Free vibration of circular plates of arbitrary thickness

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Free vibration of circular plates of arbitrary thickness is investigated using the method of initial functions. State-space approach is used to derive the governing equations of the above method. The formulation is such that theories of any desired order can be obtained by deleting higher terms in the infinite-order differential equations. Numerical results are obtained for flexural and extensional vibration of circular plates. Results are also computed using Mindlin's theory and they are in agreement with the present analysis.

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LIST OF SYMBOLS

a	radius of the plate
$2h$	thickness of the plate
u, w	displacements in r, z directions
E	modulus of elasticity
G	modulus of rigidity
T	depth radius ratio ($=2h/a$)
α	$\partial/\partial r$
β	$\partial/\partial z$
γ^2	∇^2 ($=\alpha^2+\alpha/r$)
ρ	mass density
μ	Poisson's ratio
σ_r, σ_θ	direct stresses
τ_{rz}	shear stress
η	eigenvalue of square matrix
ω	frequency of harmonic vibration
λ	frequency parameter $[=\omega a(\rho/G)^{1/2}]$

INTRODUCTION

The thin plate theory has been extensively used for the static and dynamic analysis of plates. Due to the approximations inherent in its derivation, this theory cannot be applied to thick plates. Mindlin[1] gave a rigorous theory of plates which includes the effect of shear deformation and rotatory inertia in addition to flexure and transverse inertia. Deresiewicz and Mindlin[2] used this theory for the axially symmetric vibration of circular disk with free edge. Deresiewicz[3] solved the symmetric flexural vibration of clamped circular disk. Kane and Mindlin[4] gave a theory for the high-frequency extensional vibration of circular plates. Kalnins and Dym[5] have given a brief review of the above investigations. Reismann[6] and Reismann and Greene[7] used Mindlin's theory for the study of the dynamic response of circular and annular plates with clamped edges.

Vlasov[8] proposed the method of initial functions (MIF) for elastostatic problems in rectangular region. By the application of this method, the three-dimensional problem is reduced to a two-dimensional one and the resulting differential equation is independent of the thickness coordinate. Iyengar and Raman[9] and Rao and Das[10] extended this method for three-dimensional elastodynamic problems. Iyengar and Raman[11] used the state-space approach for studying the vibrations of rectangular beams of arbitrary depth. In the present paper the same approach is used for deriving the governing equations of the MIF in the axisymmetric vibration of circular plates. The method is used to calculate the natural frequencies of flexural and extensional vibration of circular plates. The results computed using the Mindlin's theory, thin-plate theory, and Kane–Mindlin theory are also given for comparison.

I. FORMULATION

For an elastic body without body forces, the equilibrium equations and stress displacement relations in the axisymmetric case are

$$\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{(\sigma_r-\sigma_\theta)}{r}=\frac{\rho\partial^2u}{\partial t^2}\ ,$$

$$\frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=\frac{\rho\partial^2w}{\partial t^2}\ , \tag{1}$$

$$\sigma_r=\left[\frac{2G}{(1-2\mu)}\right]\left[(1-\mu)\frac{\partial u}{\partial r}+\mu\left(\frac{u}{r}+\frac{\partial w}{\partial z}\right)\right]\ ,$$

$$\sigma_\theta=\left[\frac{2G}{(1-2\mu)}\right]\left[\frac{(1-\mu)u}{r}+\mu\left(\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}\right)\right]\ ,$$

$$\sigma_z=\left[\frac{2G}{(1-2\mu)}\right]\left[(1-\mu)\frac{\partial w}{\partial z}+\mu\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)\right]\ ,$$

$$\tau_{rz}=G\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\ . \tag{2}$$

Let

$$U=Gu,\quad W=Gw,\quad X=\tau_{rz},\quad Z=\sigma_z\ ,$$

$$\frac{\partial}{\partial r}=\alpha,\quad \frac{\partial}{\partial z}=\beta,\quad \frac{(\rho/G)\partial^2}{\partial t^2}=\xi^2\ ,$$

$$\frac{\partial^2}{\partial r^2}+\frac{(1/r)\partial}{\partial r}=\gamma^2=\nabla^2\ . \tag{3}$$

Eliminating σ_r and σ_θ between Eqs. (1) and (2) the following basic equations are obtained:

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$$\beta\begin{pmatrix} U \\ W \\ Z \\ X \end{pmatrix}=\begin{pmatrix} 0 & -\alpha & 0 & 1 \\ -[\mu/(1-\mu)](\alpha+1/r) & 0 & (1-2\mu)/2(1-\mu) & 0 \\ 0 & \xi^2 & 0 & (\alpha+1/r) \\ -[2/(1-\mu)](\gamma^2-1/r^2)+\xi^2 & 0 & -[\mu/(1-\mu)]\alpha & 0 \end{pmatrix}\begin{pmatrix} U \\ W \\ Z \\ X \end{pmatrix} . \tag{4}$$

Let Y denote the state vector

$[U W Z X]^T$.

Equation (4) can be written as

$$d/dz\,[\mathbf{Y}]=[\mathbf{A}]\,[\mathbf{Y}] . \tag{5}$$

The integration of this differential equation yields

$$[\mathbf{Y}]=\exp(z\cdot\mathbf{A})\,\mathbf{Y}(0) , \tag{6}$$

where $\mathbf{Y}(0)$ correspond to Y at $z=0$. The exponential matrix is the transfer matrix that maps the initial state vector into the field. The characteristic equation of the determinant associated with the matrix $[A]$ is

$$\{\eta^2+\gamma^2-[(1-2\mu)/2(1-\mu)]\,\xi^2\}\,[\eta^2+\gamma^2-\xi^2]=0 . \tag{7}$$

The roots are

$$\eta=\pm i\delta_1,\quad \pm i\delta_2 , \tag{8}$$

where

$$\delta_1=\{\gamma^2-[(1-2\mu)/2(1-\mu)]\,\xi^2\}^{1/2} ,$$

$$\delta_2=(\gamma^2-\xi^2)^{1/2} . \tag{9}$$

Using Cayley Hamilton theorem we can write

$$\exp(z\mathbf{A})=a_0\mathbf{I}+a_1\mathbf{A}+a_2\mathbf{A}^2+a_3\mathbf{A}^3 . \tag{10}$$

Equation (10) must also be satisfied by the eigenvalues of the matrix **A**. Hence

$$\exp(z\eta)=a_0+a_1\eta+a_2\eta^2+a_3\eta^3 . \tag{11}$$

Substituting the roots of Eq. (8) into Eq. (11) and solving, the following values are obtained for a_0, a_1, a_2, and a_3:

$$a_0=[2(1-\mu)/\xi^2]\,[\delta_1^2\cos z\delta_2-\delta_2^2\cos z\delta_1] ,$$

$$a_1=[2(1-\mu)/\xi^2]\,[\delta_1^2(\sin z\delta_2)/\delta_2-\delta_2^2(\sin z\delta_1)/\delta_1] ,$$

$$a_2=[2(1-\mu)/\xi^2]\,[\cos z\delta_2-\cos z\delta_1] ,$$

$$a_3=[2(1-\mu)/\xi^2]\,[(\sin z\delta_2)/\delta_2-(\sin z\delta_1)/\delta_1] . \tag{12}$$

Substituting these values in Eq. (10) the transfer matrix $[L]$ is obtained. From Eq. (6) we get

$$\begin{pmatrix} U \\ W \\ Z \\ X \end{pmatrix}=\begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}\begin{pmatrix} U_0 \\ W_0 \\ Z_0 \\ X_0 \end{pmatrix} , \tag{13}$$

where U_0, W_0, Z_0, and X_0 are all initial unknown functions in the plane $z=0$. The coefficients L_{11}, L_{12}, etc., of the transfer matrix have the following values:

$$L_{11}=L_{44}=(1/\xi^2)\,[2\gamma^2\cos z\delta_1-(2\gamma^2-\xi^2)\cos z\delta_2] ,$$

$$L_{12}=-(\alpha/\xi^2)\,[(2\gamma^2-\xi^2)(\sin z\delta_1)/\delta_1 - 2(\gamma^2-\xi^2)(\sin z\delta_2)/\delta_2] ,$$

$$L_{13}=(\alpha/\xi^2)\,(\cos z\delta_1-\cos z\delta_2) ,$$

$$L_{14}=(1/\xi^2)\,[\gamma^2(\sin z\delta_1)/\delta_1-(\gamma^2-\xi^2)(\sin z\delta_2)/\delta_2] ,$$

$$L_{21}=-(\gamma/\xi^2)\,[2\delta_1\sin z\delta_1-(2\gamma^2-\xi^2)(\sin z\delta_2)/\delta_2] ,$$

$$L_{22}=L_{33}=-(1/\xi^2)\,[(2\gamma^2-\xi^2)\cos z\delta_1-2\gamma^2\cos z\delta_2] ,$$

$$L_{23}=-(1/\xi^2)\,[\delta_1\sin z\delta_1-\gamma^2(\sin z\delta_2)/\delta_2] ,$$

$$L_{24}=(\gamma/\xi^2)\,[\cos z\delta_1-\cos z\delta_2] ,$$

$$L_{31}=-[2(2\gamma^2-\xi^2)\gamma/\xi^2]\,[\cos z\delta_1-\cos z\delta_2] ,$$

$$L_{32}=(1/\xi^2)\{[4\gamma^2(\gamma^2-\xi^2)+\xi^4](\sin z\delta_1)/\delta_1 - 4\gamma^2(\gamma^2-\xi^2)(\sin z\delta_2)/\delta_2\} ,$$

$$L_{34}=-(\gamma/\xi^2)\,[(2\gamma^2-\xi^2)(\sin z\delta_1)/\delta_1-2(\gamma^2-\xi^2)(\sin z\delta_2)/\delta_2] ,$$

$$L_{41}=-(1/\xi^2)\,[4\gamma^2\delta_1\sin z\delta_1-(4\gamma^2\delta_2^2+\xi^4)(\sin z\delta_2)/\delta_2] ,$$

$$L_{42}=-[2(2\gamma^2-\xi^2)\alpha/\xi^2]\,[\cos z\delta_1-\cos z\delta_2] ,$$

$$L_{43}=-(\alpha/\xi^2)\,[2\delta_1\sin z\delta_1-(2\gamma^2-\xi^2)(\sin z\delta_2)/\delta_2] . \tag{14}$$

II. APPLICATION

A. Plate subjected to antisymmetrical loading

Taking $z=0$ as the reference plane, because of antisymmetry in loading (Fig. 1),

$$U_0=Z_0=0 . \tag{15}$$

On the plane $z=\pm h$

$$Z=\pm p(r,t),\quad X=0 . \tag{16}$$

Using these conditions, Eq. (13) reduces to the following two equations:

$$\begin{pmatrix} L_{32} & L_{34} \\ L_{42} & L_{44} \end{pmatrix}_{z=h}\begin{pmatrix} W_0 \\ X_0 \end{pmatrix}=\begin{pmatrix} p \\ 0 \end{pmatrix} . \tag{17}$$

The second equation of (17) is satisfied by introducing an auxiliary function F such that

$$W_0=L_{44}^{(h)}F,\quad X_0=-L_{42}^{(h)}F . \tag{18}$$

The first equation of (17) leads to the following differential equation for free vibration:

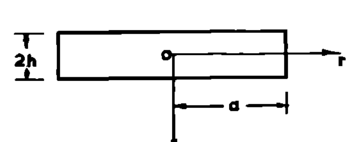

FIG. 1. Coordinate system.

$$(1/\xi^2)\,[(4\gamma^2\delta_2^2+\xi^4)\cosh h\delta_2(\sinh h\delta_1)/\delta_1 - 4\gamma^2\delta_2^2\cosh h\delta_1(\sinh h\delta_2)/\delta_2]\,F=0\;. \tag{19}$$

Equation (19) is the exact transcendental partial differential equation governing the axisymmetric flexural vibration of circular plates. Expanding the trigonometric expressions and retaining a finite number of terms, solution of a desired order can be obtained. When F is known, the values of X_0 and W_0 can be obtained from Eq. (18) and the stresses can be obtained from Eq. (13).

B. Plate subjected to symmetrical loading

Taking $z=0$ as the reference plane, because of symmetry in loading

$$W_0=X_0=0\;. \tag{20}$$

On the plane $z=\pm h$

$$Z=-p(r,t),\quad X=0\;. \tag{21}$$

Using these values of Z and X, Eqs. (13) reduce to the following two equations:

$$\begin{pmatrix} L_{31} & L_{33} \\ L_{41} & L_{43}\end{pmatrix}_{z=h}\begin{pmatrix} U_0 \\ Z_0\end{pmatrix}=\begin{pmatrix} -p \\ 0\end{pmatrix}. \tag{22}$$

By introducing an auxiliary function φ such that

$$U_0=L_{43}^{(h)}\varphi,\quad Z_0=L_{41}^{(h)}\varphi\;, \tag{23}$$

the second equation of (22) is identically satisfied and the first equation of (22) leads to the following exact partial differential equation for free vibration symmetric with respect to the middle plane:

$$(1/\xi^2)\,[4\gamma^2\delta_1^2\cosh h\delta_2(\sinh h\delta_1)/\delta_1 - (2\gamma^2-\xi^2)^2\cosh\delta_1(\sinh h\delta_2)/\delta_2]\,\varphi=0\;. \tag{24}$$

Expanding the trigonometric expressions and retaining terms up to h^3, we get a fourth-order theory. The stresses can be obtained from the auxiliary function as in the previous antisymmetric case.

C. Boundary conditions

The boundary conditions can be expressed in terms of the auxiliary function F or φ. As an illustration, the boundary conditions are given for a fourth-order theory of the antisymmetric case:

(i) Hinged edge ($w=0$, $\sigma_r=0$):

$$F=0$$
$$[\alpha^2+(\mu/r)\alpha]\,F=0\;. \tag{25}$$

(ii) Clamped edge ($w=0$, $u=0$):

$$F=0,\quad \alpha F=0\;. \tag{26}$$

(iii) Free edge ($\sigma_r=0$, $\tau_{rz}=0$).

Out of the two conditions for the free edge, one can be satisfied exactly and the remaining approximately. Assuming σ_r is satisfied exactly then

$$[\alpha^2+(\mu/r)\alpha]\,F=0\;. \tag{27a}$$

The remaining condition is obtained approximately as

$$\int_{-h}^{h}\tau_{rz}\,dr=0\;. \tag{27b}$$

D. Solution of the differential equation

As the method of solution of the differential equation for the symmetric and the antisymmetric case are identical, the method is explained for the antisymmetric case.

Expanding the trigonometric terms in Eq. (19) and retaining terms up to h^3, the following differential equation is obtained:

$$\{[2/3(1-\mu)]\,h^3\gamma^4-[2(2-\mu)/3(1-\mu)]\,h^3\gamma^2\xi^2 +[(7-8\mu)/12(1-\mu)]\,h^3\xi^4+h\xi^2\}\,F=0\;. \tag{28}$$

In the expanded form this will be

$$\{[2/3(1-\mu)]\,h^3\nabla^4-[2(2-\mu)\rho/3(1-\mu)G]\,h^3\nabla^2\,\partial^2/\partial t^2 +[(7-8\mu)\rho^2/12(1-\mu)G^2]\,h^3\,\partial^4/\partial t^4 +(h\rho/G)\,\partial^2/\partial t^2\}\,F=0\;. \tag{29}$$

For free vibration, one can assume

$$F(r,t)=F_1(r)\cos\omega t\;. \tag{30}$$

Substituting Eq. (30) in Eq. (29) and simplifying, we get the following differential equation for F_1:

$$[a^4\nabla^4+Ba^2\nabla^2+C]\,F_1=0\;, \tag{31}$$

where

$$B=(2-\mu)\lambda^2\;,$$
$$C=[(7-8\mu)/8]\,\lambda^4-[3(1-\mu)/h^2]\,a^2\lambda^2\;,$$
$$\lambda=\omega a(\rho/G)^{1/2}\;. \tag{32}$$

The solution of Eq. (31) leads to two different cases as follows:

Case 1: When $(B^2-4C)^{1/2}>B$ then the solution is

$$F_1=C_1J_0(\bar{p}r/a)+C_2I_0(qr/a)\;, \tag{33}$$

where

$$\bar{p}=\{[(B^2-4C)^{1/2}+B]/2\}^{1/2}\;,$$
$$q=\{[(B^2-4C)^{1/2}-B]/2\}^{1/2}\;. \tag{34}$$

Case 2: When $(B^2-4C)^{1/2}<B$ then the solution is

$$F_1=C_1J_0(\bar{p}r/a)+C_2J_0(\bar{q}r/a)\;, \tag{35}$$

where $\bar{p}$ is defined as in Eq. (34) and

$$\bar{q}=\{\tfrac{1}{2}[B-(B^2-4C)^{1/2}]\}^{1/2}\;. \tag{36}$$

The frequency parameter λ is obtained by substituting F_1 in the equations for the boundary conditions. In the case of eighth-order theory, the solution for F_1 is written by assuming trial values of λ. Substituting this solution in the expressions for the boundary conditions, we get a set of homogeneous simultaneous equations. For nontrivial solution the coefficient determinant should be zero. That value of λ which satisfies this condition is found. From Eq. (18) as a first approximation one obtains

$$W_0 = F \ . \tag{37}$$

In Eq. (29) if the first and last term only are retained one has

$$\{[2/3(1-\mu)]h^3\nabla^4 + (\rho h/G)\,\partial^2/\partial t^2\}F = 0 \ . \tag{38}$$

Substituting Eq. (37) in Eq. (38) and simplifying we obtain the following familiar equation of the elementary theory:

$$\{[2E/3(1-\mu^2)]h^3\nabla^4 + 2h\rho\partial^2/\partial t^2\}\bar{w} = 0 \ ,$$

where $\bar{w}$ is the transverse deflection of the middle surface.

III. NUMERICAL RESULTS

Numerical values of λ for the antisymmetric case, have been computed by the fourth and eighth order MIF theories for plates with hinged edge and clamped edge. Some of the results are given in Tables I and II for three different values of "T" the thickness radius ratio. The values computed using the thin plate theory and Mindlin's improved plate theory are also given for comparison. The values of λ for the symmetric case have been computed using the fourth-order theory for a plate with free edge. The results for the first two modes are given in Table III for five different values of T. The results computed using Kane–Mindlin theory are also given for comparison.

TABLE I. Frequency parameter λ for flexural vibration of circular plate with hinged support, Poisson's ratio = 0.3.

$T=2h/a$	Mode	Thin plate theory	Mindlin	MIF IV	MIF VIII
	1	0.1204	0.1125	0.1114	0.1125
	2	0.7251	0.7125	0.7051	0.7110
	3	1.8092	1.7500	1.7264	1.7406
	4	3.3746	3.1812	3.0362	3.1157
0.05	5	5.4215	4.6124	4.5325	4.6234
	6	7.9499	7.0312	6.9831	7.1006
	7	10.9599	9.3156	9.1653	9.3017
	8	14.4514	11.4562	11.2246	11.4115
	9	18.4246	14.4250	14.1007	14.3823
	10	22.8792	17.1749	16.6923	17.0511
	1	0.2408	0.2375	0.2371	0.2375
	2	1.4502	1.4125	1.3841	1.3972
	3	3.6184	3.0562	3.0009	3.0514
	4	6.7492	5.1562	5.1468	5.1625
0.1	5	10.8430	7.9562	7.9433	7.9641
	6	15.8999	10.7661	10.7632	10.8183
	7	21.9198	14.0249	13.6919	13.9861
	8	28.9029	17.1122	16.7314	17.1003
	9	36.8491	20.0113	19.6097	20.0027
	10	45.7585	23.3120	22.6089	23.1386
	1	0.4816	0.2562	0.2557	0.2562
	2	2.9004	2.3561	2.3143	2.3471
	3	7.2369	5.1515	5.0046	5.1194
	4	11.4985	7.8781	7.7351	7.8815
0.2	5	21.6859	10.6780	10.4863	10.6853
	6	31.7997	14.7181	13.9104	14.1812
	7	43.8396	16.5749	16.3147	16.5483
	8	57.8057	17.4874	17.0385	17.4651
	9	73.6982	19.0046	18.6059	18.9745
	10	91.5171	20.5811	19.8140	20.2900

TABLE II. Frequency parameter λ for flexural vibration of clamped circular plate, Poisson's ratio = 0.3.

$T=2h/a$	Mode	Thin plate theory	Mindlin	MIF IV	MIF VIII
	1	0.2492	0.2437	0.2402	0.2429
	2	0.9703	0.9125	0.8635	0.9109
	3	2.1739	2.0999	1.9017	2.0853
	4	3.8593	3.6297	3.3829	3.6174
0.05	5	6.0263	5.1781	5.0702	5.1369
	6	8.6750	7.6226	7.4923	7.5942
	7	11.8052	9.9750	9.7724	9.9571
	8	15.4171	12.4124	12.2751	12.3914
	9	19.5106	15.1687	14.9349	15.1199
	10	24.0856	17.7556	17.5811	17.7386
	1	0.4985	0.4875	0.4789	0.4811
	2	1.9406	1.8312	1.8057	1.8248
	3	4.3478	3.8359	3.7941	3.8254
	4	7.7186	6.2781	6.1932	6.2478
0.1	5	12.0527	9.0039	8.9105	8.8914
	6	17.3500	11.8390	11.7959	11.8186
	7	23.6105	14.8289	14.6459	14.7951
	8	30.8343	17.8381	17.6184	17.8067
	9	39.0212	20.9221	20.6815	20.8945
	10	48.1713	23.9499	23.7261	23.9137
	1	0.9970	0.9437	0.9406	0.9435
	2	3.8813	3.1812	3.1588	3.1807
	3	8.6957	5.9797	5.9420	5.9789
	4	15.4372	8.9508	8.9133	8.9520
0.2	5	24.1053	11.9335	11.8074	11.9411
	6	34.7002	14.7194	14.2779	14.5927
	7	47.2211	17.0183	17.0013	17.3362
	8	61.6686	17.3749	17.1124	17.4115
	9	78.0424	20.3684	19.9876	20.3886
	10	96.3427	20.8874	20.5431	20.9137

IV. DISCUSSION

In the case of flexural vibration, the frequency values calculated using the thin plate theory are always higher than those by MIF theory and Mindlin's theory. Though the values given by the MIF theory increase with the order of the theory, the difference between the values by the fourth order and eighth order is small. Hence

TABLE III. Frequency parameter λ for extensional vibration of a circular plate with free edge, Poisson's ratio = 0.3.

$T=2h/a$	Mode	MIF IV	Kane and Mindlin
0.5	1	3.4065	3.4285
	2	8.6943	8.7058
1.0	1	3.2972	3.3362
	2	6.7293	6.7511
1.5	1	2.7855	2.8172
	2	4.4903	4.5225
2.0	1	2.1684	2.1951
	2	4.1148	4.1416
2.5	1	1.8953	1.9136
	2	4.0089	4.0110

the fourth-order theory is accurate enough for all practical purposes. The results obtained by Mindlin's theory are slightly higher than those by the present theory. In the case of extensional vibration the values of λ calculated using Kane–Mindlin theory are slightly higher than those obtained by the fourth-order MIF theory.

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