamental form and add the Chern–Simons correction to the tangent bundle curvature term; no analogous correction is required for the vector bundle piece. Hence for $n=4$, we find the index theorem

$$\begin{aligned}\operatorname{index}(\Delta_\pm \otimes V, \partial M) &\equiv \nu_+(\partial M) - \nu_-(\partial M)\\ &= \frac{\dim V}{24\cdot 8\pi^2}\left[\int_M \operatorname{Tr}(R\wedge R) - \int_{\partial M} \operatorname{Tr}(\theta\wedge R)\right]\\ &\quad - \frac{1}{8\pi^2}\int_M \operatorname{Tr}(F\wedge F) - \frac{1}{2}[\eta_{D_V}(\Delta_\pm\otimes V, \partial M) + h_{D_V}(\Delta_\pm\otimes V, \partial M)].\end{aligned} \tag{8.8}$$

Examples 8.4

1. *Self-dual Taub–NUT metric* (Eguchi, Gilkey and Hanson [1978]). Consider the metric

$$ds^2 = \frac{r+m}{r-m}\,dr^2 + (r^2-m^2)\left[\sigma_x{}^2 + \sigma_y{}^2 + \left(\frac{2m}{r+m}\right)^2 \sigma_z{}^2\right]$$

and the product metric

$$ds_0{}^2 = \frac{r_0+m}{r_0-m}\,dr^2 + (r_0{}^2-m^2)\left[\sigma_x{}^2 + \sigma_y{}^2 + \left(\frac{2m}{r_0+m}\right)^2 \sigma_z{}^2\right].$$

The connections are

$$\omega_{01} = -\frac{r}{r+m}\sigma_x, \qquad \omega_{02} = -\frac{r}{r+m}\sigma_y, \qquad \omega_{03} = -\frac{2m^2}{(r+m)^2}\sigma_z$$

$$\omega_{23} = -\frac{m}{r+m}\sigma_x, \qquad \omega_{31} = -\frac{m}{r+m}\sigma_y, \qquad \omega_{12} = \left(\frac{2m^2}{(r+m)^2} - 1\right)\sigma_z$$

and

$$(\omega_0)_{0i} = 0, \qquad (\omega_0)_{12} = \left(\frac{2m^2}{(r_0+m)^2} - 1\right)\sigma_z$$

$$(\omega_0)_{23} = -\frac{m}{r_0+m}\sigma_x, \qquad (\omega_0)_{31} = -\frac{m}{r_0+m}\sigma_y.$$

Hence the second fundamental form at the boundary $r=r_0$ is

$$\theta_{01} = -\frac{r_0}{r_0+m}\sigma_x, \qquad \theta_{02} = -\frac{r_0}{r_0+m}\sigma_y, \qquad \theta_{03} = -\frac{2m^2}{(r_0+m)^2}\sigma_z$$

$$\theta_{23} = \theta_{31} = \theta_{12} = 0.$$

Then the Dirac index is

$$\text{index(Dirac}, r_0) = \frac{1}{24 \cdot 8\pi^2}\left(\int_{M(r_0)} \operatorname{Tr} R \wedge R - \int_{S^3 \text{ at } r_0} \operatorname{Tr} \theta \wedge R \right) - \frac{1}{12}\left[1 - 2\left(\frac{2m}{r_0 + m}\right)^2 + \left(\frac{2m}{r_0+m}\right)^4\right],$$

where we used Hitchin's formula [1974] for the η-invariant. Performing the integrals (the r-integration is from m to r_0), we find

$$\begin{aligned}\text{index(Dirac}, r_0) &= \left[\frac{4m^3(m-2r_0)}{3(r_0+m)^4} - \left(-\frac{1}{12}\right)\right] - \frac{2m^2(r_0-m)^2}{3(r_0+m)^4} - \frac{1}{12}\left[1 - \frac{8m^2}{(r_0+m)^2} + \frac{16m^4}{(r_0+m)^4}\right] \\ &= 0.\end{aligned}$$

Thus the Atiyah–Patodi–Singer index theorem states that there is *no asymmetry* between positive and negative chirality Dirac spinors obeying the appropriate boundary conditions.

2. *Index theorems for the metric of Eguchi and Hanson* (Atiyah [1978]; Hanson and Römer [1978]). We take the metric treated in example 3.3.3,

$$ds^2 = \frac{dr^2}{(1-(a/r)^4)} + r^2(\sigma_x{}^2 + \sigma_y{}^2 + (1-(a/r)^4)\sigma_z{}^2),$$

where σ_x, σ_y, σ_z range over $P_3(\mathbb{R})$, and choose the product metric at $r = r_0$ to be

$$ds_0{}^2 = \frac{dr^2}{(1-(a/r_0)^4)} + r_0{}^2(\sigma_x{}^2 + \sigma_y{}^2 + (1-(a/r_0)^4)\sigma_z{}^2).$$

The second fundamental form $\theta = \omega - \omega_0$ at the boundary $r = r_0$ is then

$$\theta_{01} = -(1-(a/r_0)^4)^{1/2}\sigma_x, \qquad \theta_{02} = -(1-(a/r_0)^4)^{1/2}\sigma_y, \qquad \theta_{03} = -(1+(a/r_0)^4)\sigma_z$$

$$\theta_{12} = \theta_{23} = \theta_{31} = 0.$$

We choose the orientation $dr \wedge \sigma_x \wedge \sigma_y \wedge \sigma_z$ to be positive.

Integrating the appropriate forms for the Euler characteristic over the manifold M and its boundary $P_3(\mathbb{R})$ with $r_0 \to \infty$, we find both a 4-volume term and a boundary correction,

$$\chi(M) = \tfrac{3}{2} - (-\tfrac{1}{2}) = 2.$$

The integral of the first Pontrjagin class for this metric is

$$P_1[M] = -\frac{1}{8\pi^2}\int_M \operatorname{Tr}(R \wedge R) = -3,$$

while the Chern–Simons boundary correction vanishes,

$$-Q_1[\partial M = P_3(\mathbf{R})] = \frac{1}{8\pi^2} \int_{P_3(\mathbf{R})} \mathrm{Tr}(\theta \wedge R) = 0.$$

The signature complex η-invariant correction for the $P_3(\mathbf{R})$ boundary is

$$\xi_S = \frac{1}{2}\cot^2\frac{\pi}{2} = 0,$$

so the signature is

$$\tau(M) = \tfrac{1}{3}P_1 + \xi_S = -1.$$

The index of the spin $\frac{1}{2}$ Dirac operator is

$$I_{1/2} = \mathrm{index}(\mathrm{Dirac}, \partial M) = -\tfrac{1}{24}P_1 + \xi_{\mathrm{Dirac}}.$$

For $P_3(\mathbf{R})$, ξ_{Dirac} is $\frac{1}{2}$ the G-index,

$$\xi_{\mathrm{Dirac}} = \frac{1}{2}\left(\frac{\mathrm{i}}{2\sin(\pi/2)} \times \frac{\mathrm{i}}{2\sin(\pi/2)}\right) = -\frac{1}{8}.$$

Thus there is no asymmetry between positive and negative chirality Dirac spinors,

$$I_{1/2} = -\tfrac{1}{24}(-3) - \tfrac{1}{8} = 0.$$

The spin $\frac{3}{2}$ Rarita–Schwinger operator index theorem reads

$$I_{3/2} = \mathrm{index}(\text{Rarita–Schwinger}, \partial M) = \tfrac{21}{24}P_1 + \xi_{\mathrm{RS}}$$

where

$$\xi_{\mathrm{RS}} = -\frac{1}{2}\frac{(2\cos\theta_1 + 2\cos\theta_2 - 1)}{(2\sin\frac{1}{2}\theta_1)(2\sin\frac{1}{2}\theta_2)}.$$

For $P_3(\mathbf{R})$ boundaries ($\theta_1 = \theta_2 = \pi$), we have

$$I_{3/2} = \tfrac{21}{24}(-3) + \tfrac{5}{8} = -2.$$

Hence

$$I_{3/2} = 2\tau$$

and there does exist an asymmetry between positive and negative chirality Rarita–Schwinger spinors for this metric.

9. Differential geometry and Yang–Mills theory

In this section, we first give a brief introduction to the path-integral method for quantizing Yang–Mills theories and then describe some of the Yang–Mills instanton solutions. The last part of the section contains a list of mathematical results concerning Yang–Mills theories whose detailed treatment is beyond the scope of this article.

9.1. Path-integral approach to Yang–Mills theory

The most useful approach to the quantization of gauge theories appears to be Feynman's path integral method. From a geometric point of view, the path integral has the advantage of being able to take the global topology of the gauge potentials into account, while the canonical perturbation theory approach to quantization is sensitive only to the local topology.

At present, a mathematically precise theory of path integration can be formulated only for spacetimes with positive signatures $(+,+,+,+)$; we refer to such spacetimes as "Euclidean" or "imaginary time" manifolds. Physically meaningful answers are obtainable by continuing the results of the Euclidean path integration back to the Minkowski regime with signature $(-,+,+,+)$.

In the Euclidean path-integral approach to quantization, each field configuration $\varphi(x)$ is weighted by the "Boltzmann factor", i.e., the exponential of minus its Euclidean action $S[\varphi]$:

$$(\text{contribution of } \phi(x)) = \exp(-S[\phi]).$$

For Yang–Mills theories, the Euclidean action is

$$S[A] = +\frac{1}{4}\int_M F^a_{\mu\nu}F^a_{\mu\nu}\, g^{1/2}\, d^4x = -\frac{1}{2}\int_M \mathrm{Tr}\, F \wedge *F, \tag{9.1}$$

which is positive definite. The contribution of each gauge potential or connection $A_\mu(x)$ to the path integral is therefore bounded and well-behaved.

The complete generating functional for the transition amplitudes of a theory is obtained by summing (or functionally integrating) over all inequivalent field configurations. Since the first-order functional variation of the action vanishes for solutions of the equations of motion, these configurations correspond to stationary points in the functional space. Therefore, in the path-integral approach, we first seek solutions to the Euclidean field equations with minimum action and then compute quantum-mechanical fluctuations around them.

The Yang–Mills field equations found by varying the action may be written as

$$d*F + A \wedge *F - *F \wedge A = 0,$$

while the Bianchi identities are

$$dF + A \wedge F - F \wedge A = 0.$$

These two equations together imply that the curvature F is *harmonic* in a suitable sense.

Minima of the action: In order to find the minimum action configurations of the Yang–Mills theory, let us consider the inequality

$$\int_M (F^a_{\mu\nu} \pm {}^*F^a_{\mu\nu})^2 g^{1/2}\, d^4x \geq 0.$$

This bound is saturated by the self-dual field configurations

$$F = \pm * F. \tag{9.2}$$

In fact, these field configurations solve the Yang–Mills field equations since the Bianchi identities imply the field equations. The action now becomes

$$S = -\tfrac{1}{2}\int \operatorname{Tr} F \wedge *F = \mp\tfrac{1}{2}\int \operatorname{Tr} F \wedge F = 4\pi|k|,$$

where

$$-C_2 = k = -\frac{1}{8\pi}\int_M \operatorname{Tr} F \wedge F \tag{9.3}$$

is the integral of the 2nd Chern class. 't Hooft [1976a] called such special field configurations "instantons" since in the case $|k| = 1$ their field strength is centered around some point in space-time and thus attains its maximum value at some "instant of time".

Physical interpretation of instantons: The instanton can be interpreted as a quantum-mechanical tunneling phenomenon in Yang–Mills gauge theories. It induces a transition between homotopically inequivalent vacua. The true ground state of Yang–Mills theory then becomes a coherent mixture of all these vacuum states. For more details on this subject, see, for example, Jackiw [1977]. One-loop quantum-mechanical fluctuations about the instanton have been explicitly calculated by 't Hooft [1977], who showed that the instanton solved the long-standing $U(1)$ problem via its coupling to the anomaly of the ninth axial current.

9.2. Yang–Mills instantons

The dominant contribution to the Euclidean path integral comes from the instanton solutions obeying the self-duality condition

$$F = \pm * F.$$

All gauge-potentials or connections satisfying the Yang–Mills equations with self-dual curvature are now, in principle, known (see section 9.3).

1. *BPST solution* (Belavin et al. [1975]) [see examples 4.3.3. and 5.5.2]. The instanton of Belavin, Polyakov, Schwarz and Tyupkin solves the Yang–Mills equations with $k = \pm 1$. Although the spacetime of the solution appears to be $\mathbb{R}^4$, the boundary conditions at ∞ allow the space to be compactified to S^4. Hence the BPST instanton is a connection with self-dual curvature on an SU(2) principal bundle over S^4 with second Chern number $C_2 = -1$. Since the action of the BPST instanton is $S = 4\pi$, it has the least action possible for a nontrivial topology and thus is the most important solution in Yang–Mills theory. We note that the BPST instanton is, in fact, a connection on the Hopf fibering π: $S^7 \to S^4$ (Trautman [1977]) and for this reason can be obtained from self-dual combinations of the standard Riemannian connections on S^4 (see example 5.5.2).

2. *Multi-center* SU(2) *solutions.* A special class of self-dual solutions of the SU(2) Yang–Mills equations for arbitrary "instanton-number" k is obtained by the following simple ansatz ('t Hooft [1976b]; Wilczek [1976]; Corrigan and Fairlie [1977]),

$$A^a_\mu = -\bar{\eta}^a_{\mu\nu}\,\partial_\nu \ln\phi\,, \tag{9.4}$$

where the constants $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ are given in appendix C. Imposing the self-duality condition, one obtains

$$\Box\phi/\phi = 0.$$

't Hooft gave the following solution to this equation,

$$\phi(x) = 1 + \sum_{i=1}^{k} \frac{\rho_i}{(x - x_i)^2}\,.$$

x_i and ρ_i are interpreted as the position and the size of the ith instanton and the solution describes the k-instanton configuration. The k-anti-instanton solution is obtained by replacing $\bar{\eta}$ by η.

This class of solutions was further generalized by Jackiw, Nohl and Rebbi [1977] who noticed that the 't Hooft solution is not invariant under conformal transformations and can, in fact, be generalized as

$$\phi(x) = \sum_{i=1}^{k+1} \frac{\lambda_i}{(x - y_i)^2}\,.$$

This solution again describes a k-instanton configuration and possesses $5k+4$ parameters (overall scale is irrelevant). Here, however, the parameters λ_i and y_i are not directly related to the size and location of the ith instanton. In the special cases of $k = 1$ and 2, the solution possesses 5 and 13 parameters, respectively, when one excludes parameters associated with gauge transformations.

9.3. *Mathematical results concerning Yang–Mills theories*

There exist a variety of mathematical results concerning Yang–Mills theories and differential geometry whose detailed treatment is beyond the scope of this work. We present here a list of assorted mathematical facts which we feel might be of relevance to physics.

1. *Parameter space for instanton solutions.* Schwarz [1977] and Atiyah, Hitchin and Singer [1977] have applied the index theorem to an elliptic complex corresponding to the Yang–Mills equations. This complex allows one to analyze small self-dual fluctuations around the instanton solution. Determination of the index of the complex then allows one to compute the *number of possible free parameters* in an instanton solution. They found that for the k-instanton SU(2) solution,

$$\text{no. of free parameters} = 8k - 3,$$

in agreement with the results of Jackiw and Rebbi [1977] and Brown, Carlitz and Lee [1977] who used physicists' methods. Thus the Jackiw–Nohl–Rebbi solution exhausts the number of available parameters only for $k = 1$ and $k = 2$.

The analysis of small self-dual oscillations around instanton solutions was then extended to include all Lie groups (Atiyah, Hitchin and Singer [1978]; Bernard, Christ, Guth and Weinberg [1977]). The dimension of the space of parameters for irreducible self-dual connections on principal G-bundles over S^4 with $C_2 = -k$ is given in table 9.1 for each G. We also list restrictions on k which must hold if there are to exist irreducible connections which are not obtained by embedding the connection of a smaller group.

Table 9.1

Group	Dimension of parameter space	Irreducibility condition
SU(n)	$4nk - n^2 + 1$	$k \geq n/2$
Spin(n)	$4(n-2)k - n(n-1)/2$	$k \geq n/4$ ($n \geq 7$)
Sp(n)	$4(n+1)k - n(2n+1)$	$k \geq n$
G_2	$16k - 14$	$k \geq 2$
F_4	$36k - 52$	$k \geq 3$
E_6	$48k - 78$	$k \geq 3$
E_7	$72k - 133$	$k \geq 3$
E_8	$120k - 248$	$k \geq 3$

Thus, for example, SU(3) solutions have $12k - 8$ parameters and for $k \geq 2$ there exist irreducible SU(3) solutions which are not obtained from SU(2) solutions.

We remark that physicists often refer to the dimension of the parameter space as the number of *zero-frequency modes*, while mathematicians may refer to the same thing as the *dimension of the moduli space*.

2. *Explicit solutions for the most general self-dual connections.* The $(5k+4)$-parameter Jackiw–Nohl–Rebbi solutions for SU(2) instantons do not exhaust the $(8k-3)$-dimensional parameter space for $k \geq 3$. The problem of finding the most general solutions (e.g., with $8k-3$ parameters for SU(2)) was attacked using twistor theory (Ward [1977]; Atiyah and Ward [1977]), and the method of universal connections and algebraic geometry (Atiyah, Hitchin, Drinfeld and Manin [1978]). It was shown that the problem of determining the most general self-dual connection for virtually any principal bundle over S^4 is reducible to a problem in algebraic geometry concerning holomorphic vector bundles over $P_3(\mathbb{C})$.

In fact, the whole procedure can be reduced to ordinary linear algebra. For example, to calculate the self-dual SU(2) connection for the bundle with Chern class $C_2 = -k$ one starts with a $(k+1) \times k$

dimensional quarternion-valued matrix

$$\Delta = a + bx.$$

(Physicists may prefer to think of a_{ij}, b_{ij} and x as having values in SU(2), so $x = x^0 - \mathrm{i}\,\boldsymbol{\lambda}\cdot\boldsymbol{x}$ etc., where $\{\boldsymbol{\lambda}\}$ are the Pauli matrices.)

Then one determines the universal connection $\omega = V^\dagger\,\mathrm{d}V$ by solving the equations

$$\begin{aligned} &V^\dagger\Delta = 0\\ &V^\dagger V = 1\\ &1 = VV^\dagger + \Delta\frac{1}{\Delta^\dagger\Delta}\Delta^\dagger\\ &\Delta^\dagger\Delta = \text{a real number}\end{aligned} \tag{9.5}$$

for V. The number of free parameters in $V^\dagger\,\mathrm{d}V$ which are not gauge degrees of freedom turns out to be exactly the required number. There are deep reasons, based on algebraic geometry, for the success of this construction (see e.g. Hartshorne [1978]). Propagators in these instanton fields were obtained by Christ, Weinberg and Stanton [1978] and Corrigan, Fairlie, Templeton and Goddard [1978] which generalized the result of Brown, Carlitz, Creamer and Lee [1977] for propagators in the 't Hooft, Jackiw–Nohl–Rebbi solution. We refer the reader to the original literature for further details.

3. *Universal connections* (Narasimhan and Ramanan [1961, 1963]; Dubois-Violette and Georgelin [1979]). In the derivation of the most general self-dual connections, the method of universal connections played an essential role. The theorem of Narasimhan and Ramanan shows that all fiber bundles with a given set of characteristic classes are viewable as particular projections of a more general bundle called a "universal classifying space". Typical classifying spaces are Grassmannian manifolds $\mathrm{Gr}(m, k)$, the space of all k-manifolds embedded in m-space, with m usually taken to approach infinity. Both the base manifold *and* the fiber of a given fiber bundle are included in the classifying space; complicated projections must be taken to describe bundles with complicated base manifolds.

One can write any connection on a fiber bundle in terms of a projection down from a universal connection on the classifying space. In particular, for sufficiently large m, the connection on a $U(k)$ principal bundle can always be written in terms of an $m \times k$ complex matrix V as

$$\omega = V^\dagger\,\mathrm{d}V$$

where

$$V^\dagger V = 1_k, \qquad VV^\dagger = P(x) = (m \times m \text{ projection}).$$

Choosing a local cross-section $V(x)$ of the classifying space gives the Yang–Mills potential in a certain gauge,

$$A(x) = V^\dagger(x)\,\mathrm{d}V(x).$$

$A(x)$ is *not* a pure gauge here because V is not a $k \times k$ matrix. The curvature

$$\begin{aligned} F &= \mathrm{d}A + A \wedge A \\ &= \mathrm{d}V^{+}(1 - P(x))\,\mathrm{d}V \end{aligned}$$

is, in general, non-trivial. Gauge transformations are obviously effected by multiplying V on the right by a $k \times k$ matrix $\Lambda(x)$,

$$V(x) \to V(x)\Lambda(x)$$

so that

$$\begin{aligned} A' &= (\Lambda^{+}V^{+})\,\mathrm{d}(V\Lambda) = \Lambda^{+}(V^{+}\,\mathrm{d}V)\Lambda + \Lambda^{+}(V^{+}V)\,\mathrm{d}\Lambda \\ &= \Lambda^{+}A\Lambda + \Lambda^{+}1_k\,\mathrm{d}\Lambda. \end{aligned}$$

The covariant derivative has a straightforward interpretation in terms of the action of the projection $P(x) = VV^{+}$ on the m-dimensional extension of the k-dimensional wave function Ψ,

$$\bar{\Psi} = V\Psi.$$

When one projects the exterior derivative of $\bar{\Psi}$, one finds the extension of the covariant derivative of the ordinary wave function Ψ:

$$P\,\mathrm{d}\bar{\Psi} = P\,\mathrm{d}V\Psi + PV\,\mathrm{d}\Psi = V(V^{+}\,\mathrm{d}V\Psi + \mathrm{d}\Psi) \equiv V\,\mathrm{D}\Psi.$$

4. *Compactifiability of finite-action Yang–Mills connections* (Uhlenbeck [1978]). Suppose $A(x)$ is a section of a connection one-form on a manifold M which is a compact manifold M lacking the origin, i.e.,

$$\hat{M} = M - \{0\}.$$

Suppose also that $F = \mathrm{d}A + A \wedge A$ is harmonic and that the Yang–Mills action is finite.

Then there exist gauge transformations near $\{0\}$ which extend A to all M. In fact, it has been shown that all Euclidean finite-action Yang–Mills solutions over $M - \{0\}$ are smoothly extended to the *compact* manifold M.

This theorem tells us that any self-dual finite-action solution to the Euclidean Yang–Mills equations must describe a bundle with a compactified spacetime base manifold.

5. *Stability of all self-dual solutions* (Bourguignon, Lawson and Simons [1979]). The stability of Yang–Mills solutions has also been studied. One can show that if the base manifold M is S^4, all stable Yang–Mills solutions are *self-dual*. Combined with Uhlenbeck's theorem given above, this theorem allows us to conclude that *all finite-action stable* Yang–Mills solutions (connections with harmonic curvatures) are self-dual.

6. *Index theorems in open spaces* (Callias [1978]; Bott and Seeley [1978]). An extension of the index theorem to Yang–Mills theories in open Euclidean spaces of odd-dimension d has been given by

Callias. This result has interesting applications to the Dirac equation in $(d+1)$-dimensional Minkowski spacetime.

7. *Meron solutions.* Besides the instantons, which are non-singular solutions to the Euclidean Yang–Mills field equations, there is a class of singular solutions called merons (Callan, Dashen and Gross [1977]) which were first discovered by De Alfaro, Fubini and Furlan [1976]. As compared with instantons whose topological charge density $\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x)$ is a smooth function of x, the topological charge density of merons vanishes everywhere except at the singular points.

For instance, the SU(2) 2-meron solution is given by

$$A = \tfrac{1}{2}g_1^{-1}\,\mathrm{d}g_1 + \tfrac{1}{2}g_2^{-1}\,\mathrm{d}g_2,$$

where

$$g_i = \frac{(t-t_i) - \mathrm{i}\boldsymbol{\lambda}\cdot(\boldsymbol{x}-\boldsymbol{x}_i)}{[(t-t_i)^2 + (\boldsymbol{x}-\boldsymbol{x}_i)^2]^{1/2}}.$$

The topological charge density of this solution is a sum of two δ-functions centered at x_1 and x_2, each of which gives $\frac{1}{2}$ unit of the quantized topological charge. Therefore, in some sense, the meron is a split instanton.

Glimm and Jaffee [1978] considered an axially-symmetric multimeron configuration and the existence of a solution for this configuration was proved by Jonsson, McBryan, Zirilli and Hubbard [1979].

8. *Absence of global gauge conditions in functional space of connections* (Singer [1978a]). The Feynman path-integral approach to the quantization of field theories is based on the use of the functional space of the field variables. In the case of Yang–Mills theories, the fields in question are the connections on the principal bundle, which are defined only up to gauge transformations. Hence the *functional space of connections* is a complicated infinite-dimensional fiber bundle whose projection carries all gauge-equivalent connections into the same point in the base space or *moduli space* of the bundle.

Physical quantities are calculated by integrating over the moduli space to avoid the meaningless infinities which would result from integrating over gauge-equivalent connections. Gribov [1977, 1978] discovered that there exist gauge-equivalent connections which obey the Coulomb gauge condition, so that defining functional integration over the moduli space could be potentially troublesome.

The mathematical nature of the problem of defining the moduli space of the functional space of connections was examined by Singer using techniques of global analysis. He has shown that for compact simply-connected spacetimes the infinite-dimensional bundle in question is nontrivial; hence a single global gauge condition could never be used to define a global section, and thus could not unambiguously define the moduli space. He showed that the manifold described by any given gauge condition eventually turned back on itself to intersect a given fiber of the functional bundle an infinite number of times. Thus the moduli space over which the path integration for gauge theories must be performed can be defined only in local patches.

9. *Natural metric on the functional space of connections and the Faddeev–Popov determinant* (Singer [1978b]; Babelon and Viallet [1979]). Before one can integrate over a functional space, one must know the *measure* of the integration element. To get the proper transformation properties of the functional measure, physicists multiply the integrand by a factor called the Faddeev–Popov determinant. It is now known that this measure follows from a *natural metric* on the moduli space of the functional space of

connections. The Faddeev–Popov determinant arises naturally as the standard $g^{1/2}$ Jacobian multiplying the naive measure.

10. *Ray–Singer torsion and the functional integral* (Singer [1978c]; Schwarz [1978, 1979a, b]). Functional determinants obtained by calculating the quadratic fluctuations around instantons are essential elements of the quantized Yang–Mills theory. Thus it is interesting to note that these functional determinants are intimately related to a mathematical construction by Ray and Singer [1971, 1973] introduced many years ago. Additional insights into the functional integral in Yang–Mills theory might be gained by the exploration of the Ray–Singer analytic torsion.

10. Differential geometry and Einstein's theory of gravitation

The intimate relationship between Einstein's theory of gravity and Riemannian geometry has been thoroughly explored over the years. Here we will attempt to outline some of the more recent ideas concerning the physics of gravitation and the relevance of modern differential geometry to gravitation. We begin with an introduction to current work on quantum gravity and gravitational instantons. We then present a list of mathematical results which are of specific interest to the study of gravity.

10.1. Path integral approach to quantum gravity

Quantization of the theory of gravitation is one of the most outstanding problems in theoretical physics. Due to the non-polynomial character of the theory the standard methods of quantization do not work for gravity. At present, Feynman's path integral approach appears to be the most viable procedure for quantizing gravity. Path integration has the advantage of being able to take into account the global topology of the space-time manifold as opposed to other quantization schemes. However, since the theory of gravity is not renormalizable in the usual sense, we always encounter the difficulties of non-renormalizable divergences in practical calculations.

As in the Yang–Mills case, we work with the Euclidean version of the theory and the Euclidean (imaginary time) path integral. Our field variables $g_{\mu\nu}$ are metrics having a Riemannian signature $(+,+,+,+)$, and the (imaginary time) gravitational action is given by

$$S[g] = -\frac{1}{16\pi G}\int_M \mathcal{R} g^{1/2}\,\mathrm{d}^4x - \frac{1}{8\pi G}\int_{\partial M} K\,\mathrm{d}^3\Sigma + C \tag{10.1}$$

where G is Newton's constant, $\mathcal{R}$ is the Ricci scalar curvature and K is the trace of the second fundamental form of the boundary in the metric g. The second term is a surface correction required when ∂M is nonempty (York [1972]; Gibbons and Hawking [1977]). C is a (possibly infinite) constant chosen so that $S[g]=0$ when the metric $g_{\mu\nu}$ is the flat space metric. Einstein's field equations in empty space are given by

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R} = 0. \tag{10.2}$$

As in the Yang–Mills theory, there exist finite action solutions to the Euclidean Einstein equations which possess interesting global topological properties. We describe these solutions in the next subsection.

Non-positive-definiteness of the Einstein action: Unlike the Yang–Mills case, the gravitational action is linear in the curvature and not necessarily positive. In particular, by introducing a rapidly varying conformal factor into a metric, one can make its action negative and arbitrarily large. This causes a divergence in the path integration over the conformal factor. To get around this difficulty, Gibbons, Hawking and Perry [1978] proposed the following procedure for the evaluation of the path integral:

- separate the functional space of metrics into conformal equivalence classes;
- in each class, choose the metric g for which the Ricci scalar $\mathcal{R} = 0$;
- rotate the contour of integration of the conformal factor λ to be parallel to the pure imaginary axis in order to achieve the convergence of the integration. Namely, we put $\lambda = 1 + i\xi$ and integrate over real ξ;
- integrate over all conformal equivalence classes.

Positive action conjecture: For the metric in a given conformal equivalence class with $\mathcal{R} = 0$, the gravitational action consists entirely of the surface term. Since the physically reasonable boundary condition for the metric is asymptotic flatness, one would hope that the action is positive in this case. This leads to *the positive action conjecture* (Gibbons, Hawking and Perry [1978]):

$S \geq 0$ for all asymptotically Euclidean positive definite metrics with $\mathcal{R} = 0$.

Asymptotically Euclidean metrics are those which approach the flat metric in all spacetime directions at ∞ and whose global topology is the same as $\mathbb{R}^4$ at ∞. It can be shown that $S = 0$ only for the flat metric on $\mathbb{R}^4$ (Gibbons and Pope [1979]). The positive action conjecture has recently been proven by Schoen and Yau [1979a].

A natural modification of the positive action conjecture was suggested by the discovery of a new type of metric (Eguchi and Hanson [1978]) which is locally flat at ∞, but has a global topology different from that of $\mathbb{R}^4$ at ∞ (Belinskii, Gibbons, Page and Pope [1978]). This class of metrics is called asymptotically locally Euclidean (ALE). *The generalized positive action conjecture* (Gibbons and Pope [1979]) states that

$S \geq 0$ for any complete non-singular positive definite asymptotically locally Euclidean metric with $\mathcal{R} = 0$; $S = 0$ if and only if the curvature is self-dual.

Spacetime foam (Hawking [1978]; Perry [1979]; Hawking, Page and Pope [1979]): Since the theory of gravity is not renormalizable, one expects strong quantum fluctuations at short distances, i.e., at the size of the Planck length. These fluctuations might be viewed as a "spacetime foam" which is the basic building block of the universe. Thus the spacetime in quantized gravity theory is expected to be highly curved at small distances, while at large distances the curvature is expected to cancel and give an almost flat spacetime. Spacetime foam is an important subject for future research in quantized gravity.

10.2. Gravitational instantons

As in the Yang–Mills theory, there also exist finite action solutions to the classical field equations in the theory of gravitation. Such solutions are called *gravitational instantons* because of the close analogy to the Yang–Mills instantons. A variety of solutions of Einstein's equations with instanton-like properties have been discovered. Those with self-dual curvature are especially appealing because they have interesting mathematical properties and bear the strongest similarity to the self-dual Yang–Mills instantons. For a review, see Eguchi and Hanson [1979].

1. *The metric of Eguchi and Hanson* [1978] [see example 3.3.3]. This is the metric which most closely resembles the Yang–Mills instanton of Belavin et al. [1975]. It has a self-dual Riemannian curvature which falls off rapidly in all spacetime directions and has $\chi = 2$, $\tau = -1$. The boundary at ∞ is $P_3(\mathbb{R}) = S^3/\mathbb{Z}_2$ (Belinskii et al. [1978]), and thus it is the simplest example of an asymptotically locally Euclidean metric. The global manifold is $T^*(P_1(\mathbb{C}))$.

2. *Multi-center self-dual metrics* (Hawking [1977]; Gibbons and Hawking [1978]). This class of metrics is given by

$$ds^2 = V^{-1}(\boldsymbol{x})(d\tau + \boldsymbol{\omega}\cdot d\boldsymbol{x})^2 + V(\boldsymbol{x})\, d\boldsymbol{x}\cdot d\boldsymbol{x},$$

where

$$\nabla V = \pm\nabla\times\boldsymbol{\omega}$$

$$V = \epsilon + 2m\sum_{i=1}^{k}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_i|}.$$

The connection and the curvature are both self-dual in this coordinate system. The case $\epsilon = 1$, $k = 1$ is the self-dual Taub–NUT metric discussed in example 3.3.2, but in a different coordinate frame. When $\epsilon = 1$ for general k, we find the multi-Taub–NUT metric. These metrics approach a flat metric in the spatial direction $|\boldsymbol{x}|\to\infty$, but are periodic in the variable τ.

When $\epsilon = 0$ the asymptotic behavior of the metric changes completely and the metric $g_{\mu\nu}$ approaches the flat metric at 4-dimensional ∞ modulo the identification of points of spacetime under the action of a discrete group. The case $\epsilon = 0$, $k = 1$ turns out to be just a coordinate transformation of the flat space metric. When $\epsilon = 0$, $k = 2$ the metric is a coordinate transformation of the Eguchi–Hanson metric discussed above (Prasad [1979]). For general k, the metric represents a $(k-1)$-instanton configuration whose boundary at ∞ is the lens space $L(k, 1)$ of S^3. ($L(k, m)$ is defined by identifying the points of S^3 = [boundary of $\mathbb{C}^2$] related by the map

$$(z_1, z_2)\to(e^{2\pi i/k} z_1, e^{2\pi i m/k} z_2).)$$

The $\epsilon = 0$ general-k metric has $\chi = k$, $|\tau| = k - 1$. The possibility of self-dual metrics on manifolds whose boundaries are given by S^3 modulo other discrete groups has been considered by Hitchin [1979] and Calabi [1979] and will be discussed below.

3. *Fubini–Study metric on* $P_2(\mathbb{C})$ (Eguchi and Freund [1976]; Gibbons and Pope [1978]) [see example 3.4.3]. The manifold $P_2(\mathbb{C})$ is closed and compact without boundary and has $\chi = 3$, $\tau = 1$. Except for the fact that $P_2(\mathbb{C})$ fails to admit well-defined Dirac spinors, the Fubini–Study metric on $P_2(\mathbb{C})$ would be an appealing gravitational instanton; this metric satisfies Einstein's equations with nonzero cosmological constant and has a self-dual Weyl tensor, rather than a self-dual curvature.

4. *K*3 *surface*. The K3 surface is the *only* compact regular simply-connected manifold without boundary which admits a nontrivial metric with self-dual curvature (Yau [1977]). While the explicit form of the metric is not known, it must exist; since its curvature is self-dual it will solve Einstein's equations

with zero cosmological constant. For the K3 surface, $\chi = 24$ and $\tau = -16$. (*Remark*: The natural structure on the K3 surface is, precisely speaking, *anti*-self-dual (see Atiyah, Hitchin and Singer [1978]).)

5. *Miscellaneous solutions.* Among other interesting solutions are the Euclidean de Sitter space metric (i.e., the standard metric on S^4), the non-self-dual Taub–NUT metric with horizon and the compact rotating metric on $P_2(\mathbb{C}) \oplus P_2(\mathbb{C})$ found by Page [1978a,b], and the rotating Taub–NUT-like metric of Gibbons and Perry [1979].

10.3. Nuts and bolts

The gravitational instantons listed above can be described in terms of interesting mathematical structures called "nuts" and "bolts" by Gibbons and Hawking [1979]. Let us examine a general Bianchi type IX metric of the following form

$$ds^2 = d\tau^2 + a^2(\tau)\,\sigma_x^2 + b^2(\tau)\,\sigma_y^2 + c^2(\tau)\,\sigma_z^2 .$$

The manifold described by this metric is regular provided the functions a, b and c are finite and nonsingular at finite proper distance τ. However, the manifold can be regular even in the presence of apparent singularities.

Let us, for simplicity, consider singularities occurring at $\tau = 0$. A metric has a removable *nut singularity* provided that near $\tau = 0$,

$$a^2 = b^2 = c^2 = \tau^2 .$$

Then this apparent singularity is nothing but a coordinate singularity of the polar coordinate system in $\mathbb{R}^4$ centered at $\tau = 0$. The singularity is removed by changing to a local Cartesian coordinate system near $\tau = 0$ and adding the point $\tau = 0$ to the manifold. Nut singularities may also be understood from the viewpoint of global topology as fixed points of the Killing vector field; by the Lefschetz fixed point theorem (see section 7), each such fixed point (or nut) adds *one unit* to the Euler characteristic of the manifold.

A metric has a removable *bolt singularity* if near $\tau = 0$,

$$a^2 = b^2 = \text{finite}$$

$$c^2 = n^2\tau^2, \qquad n = \text{integer}.$$

Here $a^2 = b^2$ implies the canonical S^2 metric $\frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)$ for the $(a^2\sigma_x{}^2 + b^2\sigma_y{}^2)$ part of the metric, while at constant (θ, ϕ), the $(d\tau^2 + c^2\sigma_z{}^2)$ part of the metric looks like

$$d\tau^2 + n^2\tau^2\tfrac{1}{4}\, d\psi^2 .$$

Provided the range of ψ is adjusted so $n\psi/2$ runs from 0 to 2π, the apparent singularity at $\tau = 0$ is just a coordinate singularity of the polar coordinate system in $\mathbb{R}^2$ at the origin. This singularity can again be removed using Cartesian coordinates. The topology of the manifold is locally $\mathbb{R}^2 \times S^2$ and the $\mathbb{R}^2$ shrinks to a point on S^2 as $\tau \to 0$. This S^2 is a fixed surface of the Killing vector field. According to the G-index theorem (see section 7), each such fixed submanifold contributes its own Euler characteristic to the Euler characteristic of the entire manifold; thus each bolt contributes *two units* to the Euler characteristic.

The self-dual Taub–NUT metric (example 3.3.2)

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}\,dr^2 + \tfrac{1}{4}(r^2-m^2)(d\theta^2 + \sin^2\theta\, d\phi^2) + m^2\left(\frac{r-m}{r+m}\right)(d\phi + \cos\theta\, d\psi)^2$$

behaves at $r = m + \epsilon$ as

$$ds^2 \approx d\tau^2 + \tau^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2),$$

where $\tau = (2m\epsilon)^{1/2}$. Thus the apparent singularity at $r = m$ is a removable nut singularity. In contrast, the Eguchi–Hanson metric (example 3.3.3),

$$ds^2 = \frac{dr^2}{1-(a/r)^4} + r^2(\sigma_x{}^2 + \sigma_y{}^2 + (1-(a/r)^4)\sigma_z{}^2),$$

behaves near $r = a$, with fixed θ and ϕ, as

$$ds^2 \approx \tfrac{1}{4}(du^2 + u^2\, d\psi^2),$$

where $u^2 = r^2[1-(a/r)^4]$. Therefore, the apparent singularity at $r = a$ is a removable bolt singularity provided that the range of ψ is chosen to be that of the usual polar coordinates on $\mathbb{R}^2$,

$$0 \le \psi < 2\pi.$$

This explains why the boundary of the manifold of this metric is $P_3(\mathbb{R}) = S^3/\mathbb{Z}_2$, rather than S^3, which would have $0 \le \psi < 4\pi$. Next, we examine the $P_2(\mathbb{C})$ metric (example 3.4.3)

$$ds^2 = \frac{dr^2 + r^2\sigma_z{}^2}{(1+\Lambda r^2/6)^2} + \frac{r^2(\sigma_x{}^2 + \sigma_y{}^2)}{1+\Lambda r^2/6}.$$

Near $r = 0$, we obviously have a nut. On the other hand, at large r and fixed θ and ϕ, the metric behaves as

$$ds^2 \approx (\Lambda/6)^{-2}(du^2 + \tfrac{1}{4}u^2\, d\psi^2),$$

where $u = 1/r$. Thus the singularity at $u = 0$ $(r \to \infty)$ is a removable bolt singularity if

$$0 \le \psi < 4\pi.$$

Finally, we note that the Gibbons–Hawking k-center metrics can be shown to have k nut singularities.

10.4. Mathematical results pertinent to gravitation

Because of the close relationship between Einstein's theory of gravitation and differential geometry, any distinction between physical knowledge about gravitation and mathematical knowledge is necessarily somewhat arbitrary. In this section we collect a variety of useful facts pertinent to gravitation which seem to us primarily mathematical in flavor.

1. *Restrictions on four-dimensional Einstein manifolds.* A number of mathematical results are known which restrict the types of four-dimensional Euclidean-signature Einstein manifolds; these are precisely the manifolds which might be expected to be important in the Euclidean path integral for gravity.

We first restrict our attention to compact simply-connected four-dimensional spin manifolds M, and note that the Euler characteristic χ and the signature τ *nearly* characterize the manifold uniquely (recall that $|\tau|$ is a multiple of 8 for a spin manifold):

Case A: $|\tau| \neq \chi - 2 \;\Rightarrow\;$ M determined up to homotopy

Case B: $|\tau| = \chi - 2 \;\Rightarrow\;$ *unknown* whether M is determined up to homotopy.

It is not known if these conditions determine M up to a homeomorphism type.

It is instructive to study a manifold's properties in terms of its Betti numbers $(b_0, b_1, b_2, b_3, b_4)$; b_2 can be broken up into two parts,

$$b_2 = b_2^+ + b_2^-,$$

where b_2^+ is the number of self-dual harmonic 2-forms and b_2^- is the number of anti-self-dual harmonic 2-forms. We know the following results:

(1) Poincaré duality for compact orientable manifolds implies $b_0 = b_4$, $b_1 = b_3$
(2) $b_0 = b_4 =$ number of disjoint pieces of M
(3) $b_1 = b_3 = 0$ if M is simply connected
(4) $\chi = b_0 - b_1 + b_2 - b_3 + b_4 = 2b_0 - 2b_1 + b_2^+ + b_2^-$
(5) $\tau = b_2^+ - b_2^-$.

Thus for M compact and simply-connected,

$$\chi = 2 - 0 + b_2^+ + b_2^-$$
$$b_2^+ = \tfrac{1}{2}(\tau + \chi - 2)$$
$$b_2^- = \tfrac{1}{2}(-\tau + \chi - 2).$$

An *Einstein manifold* is defined as a manifold which admits a metric which obeys

$$\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}.$$

We state the following theorems:

I. (Berger [1965]). $\chi \geq 0$ for a 4-dimensional compact Einstein manifold M with $\chi = 0$ only if M is flat.

II. (Hitchin [1974b]).

$$\chi \geq \tfrac{3}{2}|\tau|$$

for a 4-dimensional compact Einstein manifold M, with

$$\chi = \tfrac{3}{2}|\tau|$$

only if M is flat or its universal covering is a K3 surface.

III. (Hitchin [1974b]). If M is a compact 4-dimensional Einstein manifold with non-negative (or non-positive) sectional curvature, then

$$\chi \geq (\tfrac{3}{2})^{3/2}|\tau|$$

with equality only if M is flat.

IV. (Gibbons and Pope [1979]). Suppose M is non-compact, and its non-compactness is completely characterized by removing N asymptotically Euclidean regions from a compact manifold $\bar{M}$. Then, if M is an Einstein space,

$$\chi(M) \geq N + \tfrac{3}{2}|\tau(M)|$$
$$\chi(\bar{M}) \geq 2N + \tfrac{3}{2}|\tau(\bar{M})|.$$

Examples:

Einstein: $S^4, S^2 \times S^2, P_2(\mathbb{C}), 2P_2(\mathbb{C}), 3P_2(\mathbb{C})$

not Einstein: $S^1 \times S^3, 2T^4, nP_2(\mathbb{C})$ for $n \geq 4$.

2. *K3 surface*. The K3 surface and the four-torus T^4, are the *only* closed, compact manifolds admitting metrics with self-dual Riemann curvature. (Conversely, all Ricci flat manifolds are self-dual if they are closed and compact.) For T^4, the self-dual metric is the trivial flat metric. For the K3 surface, the self-dual metric is nontrivial but unknown, although Yau [1978] has, in principle, given a way to construct it numerically. Other approaches to finding the K3 metric have been described by Page [1978c] and by Gibbons and Pope [1979]. Only the K3 surface and the Enriques surface (whose universal covering is K3) or the quotient of an Enriques surface by a free antiholomorphic involution with $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ saturate Hitchin's bound [1974b]

$$\chi = \tfrac{3}{2}|\tau|$$

with $\chi \neq 0$. We show below that $\chi = 24$, $|\tau| = 16$ and note that K3 is a complex manifold with first Betti number $b_1 = 0$, $b_2^+ = 19$, $b_2^- = 3$, and first Chern class $c_1 = 0$.

The K3 surface is definable as the solution to $f_4(z) = 0$ where f_4 is a homogeneous polynomial of degree 4 in the homogeneous coordinates z_0, z_1, z_2, z_3 of $P_3(\mathbb{C})$. It is thus instructive to examine it in the general context of polynomials $f_m(z) = 0$ of degree m in $P_3(\mathbb{C})$ (Back, Freund and Forger [1978]). We let V be the corresponding two-dimensional complex surface in $P_3(\mathbb{C})$ and split the tangent bundle of $P_3(\mathbb{C})$ in parts normal and tangential to V:

$$T(P_3(\mathbb{C})) = T(V) \oplus N(V).$$

The Chern classes for Whitney sums of bundles and for $P_n(\mathbb{C})$ itself are given by

$$c(T(P_3(\mathbb{C}))) = c(T(V))\,c(N(V))$$
$$c(T(P_n(\mathbb{C}))) = (1+x)^{n+1},$$

where x is $c_1(L^*)$, the normalized Kähler 2-form of the Fubini–Study metric on $P_n(\mathbb{C})$. Finally, we note that if V is given by $f_m(z) = 0$, the Chern class of $N(V)$ is given by

$$c(N(V)) = 1 + mx,$$

since m is the number of Riemann sheets of $f_m(z) = 0$. Letting

$$R = \mathrm{i}*x = \text{projection of the 2-form } x \text{ onto } V,$$

we combine the equations to give

$$(1+R)^4 = c(T(V))\,(1+mR)$$

and use the splitting principle to get (with $R \to r$)

$$c(T(V)) = \frac{1 + 4r + 6r^2 + \cdots}{1 + mr}$$
$$= 1 + (4-m)r + (m^2 - 4m + 6)r^2 = 1 + c_1 + c_2.$$

Now, since

$$\int_V R \wedge R = m = \text{number of Riemann sheets}$$

and

$$p_1 = c_1{}^2 - 2c_2 = [(4-m)^2 - 2(m^2 - 4m + 6)]R \wedge R = (4 - m^2)R \wedge R,$$

we can calculate all the properties of K3 by setting $m = 4$:

(1) $c_1 = (4-m)R = 0, \qquad c_2 = (m^2 - 4m + 6)R \wedge R = 6R \wedge R$

(2) $\tau = \frac{1}{3}P_1 = \dfrac{1}{3}\displaystyle\int_V p_1 = \frac{1}{3}m(4 - m^2) = -16$

(3) $\chi = \int_V c_2 = m(m^2 - 4m + 6) = 24$

$= \frac{3}{2}|\tau|$

(4) $\hat{A} = -\frac{1}{8}\tau = -\frac{1}{24}m(4 - m^2) = +2$

(5) $I_{\bar{\partial}} = \frac{1}{4}(\chi + \tau) = \frac{1}{4}(24 - 16) = +2.$

We thus see from (4) and (5) that K3 can be a spin manifold and a complex manifold.

3. *Harmonic spinors.* A very useful result concerning the Dirac equation on curved Euclidean (positive signature) manifolds is *Lichnerowicz's theorem* (Lichnerowicz [1963]):

If the scalar curvature $\mathcal{R}$ of a compact spin manifold is positive,

$$\mathcal{R} > 0,$$

then there are *no harmonic spinors* on the manifold.

However, there is no expression for the dimension of the space of harmonic spinors in terms of the topological invariants of the manifold: Hitchin [1974a] has shown that although the dimension of the space of harmonic spinors is conformally invariant, *it depends on the metric* used to define the Dirac operator.

4. *Spin structures.* As we observed in the section on characteristic classes, one can define spinors unambiguously on a manifold only if its second Stiefel–Whitney class *vanishes*: such a manifold is called a *spin-manifold.* However, the spinor phase ambiguity which occurs for non-spin manifolds can be cancelled by introducing an additional structure such as an electromagnetic field (a $U(1)$ principal bundle). This additional structure, the spin$_c$ structure, gives a new type of more general spin manifold. For instance, although the manifold $P_2(\mathbb{C})$ does not admit a spin structure, one can still define a spin$_c$ structure by introducing magnetic monopoles with half the Dirac charge (Trautman [1977]; Hawking and Pope [1978]). Back, Freund and Forger [1978] discuss interesting physical applications of the idea of the spin$_c$ structure.

5. *Deformations of conformally self-dual manifolds.* Singer [1978d] has examined the general case of the number of conformally self-dual deformations of a compact conformally self-dual manifold. This number is interesting to a physicist because it gives the number of free parameters, or the number of *zero-frequency modes*, of a given solution of Einstein's equations. By constructing an appropriate elliptic complex, Singer applies the index theorem and finds the number of conformally self-dual deformations to be the index of the complex:

$I = \frac{1}{2}(29|\tau| - 15\chi) + \dim(\text{conformal group}) + (\text{correction for absence of vanishing theorem if scalar curvature} \leq 0).$

Note that scale factors of the metric are not included here. This is the index of the gravitational deformations (see Gibbons and Perry [1978]) taking solutions to solutions, but the value of the action is not necessarily preserved.

Examples:

A. S^4. Here $\tau = 0$, $\chi = 2$, the conformal group is 15-dimensional and since $\mathcal{R} > 0$, there is no correction:

$$I = \tfrac{1}{2}(-30) + 15 + 0 = 0.$$

Thus a conformally self-dual metric on S^4 has no zero-frequency modes aside from a scale.

B. $P_2(\mathbb{C})$. Here $\tau = 1$, $\chi = 3$, the conformal group is 8-dimensional and $\mathcal{R} > 0$, so there is no correction:

$$I = \tfrac{1}{2}(29 - 45) + 8 + 0 = 0.$$

Thus the Fubini–Study metric, which has self-dual Weyl tensor, allows no conformally self-dual deformations apart from a scale.

C. *K3 surface*. For this manifold, $|\tau| = 16$, $\chi = 24$, the conformal group is empty, but there is no vanishing theorem because the manifold is *self-dual*; it has self-dual Riemann tensor in addition to self-dual Weyl tensor. Singer has shown that there are 5 covariant constant objects in W_-, which constitute the vanishing theorem correction. Thus

$$I = \tfrac{1}{2}(29 \times 16 - 15 \times 24) + 0 + 5 = 57.$$

Including a scale, we get 58 parameters for the K3 metric, in agreement with Hawking and Pope [1978]. This same result may also be found by observing that for the K3 surface, $b_2^+ = 19$, $b_2^- = 3$, so that one may explicitly construct the required deformations from the harmonic forms. One finds

$$I = 3 \times 19 = 57$$

as before.

The basic formula given above, of course, needs modification when the manifold in question has a boundary. The number of zero-frequency modes for self-dual (Riemann tensor) asymptotically locally Euclidean spaces with boundary $L(k+1, 1)$ has been determinated directly (Hawking and Pope [1978]). The result is

$$I = 3(k+1) - 6 = 3k - 3$$

plus a scale. Thus the Eguchi–Hanson metric [1978], which has $k = 1$, possesses no self-dual deformations apart from a scale.

6. *Asymptotically locally Euclidean self-dual manifolds.* The general concept of manifolds with self-dual Riemann tensor and asymptotic regions which are lens spaces $L(k+1, 1)$ of S^3 was introduced earlier (10.2.2). Hitchin [1979] and Calabi [1979] have examined the most general possible regular self-dual manifolds with asymptotically locally Euclidean (ALE) infinities. The complete classification of the spherical forms of S^3 is well-known (Wolf [1967]); the possible spaces which correspond to ALE infinities are:

Series A_k:	*cyclic group* of order k (=lens spaces $L(k+1, 1)$)
Series D_k:	*dihedral group* of order k
T:	*tetrahedral group*
O:	*octahedral group* $\approx$ *cubic group*
I:	*icosahedral group* $\approx$ *dodecahedral group.*

A_1 corresponds to the Eguchi–Hanson metric [1978] and A_k to the multicenter generalization of Gibbons and Hawking [1978]. We note that one must actually use the binary or double-covering groups D_k^*, T^*, O^*, I^* of D_k, T, O, I to avoid singularities in physical ALE spaces.

Complex algebraic manifolds whose boundaries correspond to each spherical form have been identified as follows, where x, y and z are all complex:

Group	Algebraic 4-manifold
A_k	$\begin{cases} z^{k+1} = xy \\ z^{k+1} + x^2 + y^2 = 0 \end{cases}$
D_k	$z^{k-1} + x^2 + y^2 z = 0$
T	$x^2 + y^3 + z^4 = 0$
O	$x^2 + y^3 + yz^3 = 0$
I	$x^2 + y^3 + z^5 = 0.$

(10.3)

These equations are, in fact, prominent in algebraic geometry (Brieskorn [1968]); they are the unique set of algebraic equations of their type which possess resolvable singularities.

The Atiyah–Patodi–Singer η-invariant, the Euler characteristic, and the signature have been calculated for each of these cases by Gibbons, Pope and Römer [1979]. They find (our signs differ):

	χ	τ	$-\xi_{1/2} = \frac{1}{2}\eta_{\text{Dirac}}$
A_k	$k+1$	$-k$	$[(k+1)^2 - 1]/12(k+1)$
D_k^*	$k+1$	$-k$	$[4(k-2)^2 + 12(k-2) - 1]/48(k-2)$
T^*	7	-6	167/288
O^*	8	-7	383/576
I^*	9	-8	1079/1440

The values of the spin $\frac{1}{2}$ index all vanish, while the spin $\frac{3}{2}$ index for each case is 2τ.

7. *Proof of positivity of the energy and the action in general relativity* (Schoen and Yau [1978, 1979a, b, c]). The positivity of the gravitational mass or energy has long been conjectured on physical grounds, but until recently, mathematical proofs existed only for special cases. Recently Schoen and Yau produced a general proof of the positive energy conjecture using differential geometry and classical analysis.

By using the observation (Gibbons, Hawking and Perry [1978]) that the positivity of the energy in five dimensions is closely related to the positivity of the action in four dimensions, Schoen and Yau then succeeded in proving the (original) positive action conjecture stated in the previous section 10.1.

The Euclidean path integral approach to gravity, which depends in part on the positivity of the action, is on a much firmer mathematical footing as a consequence of these results.

8. *Applications of the index theorems to gravity*. We have already noted that the anomalous divergences of axial currents noted by physicists are, when integrated, closely related to mathematical index theorems. (The anomalous divergence of the axial vector current in an external gravitational field was first computed using physicists' methods before the relation of the anomaly to index theory was realized. See Delbourgo and Salam [1972] and Eguchi and Freund [1976].) A great deal of attention has consequently been paid to the application of index theory to operators in the presence of Euclidean

gravity, i.e., operators on Riemannian manifolds (Eguchi, Gilkey and Hanson [1978]; Römer and Schroer [1977]; Nielsen, Römer and Schroer [1977, 1978]; Pope [1978]; Christensen and Duff [1978]; Nielsen, Grisaru, Römer and Van Nieuwenhuizen [1978]; Perry [1978]; Critchley [1978]; Hawking and Pope [1978b]; Hanson and Römer [1978]; Christensen and Duff [1979]; Römer [1979]). One can, of course, also treat the case where connections on principal bundles are included. We present here a discussion of some of the major results. A tabulation of formulas and the index properties of various manifolds is given in the appendices.

Euler characteristic: The Euler characteristic χ is the index of the Euler complex, which deals with the exterior derivative mapping even-dimensional forms to odd-dimensional forms. The Euler characteristic gives the number of zeroes of vector fields on the manifold. If the manifold has a boundary, the index formula has differential geometric surface corrections (Chern [1945]), but no nonlocal or analytic corrections.

Hirzebruch signature: The Hirzebruch signature τ is the index I_S of the signature complex, which deals with the exterior derivative operator mapping self-dual forms to anti-self dual forms. The signature is nonzero in dimensions which are multiples of 4 and gives the difference between the number of harmonic self-dual forms and anti-self-dual forms of the middle dimension. The signature is one-third the Pontrjagin number P_1 in 4 dimensions,

$$I_S = \tau = \tfrac{1}{3}P_1.$$

If the manifold has a boundary, there exist both a local surface correction and a non-local Atiyah–Patodi–Singer (APS) η-invariant correction; the meaning of the signature is altered to include only (anti)-self-dual harmonic forms which obey the APS boundary conditions.

$\hat{A}$ genus (*Dirac, spin* 1/2 *index*): The $\hat{A}$ genus is the index $I_{1/2}$ of the Dirac complex, which deals with the spin $\frac{1}{2}$ Dirac operator mapping positive chirality spinors into negative chirality spinors. The $\hat{A}$ genus is an integer if the manifold is a spin manifold, and gives the difference between the number of positive chirality and negative chirality normalizable zero-frequency solutions to the Dirac equation. In 4 dimensions the Dirac index formula is related to the signature by

$$I_{1/2} = \hat{A} = -\tfrac{1}{8}\tau = -\tfrac{1}{24}P_1.$$

If the manifold has a boundary, there are both local boundary corrections and nonlocal η-invariant corrections; the corresponding zero-frequency solutions to the Dirac equation must obey the APS boundary conditions.

Rarita–Schwinger, spin 3/2 *index*: This index theorem deals with the spin $\frac{3}{2}$ Rarita–Schwinger operator mapping positive chirality spin $\frac{3}{2}$ wave functions into negative chirality spin $\frac{3}{2}$ wave functions. The spin $\frac{3}{2}$ wave functions are familiar to physicists, but the corresponding bundles are mathematically subtle; the accepted practice at present (Römer [1979]) is to define the Rarita–Schwinger ± chirality bundles as the *virtual bundles* (see section 6.5 on K theory)

$$\Delta^+_{3/2}(M) = \Delta_{1,1/2}(M) \ominus 2\Delta_{1/2,0}(M)$$

$$\Delta^-_{3/2}(M) = \Delta_{1/2,1}(M) \ominus 2\Delta_{0,1/2}(M),$$

where

$$\Delta_{m/2,n/2}(M) = S^m\Delta_+(M) \otimes S^n\Delta_-(M).$$

$\Delta_{\pm}(M)$ are the $\pm$ chirality bundles and S^r denotes the r-fold symmetric tensor product. The Rarita–Schwinger index is related to the signature by

$$I_{3/2} = \tfrac{21}{8}\tau = \tfrac{21}{24}P_1,$$

where $I_{3/2}$ is the difference between the number of positive chirality and the negative chirality zero frequency solutions of the Rarita–Schwinger equation. If the manifold has a boundary, there are both local boundary corrections and nonlocal η-invariant corrections, and the corresponding zero-frequency wave functions must obey the APS boundary conditions. The calculation of the η-invariant corrections is nontrivial; at present, they have been computed only for cases where the G-index theorem could be used to reduce the calculation to an algebraic form (Hanson and Römer [1978]; Römer [1979]; Gibbons, Pope and Römer [1979]). Direct construction of spin $\frac{3}{2}$ zero-frequency modes can be carried out using the method of Hawking and Pope [1978b], but it is difficult to show that there are no other solutions satisfying the Atiyah–Patodi–Singer boundary conditions without using the index theorem.

General spin index theorems: Christensen and Duff [1979] and Römer [1979] have examined the general-spin elliptic complexes

$$D_{m/2,n/2}\colon \Delta_{m/2,n/2}(M) \to \Delta_{n/2,m/2}(M)$$

where $\Delta_{m/2,n/2}$ was defined above and $D_{m/2,n/2}$ is an appropriate elliptic operator. They find that the index theorem takes the form

$$I_{m/2,n/2}[M] = -\frac{(m+1)(n+1)}{720}\{n(n+2)(3n^2+6n-14) - m(m+2)(3m^2+6m-14)\}\, P_1[M]. \quad (10.4)$$

In particular, one recovers the Dirac results

$$I_{1/2} \equiv I_{1/2,0} = -\tfrac{1}{24}P_1[M].$$

If the manifold has a boundary, surface corrections and η-invariant corrections must be applied. Römer [1979] has calculated the η-invariant corrections for a variety of interesting cases using G-index theory. For example, for the Eguchi–Hanson metric [1978], which has $P_3(\mathbb{R})$ as the boundary and no local surface corrections, the non-local boundary correction to the index is

$$\xi_{m/2,n/2}[P_3(\mathbb{R})] = \tfrac{1}{32}(m+1)(n+1)[(-1)^m - (-1)^n].$$

When one includes the effect of a principal G-bundle or vector bundle V_G with structure group G for a 4-dimensional manifold with no boundary corrections, Römer [1979] finds the full index

$$I^G_{m/2,n/2} = \dim V_G \cdot I_{m/2,n/2}[M] + \tfrac{1}{6}(m+1)(n+1)[m(m+2) - n(n+2)]\, \mathrm{ch}_2(V_G[M]), \quad (10.5)$$

where ch_2 denotes the Chern character on V_G integrated over its Λ^4 component.

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Appendix A: Miscellaneous formulas

1. *Manifolds.*

Tangent frame basis: $E_\mu = \dfrac{\partial}{\partial x^\mu}$; $E'_\nu = E_\mu \phi^\mu{}_\nu$

Cotangent frame basis: $e^\mu = \mathrm{d}x^\mu$; $e'^\mu = (\phi^{-1})^\mu{}_\nu e^\nu$

Transition function: $(\phi_{UU'})^\mu{}_\nu = \partial x^\mu / \partial x'^\nu$

Inner product: $\langle \partial/\partial x^\mu, \mathrm{d}x^\nu \rangle = \delta_\mu{}^\nu$

Vector field: $V = v^\mu \, \partial/\partial x^\mu$

Covector field: $P = p_\mu \, \mathrm{d}x^\mu$

Boundary: If $\dim(M) = n$, then $\dim(\partial M) = n - 1$.

$\partial\partial M = \emptyset$ (empty).

2. *Differential forms.* n = dimension of manifold. ω_p = p-form.

Wedge product: $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$, $\mathrm{d}x \wedge \mathrm{d}x = 0$

p form: $\omega_p = f_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$

Exterior derivative: $\mathrm{d}\omega_p = \mathrm{d}(f_{\mu\nu\ldots}(x) \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ldots)$

$= \partial_\lambda f_{\mu\nu\ldots}(x) \, \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \cdots = (p+1)\text{-form}$

$\mathrm{d}\mathrm{d}\omega_p = 0$

Dual: $*(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) = \dfrac{|g|^{1/2}}{(n-p)!} \varepsilon^{\mu_1, \ldots, \mu_p}{}_{\mu_{p+1}, \ldots, \mu_n} \, \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$

General forms: $\omega_p \wedge \omega_q = (-1)^{qp} \omega_q \wedge \omega_p$

$\mathrm{d}(\omega_p \wedge \omega_q) = \mathrm{d}\omega_p \wedge \omega_q + (-1)^p \omega_p \wedge \mathrm{d}\omega_q$

$** \, \omega_p = (-1)^{p(n-p)} \omega_p$

$\omega_p \wedge * \omega_q = \omega_q \wedge * \omega_p$

Coderivative: $\delta\omega_p = (-1)^{np+n+1} * \mathrm{d} * \omega_p = (p-1)\text{-form}$
(for positive signature metrics)

$\delta\delta\omega_p = 0$

Inner product: Let α_p and β_p be p-forms, M compact, $\partial M = \emptyset$.

$$(\alpha_p, \beta_p) = (\beta_p, \alpha_p) = \int_M \alpha_p \wedge * \beta_p$$

$$(\mathrm{d}\alpha_p, \beta_{p+1}) = (\alpha_p, \delta\beta_{p+1}); (\alpha_p, \mathrm{d}\beta_{p-1}) = (\delta\alpha_p, \beta_{p-1})$$

Laplacian: $\Delta\omega_p = (\mathrm{d}+\delta)^2\omega_p = (\mathrm{d}_{p-1}\delta_p + \delta_{p+1}\,\mathrm{d}_p)\omega_p = p\text{-form}$

Coordinate Laplacian: $\Delta\phi(x) = -|g|^{-1/2}\,\partial_\mu(g^{\mu\nu}|g|^{1/2}\,\partial_\nu)\phi(x)$

Stokes' Theorem: $\displaystyle\int_M \mathrm{d}\omega_{p-1} = \int_{\partial M} \omega_{p-1}$, where $\dim(M) = p$.

Hodge's theorem: $\omega_p = \mathrm{d}\alpha_{p-1} + \delta\beta_{p+1} + \gamma_p, \qquad \Delta\gamma_p = 0$

3. *Homology and cohomology.*

Homology: Z_p = cycles (p-chains a_p, with $\partial a_p = \emptyset$)

B_p = boundaries (p-chains b_p, with $b_p = \partial a_{p+1}$ for some a_{p+1})

$H_p = Z_p/B_p$ (homology = cycles modulo boundaries)

Cohomology: Z^p = closed forms (p-forms ω_p, with $\mathrm{d}\omega_p = 0$)

B^p = exact forms (p-forms ω_p, with $\omega_p = \mathrm{d}\alpha_{p-1}$ for some α_{p-1})

$H^p = Z^p/B^p$ (cohomology = closed modulo exact forms)

de Rham's theorem: H^p (de Rham) $\approx H^p$ (simplicial) $\approx H_p$ (simplicial)

Poincaré duality: $\dim H^p(M;\mathbb{R}) = \dim H^{n-p}(M;\mathbb{R})$, M orientable

Betti numbers: $b_p = \dim H^p = \dim H_p$ = number of harmonic p-forms γ_p, $\Delta\gamma_p = 0$

4. *Riemannian manifolds.* $g_{\mu\nu}$ = curved metric on M, η_{ab} = flat metric

Metric: $\mathrm{d}s^2 = \mathrm{d}x^\mu g_{\mu\nu}\,\mathrm{d}x^\nu = e^a\eta_{ab}e^b$

Vierbein basis of $T^*(M)$: $e^a = e^a{}_\mu\,\mathrm{d}x^\mu$

$$T(M):\quad E_a = E_a{}^\mu \frac{\partial}{\partial x^\mu} = \eta_{ab}g^{\mu\nu}e^b{}_\nu \frac{\partial}{\partial x^\mu}$$

Connection one-form: $\omega^a{}_b = \omega^a{}_{b\mu}\,\mathrm{d}x^\mu$

Cartan structure equations:

$$\text{torsion} = T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b$$

$$\text{curvature} = R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

Cartan identities: $\mathrm{d}T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b$

$\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0$ (*Bianchi identity*)

Frame change: $\eta_{ab}\Phi^a{}_c\Phi^b{}_d = \eta_{cd}$

$e'^a = \Phi^a{}_b e^b$

$\omega'^a{}_b = (\Phi\omega\Phi^{-1} + \Phi\,\mathrm{d}\Phi^{-1})^a{}_b$

$T'^a = \Phi^a{}_b T^b$

$R'^a{}_b = (\Phi R \Phi^{-1})^a{}_b$

Levi–Civita
connection: 1. $T^a = 0$ (torsion free)

2. $\omega_{ab} = -\omega_{ba}$ (covariant constant metric)

These imply the *cyclic identity*, $R^a{}_b \wedge e^b = 0$.

5. *Complex manifolds.* $z_k = x_k + \mathrm{i}y_k$, $\bar{z}_k = x_k - \mathrm{i}y_k$

$$\partial f = \frac{\partial f}{\partial z_k}\,\mathrm{d}z^k = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} - \mathrm{i}\frac{\partial f}{\partial y_k}\right)\mathrm{d}z^k$$

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}_k}\,\mathrm{d}\bar{z}_k = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} + \mathrm{i}\frac{\partial f}{\partial y_k}\right)\mathrm{d}\bar{z}_k$$

Exterior derivative: $\mathrm{d} = \partial + \bar{\partial}$

Hermitian metric: $\mathrm{d}s^2 = g_{jk}\,\mathrm{d}z^j\,\mathrm{d}\bar{z}^k$, g_{jk} = hermitian

Kähler form: $K = \bar{K} = \frac{\mathrm{i}}{2} g_{jk}\,\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k$, g_{jk} = hermitian

6. *Some useful differential forms for practical calculations.*
Two dimensions: $x = r\cos\theta$, $y = r\sin\theta$ $0 \le \theta < 2\pi$

$$\begin{pmatrix}\mathrm{d}r\\ r\,\mathrm{d}\theta\end{pmatrix} = \begin{pmatrix} x/r & y/r\\ -y/r & x/r\end{pmatrix}\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix}, \qquad \mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}\theta$$

Three dimensions: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$\rho^2 = x^2 + y^2 = r^2\sin^2\theta \qquad 0 \le \theta < \pi,\quad 0 \le \phi < 2\pi$$

$$\begin{pmatrix}\mathrm{d}r\\ r\,\mathrm{d}\theta\\ r\sin\theta\,\mathrm{d}\phi\end{pmatrix} = \begin{pmatrix} x/r & y/r & z/r\\ xz/r\rho & yz/r\rho & -\rho/r\\ -y/\rho & x/\rho & 0\end{pmatrix}\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\\ \mathrm{d}z\end{pmatrix}$$

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2\sin\theta\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$$

$$r^{-3}(x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y) = \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi$$

Four dimensions:

(Instead of using the ordinary polar coordinates, we exploit the relationship between S^3 and SU(2))

$$z_1 = x + \mathrm{i}y = r\cos\frac{\theta}{2}\exp\frac{\mathrm{i}}{2}(\psi + \phi)$$

$$z_2 = z + \mathrm{i}t = r\sin\frac{\theta}{2}\exp\frac{\mathrm{i}}{2}(\psi - \phi)$$

$$0 \le \theta < \pi, \quad 0 \le \phi < 2\pi, \quad 0 \le \psi < 4\pi$$

$$\begin{pmatrix} \mathrm{d}r \\ r\sigma_x \\ r\sigma_y \\ r\sigma_z \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x & y & z & t \\ -t & -z & y & x \\ z & -t & -x & y \\ -y & x & -t & z \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \\ \mathrm{d}t \end{pmatrix} = \frac{1}{2r}\begin{pmatrix} \bar z_1 & \bar z_2 & z_1 & z_2 \\ \mathrm{i}z_2 & -\mathrm{i}z_1 & -\mathrm{i}\bar z_2 & \mathrm{i}\bar z_1 \\ z_2 & -z_1 & \bar z_2 & -\bar z_1 \\ -\mathrm{i}\bar z_1 & -\mathrm{i}\bar z_2 & \mathrm{i}z_1 & \mathrm{i}z_2 \end{pmatrix}\begin{pmatrix} \mathrm{d}z_1 \\ \mathrm{d}z_2 \\ \mathrm{d}\bar z_1 \\ \mathrm{d}\bar z_2 \end{pmatrix}$$

$\mathrm{d}\sigma_x = 2\sigma_y \wedge \sigma_z$, cyclic (Maurer–Cartan structure equation)

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t = r^3\mathrm{d}r \wedge \sigma_x \wedge \sigma_y \wedge \sigma_z = \tfrac{1}{4}\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar z_1 \wedge \mathrm{d}\bar z_2$$

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}t^2 = \mathrm{d}r^2 + r^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2) = \mathrm{d}z_1\,\mathrm{d}\bar z_1 + \mathrm{d}z_2\,\mathrm{d}\bar z_2$$

Minkowski space: $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$, $\quad \epsilon_{0123} = +1$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}$$

Hodge $*$: $\quad *\mathrm{d}t = -\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$

$*(\mathrm{d}x^1 \wedge \mathrm{d}t) = +\mathrm{d}x^2 \wedge \mathrm{d}x^3$, cyclic

$*(\mathrm{d}x^2 \wedge \mathrm{d}x^3) = -\mathrm{d}x^1 \wedge \mathrm{d}t$, cyclic

Laplacian: $\quad \Delta = \mathrm{d}\delta + \delta\mathrm{d} = +\dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial\boldsymbol{x}\cdot\partial\boldsymbol{x}}$

Maxwell's equations: $\quad A = -A^0\,\mathrm{d}t + \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}$

$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \quad E^i = F^{0i} = -(\partial A^i/\partial t + \partial A^0/\partial x^i)$

$F = \mathrm{d}A = \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{x} \wedge \mathrm{d}t + \tfrac{1}{2}B_i\epsilon_{ijk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k$

$**F = -F; \quad *F = \pm\mathrm{i}F \to \boldsymbol{E} = \pm\mathrm{i}\boldsymbol{B}.$

7. *Determining the Levi-Civita connection*. Let $\omega_{ab} = -\omega_{ba}$ and $\mathrm{d}e^a = c^a_{0i}e^0 \wedge e^i + c^a_{23}e^2 \wedge e^3 + c^a_{31}e^3 \wedge e^1 + c^a_{12}e^1 \wedge e^2 = -\omega^a{}_b \wedge e^b$. Then

$$\omega^0{}_1 = e^0[-c^0_{01}] + e^1[-c^1_{01}] + e^2(\tfrac{1}{2})(c^0_{12} - c^1_{02} - c^2_{01}) + e^3(\tfrac{1}{2})(-c^0_{31} - c^3_{01} - c^1_{03})$$

$$\omega^0{}_2 = e^0[-c^0_{02}] + e^1(\tfrac{1}{2})(-c^0_{12} - c^1_{02} - c^2_{01}) + e^2(-c^2_{02}) + e^3(\tfrac{1}{2})(c^0_{23} - c^2_{03} - c^3_{02})$$

$$\omega^0{}_3 = e^0[-c^0_{03}] + e^1(\tfrac{1}{2})(c^0_{31} - c^3_{01} - c^1_{03}) + e^2(\tfrac{1}{2})(-c^0_{23} - c^2_{03} - c^3_{02}) + e^3(-c^3_{03})$$

$$\omega^2{}_3 = e^0(\tfrac{1}{2})(c^3_{02} - c^0_{23} - c^2_{03}) + e^1(\tfrac{1}{2})(c^2_{31} + c^3_{12} - c^1_{23}) + e^2(-c^2_{23}) + e^3(-c^3_{23})$$

$$\omega^3{}_1 = e^0(\tfrac{1}{2})(c^1_{03} - c^0_{31} - c^3_{01}) + e^1(-c^1_{31}) + e^2(\tfrac{1}{2})(c^3_{12} + c^1_{23} - c^2_{31}) + e^3(-c^3_{31})$$

$$\omega^1{}_2 = e^0(\tfrac{1}{2})(c^2_{01} - c^0_{12} - c^1_{02}) + e^1(-c^1_{12}) + e^2(-c^2_{12}) + e^3(\tfrac{1}{2})(c^1_{23} + c^2_{31} - c^3_{12})$$

8. *n-sphere metrics.* $R^2 = \Sigma_{i=1}^{n+1} X_i^2 =$ the (constant) radius of the sphere.

S^2: $\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$

$$e^i = (R\,\mathrm{d}\theta, R\sin\theta\,\mathrm{d}\phi); \quad \omega^1{}_2 = -\cos\theta\,\mathrm{d}\phi; \quad R^1{}_2 = \frac{1}{R^2} e^1 \wedge e^2$$

S^3: $\mathrm{d}s^2 = R^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2)$

$$e^i = (R\sigma_x, R\sigma_y, R\sigma_z); \quad \omega^2{}_3 = \sigma_x, \quad \omega^3{}_1 = \sigma_y, \quad \omega^1{}_2 = \sigma_z$$

$$R^2{}_3 = \frac{1}{R^2} e^2 \wedge e^3, \text{ cyclic}$$

S^4: $\mathrm{d}s^2 = (\mathrm{d}r^2 + r^2[\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2])/[1 + (r/2R)^2]^2$

$$[1 + (r/2R)^2]e^a = (\mathrm{d}r, r\sigma_x, r\sigma_y, r\sigma_z)$$

$$\omega_{i0} = \sigma_i(1 - (r/2R)^2)/(1 + (r/2R)^2)$$

$$\omega_{23} = \sigma_x, \quad \omega_{31} = \sigma_y, \quad \omega_{12} = \sigma_z$$

$$R^{ab} = \frac{1}{R^2} e^a \wedge e^b$$

S^n Cartesian metric: $r^2 = \sum_{i=1}^{n} (x^i)^2$

$$\mathrm{d}s^2 = \mathrm{d}x^i\,\mathrm{d}x^i/[1 + (r/2R)^2]^2$$

$$e^i = \mathrm{d}x^i/[1 + (r/2R)^2]$$

volume element $= e^1 \wedge e^2 \cdots \wedge e^n = \mathrm{d}^n x/[1 + (r/2R)^2]^n$

$$V(S^n) = \text{volume} = 2\pi^{(n+1)/2} R^n / \Gamma(\tfrac{1}{2}(n+1))$$

$$V(S^0, S^1, \ldots) = \left(2, 2\pi R, 4\pi R^2, 2\pi^2 R^3, \frac{8\pi^2}{3} R^4, \ldots\right)$$

$$\omega^i{}_j = \frac{x^i\,\mathrm{d}x^j - x^j\,\mathrm{d}x^i}{2R^2[1 + (r/2R)^2]}$$

$$R^i{}_j = \frac{1}{R^2} e^i \wedge e^j; \quad R_{ijkl} = \frac{1}{R^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

$$\mathcal{R}_{ij} = \frac{N-1}{R^2}\delta_{ij}, \qquad \mathcal{R} = \frac{N(N-1)}{R^2}, \qquad W_{ijkl} = 0.$$

9. *$P_n(\mathbb{C})$ metrics.*

Kähler form: $K = \frac{i}{2}\partial\bar{\partial}\ln\left(1 + \sum_{\alpha=1}^{n} z^\alpha \bar{z}^\alpha\right) = \pi c_1(L^*)$

Metric: $ds^2 = \frac{dz^\alpha\, d\bar{z}^\beta}{(1 + z^\gamma \bar{z}^\gamma)^2}[\delta_{\alpha\beta}(1 + z^\gamma \bar{z}^\gamma) - \bar{z}^\alpha z^\beta]$

Appendix B: Index theorem formulas

1. *Index theorems for Yang–Mills theory.*
Characteristic classes; $\dim(M) = 2, 4$; bundle V with curvature F.

$$C_1[V] = -\frac{1}{2\pi}\int_{M_2} \mathrm{Tr}\, F$$

$$C_2[V] = -k = +\frac{1}{8\pi^2}\int_{M_4} \mathrm{Tr}\, F \wedge F$$

Self-dual Yang–Mills index:

SU(2): $I_{YM} = 8k - 3$

SU(3): $I_{YM} = 12k - 8, \qquad k \geq 2\ (k = 1 \text{ is } \approx \mathrm{SU}(2))$

Spin $\frac{1}{2}$ index for $(2t + 1)$-dimensional representation of SU(2):

$I_{1/2}(t) = \frac{2}{3}t(t + 1)(2t + 1)k$

$I_{1/2}(1/2) = k$

2. *Index theorems for gravity.*
Characteristic classes, $\dim(M) = 4$;

$$P_1[M] = -\frac{1}{8\pi^2}\int_M \mathrm{Tr}\, R \wedge R$$

$$Q_1[\partial M] = -\frac{1}{8\pi^2}\int_{\partial M} \mathrm{Tr}(\theta \wedge R)$$

$\theta = \omega - \omega_0 =$ 2nd fundamental form, a connection with only normal components on ∂M

η-invariant:

$$\eta[\partial M, g] = \sum_{\{\lambda_i \neq 0\}} \mathrm{sign}(\lambda_i)|\lambda_i|^{-s}\big|_{s=0}$$

Topological invariants:

Signature:

$$\tau = \tfrac{1}{3}(P_1 - Q_1) + \xi_S, \quad \xi_S = -\eta_S$$

Euler characteristic:

$$\chi = \frac{1}{32\pi^2}\left[\int_M \epsilon_{abcd} R^a{}_b \wedge R^c{}_d - \int_{\partial M} \epsilon_{abcd}\left(2\theta^a{}_b \wedge R^c{}_d - \tfrac{4}{3}\theta^a{}_b \wedge \theta^c{}_e \wedge \theta^e{}_d\right)\right]$$

Spin $\frac{1}{2}$ index:

$$I_{1/2}[M, g] = -\tfrac{1}{24}(P_1[M] - Q_1[\partial M]) + \xi_{1/2}$$

$$\xi_{1/2} = -\tfrac{1}{2}[\eta_{1/2} + h_{1/2}]$$

$h_{1/2}$ = dimension of harmonic space

Spin $\frac{3}{2}$ index:

$$I_{3/2}[M, g] = +\tfrac{21}{24}(P_1[M] - Q_1[\partial M]) + \xi_{3/2}$$

Index of conformally self-dual gravitational perturbations; self-dual ALE metrics with infinity = $L(k+1, 1)$:

$$I_G = 3k - 3 + (\text{scale}) = 3k - 2.$$

3. *Combined Yang–Mills and gravity index.*

Let V be a bundle over a 4-manifold M, $\partial M = \emptyset$.

Spin $\frac{1}{2}$ index:

$$I_{1/2} = -\tfrac{1}{24}\dim(V)\, P_1[M] - C_2[V]$$

Spin $\frac{3}{2}$ index:

$$I_{3/2} = \tfrac{21}{24}\dim(V)\, P_1[M] + 3C_2[V]$$

If $\partial M \neq \emptyset$, replace P_1 by $P_1 - Q_1$ and add the appropriate η-invariant term.

Appendix C: Yang–Mills instantons

Yang–Mills potentials; $A = A^a_\mu \dfrac{\lambda_a}{2\mathrm{i}}\,\mathrm{d}x^\mu$

Yang–Mills field strengths: $F = \mathrm{d}A + A \wedge A = \dfrac{1}{2} F^a_{\mu\nu} \dfrac{\lambda_a}{2\mathrm{i}}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$

Yang–Mills equations: $d({}^*F)+A\wedge({}^*F)-({}^*F)\wedge A=J$

Bianchi identities: $dF+A\wedge F-F\wedge A\equiv 0$

1. *Belavin, Polyakov, Schwarz and Tyupkin* [1975] SU(2) *solution*. We take a, i, j to range from 1 to 3, μ, ν to range from 0 to 3, and define

Pauli matrices: $\lambda_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\lambda_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\lambda_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

't Hooft matrices:
$$\eta_{a\mu\nu}=\eta_{aij}=\epsilon_{aij}\qquad a,i,j=(1,2,3)$$
$$\eta_{ai0}=\delta_{ai}\qquad a,i=(1,2,3)$$
$$\eta_{a\mu\nu}=-\eta_{a\nu\mu}$$
$$\bar\eta_{a\mu\nu}=(-1)^{\delta_{\mu0}+\delta_{\nu0}}\eta_{a\mu\nu}$$

$O(4)$ matrices: self-dual, $\sigma_{\mu\nu}=\lambda_a\eta_{a\mu\nu}$: $\sigma_{ij}=\frac{1}{2}\epsilon_{ijk}\lambda_k$, $\sigma_{0i}=\frac{1}{2}\lambda_i$

and self-dual, $\bar\sigma_{\mu\nu}=\lambda_a\bar\eta_{a\mu\nu}$: $\bar\sigma_{ij}=\sigma_{ij}$, $\bar\sigma_{0i}=-\frac{1}{2}\lambda_i$

If we set $g(x)=(t-i\boldsymbol{\lambda}\cdot\boldsymbol{x})/r$, $r^2=t^2+\boldsymbol{x}^2$, then
$$g^{-1}\,dg=i\lambda_a\sigma_a=i\lambda_a\eta_{a\mu\nu}x^\mu\,dx^\nu/r^2$$
$$dg\,g^{-1}=-i\lambda_a\bar\sigma_a=-i\lambda_a\bar\eta_{a\mu\nu}x^\mu\,dx^\nu/r^2,$$

where
$$\sigma_x=\frac{1}{r^2}(y\,dz-z\,dy+x\,dt-t\,dx),\qquad d\sigma_x=+2\sigma_y\wedge\sigma_z,\qquad \text{cyclic in }(x,y,z)$$
$$\bar\sigma_x=\frac{1}{r^2}(y\,dz-z\,dy-x\,dt+t\,dx),\qquad d\bar\sigma_x=+2\bar\sigma_y\wedge\bar\sigma_z,\qquad \text{cyclic in }(x,y,z).$$

Then the BPST solutions are

Instanton $(k=1, F={}^*F)$.
first gauge:
$$A=\frac{r^2}{r^2+a^2}\,i\lambda_b\sigma_b=\frac{\lambda_b}{2i}\,dx^\mu\left(-2\frac{\eta_{b\nu\mu}x^\nu}{r^2+a^2}\right)$$
$$=g^{-1}\hat A g+g^{-1}\,dg$$
$$F=\frac{2ia^2\lambda_b}{(r^2+a^2)^2}(dr\wedge r\sigma_b+\tfrac{1}{2}r^2\epsilon_{bcd}\sigma_c\wedge\sigma_d)$$

second gauge:

$$\hat{A} = \frac{a^2}{r^2+a^2}\,\mathrm{i}\lambda_b\bar{\sigma}_b = \frac{-\lambda_b}{2\mathrm{i}}\,\mathrm{d}x^\mu\left(+2\frac{\bar{\eta}_{b\nu\mu}x^\nu}{r^2+a^2}\left(\frac{a^2}{r^2}\right)\right)$$

$$= \frac{-\lambda_b}{2\mathrm{i}}\,\bar{\eta}_{b\mu\nu}\,\mathrm{d}x^\mu\partial^\nu \ln\left(1+\frac{a^2}{r^2}\right)$$

Anti-instanton $(k=-1, F=-*F)$.
first gauge:

$$A = \frac{r^2}{r^2+a^2}\,\mathrm{i}\lambda_b\bar{\sigma}_b = \frac{-\lambda_b}{2\mathrm{i}}\,\mathrm{d}x^\mu\left(+2\frac{\bar{\eta}_{b\nu\mu}x^\nu}{r^2+a^2}\right)$$

second gauge:

$$\hat{A} = +\frac{a^2}{r^2+a^2}\,\mathrm{i}\lambda_b\sigma_b = \frac{\lambda_b}{2\mathrm{i}}\,\mathrm{d}x^\mu\left(-2\frac{\eta_{b\nu\mu}x^\nu}{r^2+a^2}\left(\frac{a^2}{r^2}\right)\right)$$

$$= g^{-1}Ag + g^{-1}\,\mathrm{d}g$$

$$\hat{F} = \frac{2\mathrm{i}a^2\lambda_b}{(r^2+a^2)^2}\left(-\mathrm{d}r\wedge r\sigma_b + \tfrac{1}{2}r^2\epsilon_{bcd}\sigma_c\wedge\sigma_d\right)$$

2. *'t Hooft* [1976b] *and Jackiw–Nohl–Rebbi* [1977] SU(2) *solutions*. Let

$$A^{(+)} = \frac{-\lambda^a}{2\mathrm{i}}\,\mathrm{d}x^\mu\,\bar{\eta}_{a\mu\nu}\,\partial^\nu \ln\phi(x)$$

and

$$A^{(-)} = \frac{-\lambda^a}{2\mathrm{i}}\,\mathrm{d}x^\mu\,\eta_{a\mu\nu}\,\partial^\nu \ln\phi(x).$$

Then if

$$\Box\phi/\phi = 0,$$

where

$$\Box = \sum_{\mu=0}^{3}(\partial^2/\partial x^\mu\,\partial x^\mu),$$

we find that

$A^{(+)}$ has $F=+*F$ (instantons)

$A^{(-)}$ has $F=-*F$ (anti-instantons).

The solutions for ϕ yielding instanton number $|k|$ are:

$$\phi = 1 + \sum_{j=1}^{k} \frac{\rho_j}{(x - x_j)^2}: \qquad \text{'t Hooft}$$

$$\phi = \sum_{j=1}^{k+1} \frac{\rho_j}{(x - x_j)^2}: \qquad \text{Jackiw–Nohl–Rebbi.}$$

Note that the $k = 1$ 't Hooft solution is obviously equal to the BPST instanton in the second gauge.

3. *Other explicit instanton solutions.* We refer the reader to Christ, Weinberg and Stanton [1978] and Corrigan, Fairlie, Templeton and Goddard [1978] for explicit applications of the results of Atiyah, Hitchin, Drinfeld and Manin [1978].

Appendix D: Gravitational instantons

Metric: $\mathrm{d}s^2 = \mathrm{d}x^\mu\, g_{\mu\nu}(x)\, \mathrm{d}x^\nu = e^a \eta_{ab} e^b$

Vierbein: $e^a = e^a{}_\mu\, \mathrm{d}x^\mu, \qquad \eta_{ab} = \text{flat}$

Levi–Civita connection: $\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$

$$\omega_{ab} = -\omega_{ba} = \omega_{ab\mu}\, \mathrm{d}x^\mu$$

Curvature: $R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d$

Cyclic identity: $R^a{}_b \wedge e^b = 0$

Bianchi identities: $\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0$

Empty-space Einstein equations: $(\mathcal{R}_{ab} = R_{ambn}\eta^{mn}, \quad \mathcal{R} = \mathcal{R}_{ab}\eta^{ab})$

$$\mathcal{R}_{ab} - \tfrac{1}{2}\eta_{ab}\mathcal{R} = 0 \qquad (\text{alternate form: } \tilde{R}^a{}_b \wedge e^b = 0, \text{ where } \tilde{R}^a{}_b = \tfrac{1}{4}\epsilon^a{}_{bef} R^{ef}{}_{cd} e^c \wedge e^d).$$

Einstein equations with matter and a cosmological constant

$$\mathcal{R}_{ab} - \tfrac{1}{2}\eta_{ab}\mathcal{R} = T_{ab} - \Lambda \eta_{ab}.$$

We list a variety of explicitly known metrics and give a table of the properties of the metrics and their corresponding manifolds.

1. *Metric of Eguchi and Hanson* [1978].

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{[1 - (a/r)^4]} + r^2(\sigma_x{}^2 + \sigma_y{}^2 + [1 - (a/r)^4]\sigma_z{}^2)$$

curvature is self-dual.

Table D.1
Properties of four-dimensional gravitational instantons
* Denotes entries which are unavailable or involve issues too complex to be abbreviated in the table. – Denotes undefined items. "No. param" gives the number of parameters of the metric. (The number of actual zero-frequency modes may be larger.)

Metric	M	∂M	Λ	Self-dual: Riemann Weyl Neither	Kähler: Yes No	χ	τ	$I_{1/2}$	$I_{3/2}$	No. param	Action
Flat space	$\mathbf{R}^4$	S^3	0	R = 0	Y	1	0	0	0	0	0
Torus	T^4	$\emptyset$	0	R = 0	Y	0	0	0	0	0	0
de Sitter	S^4	$\emptyset$	>0	W = 0	N	2	0	0	0	1	$3\pi/\Lambda$
Page	$P_2+\bar{P}_2$	$\emptyset$	>0	N	N	4	0	–	–	1	$1.8\pi/\Lambda$
$S^2\times S^2$	$S^2\times S^2$	$\emptyset$	>0	N	Y	4	0	0	0	1	$2\pi/\Lambda$
Schwarzschild	$\mathbf{R}^2\otimes S^2$	$S^1\times S^2$	0	N	N	2	0	0	*	1	$4\pi M^2$
Kerr	$\mathbf{R}^2\times S^2$	$S^1\times S^2$	0	N	N	2	0	0	*	2	$\pi M/\kappa$
Eguchi–Hanson	$T^*(P_1(\mathbf{C}))$	$P_3(\mathbf{R})$	0	R	Y	2	−1	0	−2	1	0
Taub–NUT	$\mathbf{R}^4$	distorted S^3	0	R	N	1	0	0	*	1	$4\pi M^2$
Fubini–Study	$P_2(\mathbf{C})$	$\emptyset$	>0	W	Y	3	1	–	–	1	$9\pi/4\Lambda$
$\epsilon=1$ Gibbons–Hawking	*	*	0	R	N	k	$\pm k$	0	*	*	$4\pi kM^2$
A_k ($\epsilon=0$ Gibbons–Hawking)	*	$S^3/\mathbf{Z}_{k+1}=L(k+1,1)$	0	R	Y	$k+1$	$-k$	0	$-2k$	$3k-2$	0
D_k^*	*	S^3/D_k^*	0	R	Y	$k+1$	$-k$	0	$-2k$	*	0
T^*	*	S^3/T^*	0	R	Y	7	−6	0	−12	*	0
O^*	*	S^3/O^*	0	R	Y	8	−7	0	−14	*	0
I^*	*	S^3/I^*	0	R	Y	9	−8	0	−16	*	0
Taub-bolt	$P_2(\mathbf{C})-\{0\}$	distorted S^3	0	N	N	2	−1	–	–	1	$\frac{4}{5}\cdot 4\pi M^2$
Rotating Taub-bolt	*	*	0	N	N	2	−1	–	–	2	$4\pi\|N\|M$
K3 (unknown)	K3	$\emptyset$	0	R	Y	24	−16	+2	−42	58	0

2. *Euclidean Taub–NUT metric* (*Hawking* [1977]).

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}\,dr^2 + (r^2-m^2)(\sigma_x{}^2+\sigma_y{}^2) + 4m^2\frac{r-m}{r+m}\,\sigma_z{}^2$$

curvature is self-dual.

3. *Fubini–Study metric on* $P_2(\mathbf{C})$.

$$ds^2 = \frac{dr^2 + r^2\sigma_z{}^2}{(1+\Lambda r^2/6)^2} + \frac{r^2(\sigma_x{}^2+\sigma_y{}^2)}{1+\Lambda r^2/6}$$

self-dual Weyl tensor
cosmological term Λ.

4. *Taub–NUT–De Sitter metrics* (include 1, 2, 3 in appropriate limits).

$$ds^2 = \frac{L^2-\rho^2}{4\Delta}\, d\rho^2 + (\rho^2 - L^2)(\sigma_x{}^2 + \sigma_y{}^2) + \frac{4L^2\Delta}{\rho^2 - L^2}\cdot \sigma_z{}^2$$

$$\Delta = \rho^2 - 2m\rho + l^2 + \frac{\Lambda}{4}(l^4 + 2l^2\rho^2 - \tfrac{1}{3}\rho^4)$$

these metrics are not necessarily regular
cosmological term Λ.

5. *Gibbons–Hawking multi-center metrics* [1978].

$$ds^2 = V^{-1}(\boldsymbol{x})(d\tau + \boldsymbol{\omega}\cdot d\boldsymbol{x})^2 + V(\boldsymbol{x})\, d\boldsymbol{x}\cdot d\boldsymbol{x}$$

$$\nabla V = \pm \nabla \times \boldsymbol{\omega}$$

$$V = \epsilon + 2m \sum_{i=1}^{k} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_i|}$$

$\epsilon = 1$ multi-Taub–NUT ($k = 1 \rightarrow$ Taub–NUT)

$\epsilon = 0$ multi-asymptotically locally Euclidean $\begin{cases} k = 1 \rightarrow \text{flat} \\ k = 2 \rightarrow \text{Eguchi–Hanson} \end{cases}$

self-dual or anti-self-dual curvature.

6. *Euclidean Schwarzschild metric.* (t has period $8\pi M$.)

$$ds^2 = (1 - 2M/R)\, dt^2 + \frac{1}{1 - 2M/R}\, dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

7. *Euclidean Kerr metric.* (t has period $2\pi/\kappa$, ϕ has period $2\pi\alpha/\sqrt{M^2+\alpha^2}$.)

$$ds^2 = (r^2 - \alpha^2\cos^2\theta)\left(\frac{dr^2}{r^2 - 2Mr - \alpha^2} + d\theta^2\right) + \frac{\sin^2\theta}{r^2 - \alpha^2\cos^2\theta}(\alpha\, dt - (r^2 - \alpha^2)\, d\phi)^2$$

$$+ \frac{r^2 - 2Mr - \alpha^2}{r^2 - \alpha^2\cos^2\theta}(dt - \alpha\sin^2\theta\, d\phi)^2$$

$\alpha = J/M$, Kerr parameter $\kappa = \sqrt{M^2+\alpha^2}/\{2M(M + \sqrt{M^2+\alpha^2})\}$

8. *de Sitter metric on S^4.*

$$ds^2 = [1 + (r/2R)^2]^{-2}(dr^2 + r^2\sigma_x{}^2 + r^2\sigma_y{}^2 + r^2\sigma_z)$$

curvature is not self-dual
Weyl tensor vanishes.

9. $S^2 \times S^2$ *metric.*

$$ds^2 = (1 - \Lambda r^2)\,d\tau^2 + \frac{dr^2}{(1 - \Lambda r^2)} + \frac{1}{\Lambda}(d\theta^2 + \sin^2\theta\, d\phi^2)$$

curvature is not self-dual
cosmological term Λ.

10. *Page metric* [1978b] on $P_2(\mathbb{C}) \oplus \overline{P_2(\mathbb{C})}$.

$$ds^2 = 3\Lambda^{-1}(1+\nu^2)\left\{\frac{(1-\nu^2x^2)}{[3-\nu^2-\nu^2(1+\nu^2)x^2]}\frac{dx^2}{1-x^2}\right.$$
$$\left. + 4(\sigma_x{}^2 + \sigma_y{}^2)\frac{1-\nu^2x^2}{3+6\nu^2-\nu^4} + 4\sigma_z{}^2(1-x^2)\frac{[3-\nu^2-\nu^2(1+\nu^2)x^2]}{(3+\nu^2)^2(1-\nu^2x^2)}\right\}$$

curvature is not self-dual
cosmological term Λ.

11. *Taub-bolt metric* (Page [1978a]).

$$ds^2 = \frac{r^2 - N^2}{r^2 - 2.5Nr + N^2}\cdot dr^2 + 16N^2\frac{r^2 - 2.5Nr + N^2}{r^2 - N^2}\cdot \sigma_z{}^2 + 4(r^2 - N^2)(\sigma_x{}^2 + \sigma_y{}^2)$$

curvature is not self-dual.

12. *Rotating Taub-bolt metric* (Gibbons and Perry [1979]).

$$ds^2 = \Xi(r,\theta)\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2\theta}{\Xi(r,\theta)}(\alpha\, dt + P_r\, d\phi)^2$$
$$+ \frac{\Delta}{\Xi(r,\theta)}(dt + P_\theta\, d\phi)^2$$

$$\Delta = r^2 - 2Mr + N^2 - \alpha^2$$

$$P_\theta = -\alpha\sin^2\theta + 2N\cos\theta - \frac{\alpha N^2}{N^2 - \alpha^2}$$

$$P_r = r^2 - \alpha^2 - \frac{N^4}{N^2 - \alpha^2}$$

$$\Xi(r,\theta) = P_r - \alpha P_\theta = r^2 - (\alpha\cos\theta + N)^2$$

curvature is not self-dual.

13. *K3 metric.* The K3 metric with self-dual curvature is not known. For a discussion of approximations to the K3 metric, see Page [1979c].

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