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A UNIFIED APPROACH TO THE THEORY OF SPACE-CHARGE-LIMITED CURRENTS IN AN INSULATOR WITH TRAPS

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Abstract—A direct mathematical procedure is presented for obtaining the volt–ampere characteristic in an insulating crystal having spatially homogeneous trapping centers under the condition of space-charge-limited emission into the crystal. The procedure is demonstrated in three cases. (1) The entire voltage range for a crystal with a discrete level is considered. (2) The characteristic is obtained for a crystal with traps distributed exponentially in energy. (3) The behavior with traps distributed uniformly in energy is derived as a special case of the exponentially varying distribution.

Résumé—On présente une méthode mathématique directe pour obtenir la caractéristique Volt–Ampère dans un cristal isolant ayant des centres de trappe homogènes dans l'espace sous des conditions d'émission de charge d'espace limitée dans ce cristal. La méthode est démontrée dans les trois cas suivants: (1) La gamme entière de tension pour un cristal ayant un niveau discret est obtenue. (2) La caractéristique est obtenue pour un cristal ayant des trappes dont l'énergie est distribuée exponentiellement. (3) Le comportement sous des conditions de trappes ayant une distribution d'énergie uniforme est derivé comme cas spécial de la distribution à variation exponentielle.

Zusammenfassung—Die Volt-Ampere-Kennlinie eines isolierenden Kristalls, in dem sich Einfangszentren mit räumlich gleichförmiger Verteilung befinden, wird durch ein direktes mathematisches Verfahren dargestellt unter der Bedingung einer durch die Raumladung begrenzten Elektronenemission in den Kristall. Das Verfahren wird für drei Fälle durchgeführt: (1) Der gesamte Spannungsbereich für einen Kristall mit einem diskreten Energieniveau wird in Betracht gezogen. (2) Die Kennlinie wird für einen Kristall hergestellt, bei dem die Dichte der Fallen mit der Energie exponentiell ansteigt. (3) Als Sonderfall der exponentiell variierenden Verteilung wird der Fall einer gleichförmigen Verteilung der Fallen erörtert.

INTRODUCTION

THIS paper provides a unified theoretical basis for obtaining the properties of space-charge-limited current flow in an insulator having an arbitrary distribution of traps in energy. Its aim is not a mathematically rigorous treatment, because rigor in this case leads to implicit equations between voltage and current and to a loss of the physical picture which underlies the observed phenomena. Instead, the current–voltage relationships are obtained by extending an approach suggested by LAMPERT[1] and used by him to handle the discrete-energy trapping level at low voltages. Analysis of energetically distributed traps is appropriate because of the correspondence observed between the theory developed and experiments[2] involving insulating crystals, principally CdS. This experimental evidence has prompted the choice of the various trapping distributions which are treated mathematically in this paper. Space-charge-limited currents in crystals having traps distributed in energy were first treated in an heuristic fashion by ROSE.[3] The results, obtained by the method presented here, agree with ROSE's conclusions. The usefulness of the approach in this paper lies chiefly in the unity of concept it brings to the general case of space-charge-limited currents in crystals containing traps.

The problem considered is the determination of the volt–ampere characteristic for space-charge-limited currents in an insulating crystal with traps. The analysis is one-dimensional, with x denoting distance. The only free-charge carriers considered

to be present are electrons, and recombination is taken to be negligible. The crystal is assumed to have an electron-injecting contact called the cathode, which exists at the plane $x = 0$. The collecting electrode is called the anode and is situated at the plane $x = a$. Positive bias is defined by making the anode positive with respect to the cathode. The cathode provides charge at a sufficient rate to maintain space-charge-limited flow in the crystal. Thus, the boundary condition at this contact is always a zero value for the electric field. By "insulating" is meant that the free-electron density remote from the injecting contact at zero applied voltage is negligible when compared with the free-electron density under applied positive bias.

The trapping states in the crystal are assumed to be uniformly distributed in space, but to have an arbitrary dependence upon energy. A quasi-thermal* equilibrium is postulated to exist between the free and trapped charge at every point in space.

Finally, it is assumed that only drift processes contribute to the current flow. This assumption has been proved not to lead to inaccuracies in more complete mathematical analyses which have taken account both of drift and diffusion. SHOCKLEY and PRIM,[4] for example, considered the effect of diffusion in a trap-free semiconductor under space-charge-limited conditions, and showed that, for applied voltages in excess of kT/e, diffusion was relatively unimportant in determining volt–ampere behavior. SUITS[5] also considered the complete solution for a variable-width space-charge region including traps, and reached the same conclusion.

The constraints just stated on the scope of the treatment are applicable in a sufficient number of actual cases to justify the detailed mathematical treatment.

METHOD OF ANALYSIS

The procedure in the mathematical analysis of space-charge-limited currents in a solid is straightforward. The exact treatment stems from the simultaneous solution of three integro-differential equations: two of them express the definitions of potential and of drift current. The third is Poisson's equation. The interdependences, however, are such that, in all but the simple, trap-free case and the case of a discrete-energy trapping level at low applied voltages, it is impossible to obtain from these an explicit relationship between current and voltage.

LAMPERT[1] treats exactly the case of discrete-energy traps at all applied voltages in an appendix to his work, the complexity of which points up the need for a more illustrative approach. The main body of LAMPERT's paper consists of the development of an illustrative approach and its application to the derivation of the volt–ampere characteristic for discrete trapping centers at voltages much below the trap-filling value.

The basis for the simplification proposed by LAMPERT lies in shifting attention from, on the one hand, current as a function of voltage to, on the other hand, anode charge as a function of voltage. As in the analyses of transistors and other semiconductor devices, this charge-oriented viewpoint succeeds both in providing a means for the derivation of manageable equations and in leading to a physical picture which may be readily grasped and understood. The postulate of a pure drift current at the anode ($J = -e\mu n_a E_a$), coupled with the derivation of an easily established range in magnitude for the anode field E_a, serves to focus attention on anode charge as the significant voltage-dependent variable. It is easily demonstrated,[1] under the conditions specified at the beginning of this paper: first, that the magnitude of the electric field $|E|$ increases monotonically from zero at the cathode to $|E_a|$ at the anode; and second, that:

$$E_a = -\alpha V/a \quad \text{with} \quad 1 \leqslant \alpha \leqslant 2 \tag{1}$$

where E_a is the anode field, V is the applied voltage and a is the anode-to-cathode spacing.

Therefore the actual current may always be computed within a factor of 2 from the relationship:

$$J \simeq e\mu n_a(V/a) \tag{2}$$

where J is the current density, e is the electronic charge, μ is the electron mobility and n_a is the anode charge density.

Equation (2) emphasizes the fact that the complex and varied behavior of insulators under

* By "quasi-thermal" it is meant that, at each point in the crystal, the free- and trapped-charge densities are in the proportions specified by the state densities and by the Fermi–Dirac statistical function.

conditions of charge injection is contained in the dependence of the anode free-electron density upon the applied voltage. In fact, if n_a be obtained as an explicit function of voltage, its insertion into equation (2) will yield an explicit volt–ampere characteristic, valid within a factor of 2 at all voltages.

The problem is thus to find the dependence of n_a on voltage. An approximate, but very useful method of accomplishing this follows from the application of Gauss's law to the crystal under conditions of charge injection from the cathode. Since the cathode field is defined to be zero (space-charge-limited injection), one may write:

$$E_a = -\frac{ea}{\epsilon\epsilon_0}(\bar{n}_i + \bar{n}_{ti}) \tag{3}$$

where $\epsilon\epsilon_0$ is the dielectric constant, $\bar{n}_i$ is the average value of the injected free-charge density

$$\left(\bar{n}_i = \frac{1}{a}\int_0^a n_i\,dx\right)$$

and $\bar{n}_{ti}$ is the average value of the injected trapped-charge density

$$\left(\bar{n}_{ti} = \frac{1}{a}\int_0^a n_{ti}\,dx\right)$$

Use of equation (1) to specify E_a incorporates voltage as the independent variable in equation (3) and leads to the form

$$\frac{\alpha V}{a} = \frac{ea}{\epsilon\epsilon_0}(\bar{n}_i + \bar{n}_{ti}) \tag{4}$$

To convert equation (4) to a form useful for calculation of the current–voltage dependence, however, the injected charge at the anode must be related to the average injected charge. If this relationship is expressed by the following equation

$$(n_a + n_{ta}) = \beta(\bar{n}_i + \bar{n}_{ti}) \tag{5}$$

some general comments may be made concerning β. An upper limit on β is established as follows. Since $|E|$ increases monotonically with x and the current density J is a constant with respect to x, n is monotonically decreasing from cathode to anode. The two assumptions of a constant trap density in space and a quasi-thermal equilibrium throughout the crystal assure that the total injected charge at the anode is a minimum. Hence, β is less than unity.

In the Appendix, an heuristic proof is given that β is always equal to or greater than $\frac{1}{2}$, and that its voltage dependence is slight. The value $\beta = \frac{1}{2}$ applies both to the trap-free insulator case and to the case of an insulator with traps whose occupancy is described by Maxwell–Boltzman statistics. In both these cases, an exact calculation for the spatial configuration of the injected charge is possible. The treatment which follows will neglect the slight variation with voltage possible in β in all cases. The chief justification for this step as well as for any other simplification proposed is, of course, the eventual correspondence between theory and experiment.

Combining equations (4) and (5) gives an over-all equation for the anode-charge density in terms of voltage:

$$\frac{\alpha V}{a} = \frac{ea}{\beta\epsilon\epsilon_0}(n_a + n_{ta}) \tag{6}$$

The incorporation of the relationship between n_a and n_{ta} into equation (6) permits solving for n_a in terms of V. This solution is then inserted into

$$J = e\mu n_a \alpha V/a \tag{7}$$

to obtain the volt–ampere behavior.

The relationship between n_a and n_{ta} is derived from the statistical formulae specifying these densities. Almost all cases of interest can be represented through the use of Maxwell–Boltzman statistics to describe the free-charge population density, and Fermi–Dirac statistics to describe the trapped-charge density. Under these conditions, a general form can be written for n_a in terms of n_{ta}. If the traps are distributed in energy according to the function $\rho(W)$ between the limiting energies W_{tl} and W_{tu}, this general form is

$$\left.\begin{aligned} n_{ta} &= \int_{W_{tl}}^{W_{tu}} \frac{\rho(W)dW}{1+\exp[(W-W_f)/kT]} \\ &= \int_{W_{tl}}^{W_{tu}} \frac{\rho(W)dW}{1+(N_c/n_a)\exp-[(W_c-W)/kT]} \end{aligned}\right\} \tag{8}$$

where N_c is the effective density of available states in the crystal conduction band, W_f is the Fermi-level energy at the anode, W_c is the energy of the conduction-band edge, k is Boltzman's constant and T is the absolute temperature.

Using equation (6–8), one can obtain an explicit volt–ampere characteristic for an insulating crystal having a general trapping-state configuration in energy. For application to a specific case, one first uses equation (8) in the form determined by the applicable $\rho(W)$, then integrates and employs equation (6) to obtain the dependence of n_a on V. The result of that calculation is used in equation (7) to write the explicit volt–ampere characteristic.

To proceed further, it is most advantageous to consider specific examples.

THE DISCRETE TRAPPING LEVEL

To illustrate the application of the theory that has been presented, the first case to be examined is that of the discrete-energy trapping level. For the discrete level, $\rho(W)$ in equation (8) is given by

$$\rho(W) = N_t\delta(W-W_t) \tag{9}$$

where N_t is the density of traps and $\delta(W-W_t)$ is the Dirac delta-function. Hence, equation (8) becomes

$$n_{ta} = \frac{N_t}{1+\theta N_t/n_a} \tag{10}$$

where θ is defined by the equation

$$\theta = (N_c/N_t)\exp-[(W_c-W_t)/kT] \tag{11}$$

and (W_c-W_t) is the depth of the trapping level below the conduction band.

As can be seen, θ has a special significance: it is the ratio of the free-charge density to the trapped-charge density at the anode at applied voltages low enough so that Maxwell–Boltzman statistics may be used to describe both populations. In actual crystals, θ will be much less than unity, and for materials such as CdS it is typically of the order 10^{-7}.[2]

Insertion of equation (11) into equation (10) leads to

$$\frac{\epsilon\epsilon_0\alpha\beta V}{ea^2} = n_a+\frac{n_aN_t}{n_a+\theta N_t} \tag{12}$$

If one considers first the low-voltage case where charge injection is slight and $\theta N_t \gg n_a$, then equation (12) takes the special form

$$\frac{\epsilon\epsilon_0\alpha\beta V}{ea^2} = n_a\left(1+\frac{1}{\theta}\right) \simeq \frac{n_a}{\theta} \tag{13}$$

with the approximation true in the case of $\theta \ll 1$. When the approximate form of equation (13) is inserted into equation (7), one obtains the familiar square-law solution of Lampert:

$$J = \frac{\mu\theta\epsilon\epsilon_0\alpha^2\beta V^2}{a^3} \qquad \begin{matrix} 1 < \alpha < 2 \\ \frac{1}{2} < \beta < 1 \end{matrix} \tag{14}$$

Equation (14) becomes identical with the exact form derived by Lampert with the substitutions $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. These magnitudes may be established directly for this simple case by a direct mathematical integration of the Poisson equation.

At voltages for which the approximation $\theta N_t \ll n_a$ is not valid, a complete solution of the quadratic equation for n_a in terms of V (equation 12) is necessary. A graph, drawn to logarithmic scales, of the solution of equation (12) for n_a/N_t vs. applied voltage is given in Fig. 1. The value assumed for θ in Fig. 1 was 10^{-2} to simplify graphical presentation, although such a large θ is highly unlikely in the crystals available at present.

The steep rise in n_a/N_t as V approaches the value $ea^2N_t/\alpha\beta\epsilon\epsilon_0$ is evident in Fig. 1. The absolute location of this voltage, denoted by the symbol V_T in Fig. 1, is uncertain owing to the uncertainty in the product $\alpha\beta$ appearing in its definition ($\frac{1}{2} < \alpha\beta < 2$). The bounds which n_a/N_t vs. V can have are easily located by reproducing Fig. 1 at the extreme permitted values for V_T. In previous work, V_T has been taken to be the injecting potential which provides enough charge to fill all traps, and has been denoted by the symbol V_{TFL}. Under the assumption of a uniform charge distribution residing in the traps, V_T can be evaluated separately from consideration of the geometrical capacitance. This assumption would imply that $\alpha\beta = 2$ if $V_T = V_{TFL}$.

The complete calculated curve in Fig. 1 shows that n_a is linear with voltage at both low and high positive biases. The deviation from linearity and high-power-law behavior is apparent over roughly two decades in voltage centered at $V = V_T$. The dependence on θ for the location of the lower asymptote in Fig. 1 emphasizes the increase in

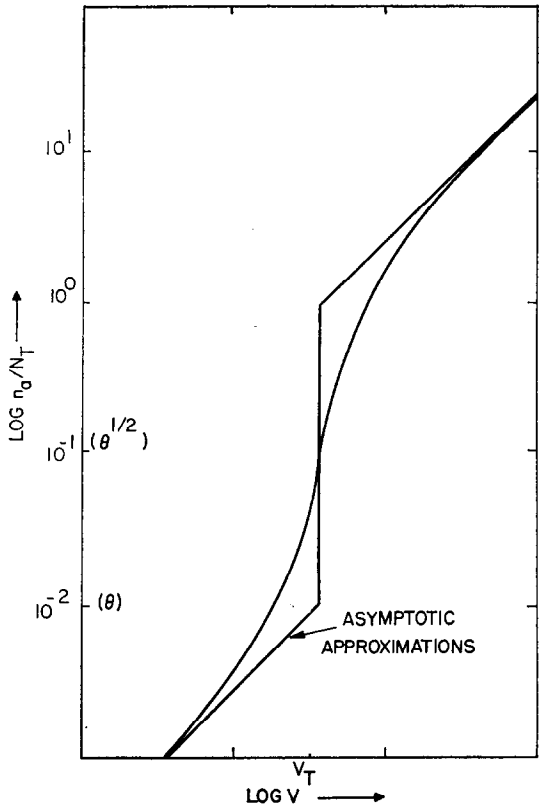

FIG. 1. The behavior of the anode free-charge density vs. voltage in a crystal with a discrete trapping center in the neighborhood of the trap-filling voltage ($V = V_T$). The plot is a solution of equation (12) under variation of the applied voltage V. The asymptotic approximations are those of LAMPERT.[1]

steepness in the variation of n_a vs. voltage with a decrease in θ. A quantitative measure of this dependence can be calculated in a fairly straightforward manner.[2]

Although the salient features of the current–voltage behavior near to trap-filling can almost be deduced by inspection of Fig. 1, for completeness, the actual solution for n_a vs. V should be inserted into equation (7) to obtain an explicit characteristic. This step leads to

$$J = \frac{e\mu\alpha V N_t}{2a}\left\{\left(\frac{V}{V_T}-1\right) + \sqrt{\left[\left(\frac{V}{V_T}-1\right)^2+\frac{4V\theta}{V_T}\right]}\right\} \qquad (15)$$

with

$$V_T = \frac{ea^2N_t}{\epsilon\,\epsilon_0\alpha\beta} \qquad 1<\alpha<2 \qquad \tfrac{1}{2}<\beta<1$$

The asymptotic square-law forms of equation (15) at both low and high voltages are easily obtained.

TRAPS VARYING IN ENERGY

Experimental evidence exists to indicate that some crystals are characterized by a trap density which increases exponentially with energy over a certain range, while others exhibit an apparently uniform trap density with energy.[2] It is possible to obtain the volt–ampere characteristics for both of these cases in a single mathematical treatment, as will now be demonstrated.

The first case to be considered is that of an insulating crystal having a distribution of traps which are exponentially varying with energy in a continuous band stretching between the two energies W_{tu} and W_{tl}. The exponential variation in density is defined (as by ROSE[3]) through a "temperature" T_c, which characterizes ρ_t, the trap-density variation with energy, by the equation

$$\rho_t = \rho_0 \exp(W-W_{tl})/kT_c \qquad (16)$$

where ρ_0 is a constant density of states per unit increment of energy.

Then, in an energy interval dW there are dN_t traps given by

$$dN_t = \rho_0 \exp[(W-W_{tl})/kT_c]\,dW \qquad (17)$$

The total number of traps per unit volume N_t is given by

$$N_t = \rho_0 kT_c\{\exp[(W_{tu}-W_{tl})/kT_c]-1\} \qquad (18)$$

and, provided $(W_{tu}-W_t) \gg |kT_c|$ and T_c is positive so that the trap density increases with W, only the first term is necessary.

From equations (8) and (17), the number of filled traps are

$$n_t = \int_{W_{tl}}^{W_{tu}} \frac{\rho_0 \exp[(W-W_{tl})/kT_c]}{1+\exp[(W-W_f)/kT]}\,dW \qquad (19)$$

This integral is not easily evaluated exactly, but an approximate form will preserve the important physical features. The approximate form is

derived simply by assuming the Fermi distribution to be unity for $W_{tl} < W < W_f$, and zero above this value, a procedure frequently used in analysis and increasingly correct as the temperature is lowered. This may be done correctly, however, only provided $|T_c| > T$ so that the numerator in equation (19) is varying more slowly than the denominator. Whether or not this is the case in practice will have to be decided on the basis of correspondence between the conclusions implied by this step and experimental data. Under this approximation, the solution for the integral, valid for $|T_c| > T$, is

$$n_t = \rho_0 kT_c\{\exp[(W_f - W_{tl})/kT_c] - 1\} \qquad (20)$$

Again, if $(W_f - W_{tl}) \gg |kT_c|$ and T_c is positive, the first term is sufficient. Combining the forms of equations (18) and (20) which use the approximations stated, one obtains

$$n_t \simeq N_t \exp[-(W_{tu} - W_f)/kT_c] \qquad (21)$$

To derive the dependence of n_t on n, one first rewrites the statistical expression for n_a

$$n_a = N_c \exp[-(W_c - W_f)/kT]$$

in the form

$$\exp(W_f/kT_c) = [(n_a/N_c)\exp(W_c/kT)]^{T/T_c} \qquad (22)$$

By use of equation (22), it is possible to eliminate W_f from equation (21), and to obtain thereby

$$n_{ta} = N_t(n_a/N_c)^{T/T_c}\exp[(W_c - W_{tu})/kT_c] \qquad (23)$$

Use of equation (6) once more to obtain n_a in terms of V under the assumption $n_a \ll n_{ta}$ leads to the form

$$n_a = \left(\frac{\alpha\beta\epsilon\,\epsilon_0 V}{ea^2 N_t}\right)^{T_c/T} N_c \exp[-(W_c - W_{tu})/kT] \qquad (24)$$

Hence, from equation (7), the dependence of current on voltage is

$$J = -\frac{e\mu N_c}{a}\left\{\left(\frac{\alpha\beta\epsilon\epsilon_0}{ea^2 N_t}\right)^{T_c/T} \exp[-(W_c - W_{tu})/kT]\right\} V^{[(T_c/T)+1]}$$

$$= -\frac{e\mu N_c}{a(V_T)^{T_c/T}} \times\{\exp[-(W_c - W_{tu})/kT]\} V^{[(T_c/T)+1]} \qquad (25)$$

where the second form expresses some of the constants in terms of V_T, defined in the first analysis.

Equation (25) shows that a trap density increasing exponentially with energy (or, therefore, a trap density which decreases when moving downward energetically from the conduction band) leads to a power-law behavior that is greater than square law. The actual power for the voltage is temperature dependent. This voltage dependence was first derived heuristically by Rose.[3]

Most of the approximations used to derive equation (25) will be met in practice. Perhaps the poorest of these is, however, the neglect of unity in the exact expressions for equations (18) and (20). In order to derive equation (25), the only specification on the exponential distribution was that $|T_c|$ needs to be greater than T. No constraints were put on the actual value for T_c which, of course, depends on the crystal properties. An increasing $|T_c|$ tends to invalidate the approximate forms for equation (18) and (20), used in deriving equation (21). Likewise, if the trap density is exponentially decreasing in energy (T_c negative), the neglect of unity with respect to the exponential terms in equations (18) and (20) is, of course, wholly unjustified. A solution is also possible, however, without these approximating steps. If the exact expressions for equations (18) and (20) are used in a series of calculations directly analogous to those used to derive equations (21) and (23), one obtains, instead of equation (25) the form

$$J = \frac{e\mu N_c}{a}\{\exp[-(W_c - W_{tl})/kT]\} V\left[1 + \frac{N_t V}{\rho_0 kT_c V_T}\right]^{T_c/T} \qquad (26)$$

Equation (26) is then, a more general solution than (25), useful when the trap distribution is characterized either by a negative T_c or by a large value for T_c. Equations (25) and (26) are

both limited in validity to voltages less than V_T by the specification that the Fermi level lie within the range of distributed traps. For T_c negative, there are further restrictions on the validity of equation (26) that stem from the approximation that all charge is trapped. This approximation was used in the derivation of equation (24). Since no experimental observations of a behavior characteristic of negative T_c have been reported, however, this case will not be considered further here.

As mentioned already, equation (26) is also valid for large T_c, i.e. for a more uniform distribution in energy of the traps (see equation 16). Therefore, equation (26) can be used to obtain the volt-ampere behavior in the case of traps distributed uniformly in energy. To do this, one recognizes the identity

$$\lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m \equiv \exp(x) \tag{27}$$

in equation (26) after allowing T_c to approach infinity. For a uniform charge density, ρ_0 is just equal to $N_t/(W_{tu}-W_{tl})$. Hence the over-all current–voltage dependence is

$$J = \frac{e\mu N_c}{a} V \exp[-(W_c-W_{tl})/kT] \exp\left[\left(\frac{W_{tu}-W_{tl}}{kT}\right)\frac{V}{V_T}\right] \tag{28}$$

The exponential dependence of current on voltage for this situation of the Fermi level traversing a uniform density of traps checks with ROSE's work.[3]

Equations (25–28) are dependent upon T_c being greater than T; this fact is used to obtain a simplified form for the integral in equation (19). For $T_c < T$, the trap density becomes much more peaked in energy, provided that the total number of traps is limited, as is necessary from considerations of physical realizability. Hence the distribution becomes more and more like a single level and is therefore characterized by the discrete-level treatment. Thus, for $T_c < T$, J tends toward a square-law behavior with voltage.

CONCLUSIONS

The unified mathematical approach developed at the beginning of this paper has been shown to yield the current–voltage characteristics in three distinct cases. The results of the calculations, which have previously been derived by other techniques, check with the work presented here. Any distribution of traps with energy may, in principle, be handled by the analysis technique discussed in this paper. In complicated cases, the integral in equation (8) may, however, become difficult or impossible to solve. Approximate methods can nonetheless be used in its evaluation, and one can thereby obtain an explicit current–voltage relationship in all cases — a distinct advantage over the direct mathematical approach. All calculations are, however, subject to whatever degree of uncertainty may exist in α or β.

The ease of calculation, when the simplified analysis presented in this paper is employed, becomes especially apparent if an attempt is made to handle the continuous trap-density cases by writing the exact forms of the Poisson equation and solving these. Besides the mathematical simplification, the conceptual value, which stems directly from the focus of viewpoint on the voltage-dependent behavior of the anode-charge density, also favors the calculation method presented here. It should, however, be realized that the complete mathematical treatment is necessary if field, potential or electron density is desired as a function of distance.

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APPENDIX

The Lower Limit in the Range of β

The lower limit for β is obtained by recognizing that the ratio of average charge to anode charge is a minimum in the case when all injected charge is free (no traps in the crystal). This can be seen by realizing that the guiding physical principle governing the ultimate charge configuration is the minimization of total energy. Thus, the charge configuration which exists when all electrons are free is that one which minimizes the

energy stored in the electric field under the constraints of a constant current and of the applied-voltage boundary conditions. If spatially uniform trapping states are available for free electrons, these traps will act to provide sites of minimum potential energy in the structure of the host lattice. Electrons tend to fill the lowest-allowed energy levels so that uniformly distributed traps result in a tendency to homogenize the space-charge configuration. A more even distribution of charge leads to an increased value for β. Hence, the existence of homogeneous immobile energy sites at energies lower than those of the conduction band will tend to spread the charge more uniformly throughout the crystal, and thereby act to increase β.

The trap-free space-charge-limited current case is one that may be solved completely for the spatial distribution of charge. This is done in Ref. 2 and the minimum value ($\beta = \frac{1}{2}$) is thereby established.

An exact statement about the voltage dependence of β is difficult to make in the absence of a specific model. As a consequence of two facts, however, it is possible to argue that this dependence will be slight. First, the narrow range permissible in the value of β precludes any large functional dependence on V other than some sort of physically unsatisfying oscillatory behavior. The second reason for the expected slight dependence on voltage stems from the way in which a voltage-dependent variable enters the applicable equations. In the exact formulation of the Poisson equation, voltage becomes a parameter in the solution for $n(x)$ and $n_t(x)$ upon the insertion of Fermi–Dirac statistics into the specification of the trapped-charge density. Since Fermi level and charge density are related only logarithmically, the over-all functional change in the equations will be slight for an incremental voltage change.